POINCARE'S CONJECTURE AND THE HOMEOTOPY GROUP OF A CLOSED, ORIENTABLE 2-MANIFOLD

Dedicated to the memory of Hanna Neumann

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1. Introduction

In 1904 Poincaré [11] conjectured that every compact, simply-connected closed 3-dimensional manifold is homeomorphic to a 3-sphere. The corresponding result for dimension 2 is classical; for dimension $\geqq 5$ it was proved by Smale [12] and Stallings [13], but for dimensions 3 and 4 the question remains open. It has been discovered in recent years that the 3-dimensional Poincaré conjecture could be reformulated in purely algebraic terms [6, 10, 14, 15] however the algebraic problems which are posed in the references cited above have not, to date, proved tractable.

We concern ourselves herc with a new and more explicit reduction of the Poincaré conjecture to an algebraic problem. Our approach is to regard an arbitrary 3-manifold as the union of two solid handlebodies, which are sewn together along their surfaces. This identification of the surfaces is via a surface homeomorphism, which in turn corresponds to an element in the homeotopy group of the surface. [The homeotopy group of a surface is the group of outer automorphisms of the fundamental group of the surface.] Thus a correspondence can be set up between 3-manifolds and elements in the homeotopy group of a surface.

We begin in section 2 by making this correspondence explicit. We then examine how the fundamental group of the 3-manifold depends on the choice of the surface automorphism (Theorem 1). In section 3 we delineate, in the homeotopy group, the class of elements which corresponds to 3-manifolds which are homology 3-spheres, that is, their abelianized fundamental group is trivial (Theorem 2). A second subset of the homeotopy group, studied in Section 4, consists of those surface automorphisms which correspond to 3-manifolds which are homeomorphic to the 3-sphere (Theorem 3). Putting together Theorems 1, 2 and 3, we are able to reformulate the Poincaré conjecture as an explicit statement about certain subsets of the homeotopy group $H(T_g)$ of a closed, orientable surface T_g of genus g (Corollary 1). The final section of the paper, Section 5, discusses the algebraic problems which remain.

2. Constructing 3-manifolds from surface homeomorphisms

Let X_g and X'_g be handlebodies of genus g with boundaries T_g and T'_g respectively. Choose a common base point $z_0 \in T_g$ for $\pi_1 T_g$ and $\pi_1 X_g$, and a common base point z'_0 for $\pi_1 T'_g$ and $\pi_1 X'_g$. Choose canonical generators for $\pi_1 T_g$, $\pi_1 T'_g$, $\pi_1 T_g$ and $\pi_1 X'_g$ in such a way that:

(1) $$\pi_1 T_g = \langle a_1, \cdots a_g, b_1, \cdots b_g; \prod_{i=1}^{g} [a_i, b_i] \rangle$$

(2) $$\pi_1 T'_g = \langle b'_1 \cdots b'_g, a'_1, \cdots a'_g; \prod_{i=1}^{g} [b'_i, a'_i] \rangle$$

(3) $$\pi_1 X_g = \langle \hat{a}_1 \cdots, \hat{a}_g \rangle$$

(4) $$\pi_1 X'_g = \langle \hat{b}_1 \cdots \hat{b}_g \rangle$$

Let Φ be the natural homomorphism from $\pi_1 T_g$ to $\pi_1 X_g$ which is induced by the inclusion map. Similarly, let Φ' be the natural homomorphism from $\pi_1 T'_g$ to $\pi_1 X'_g$. Suppose that the action of Φ and Φ' are given by:

(5) $$(a_i)\Phi = \hat{a}_i, \qquad (b_i)\Phi = 1$$

(6) $$(a'_i)\Phi' = 1, \qquad (b'_i)\Phi' = \hat{b}_i \qquad (i = 1 \cdots, g)$$

Let $\eta : (T_g, z_0) \to (T'_g, z'_0)$ be any homeomorphism which takes representatives of a_i, b_i onto representatives of $b'_i a'_i b'^{-1}_i, b'^{-1}_i$ $1 \leqq i \leqq g$. Let $\tau : (T_g, z_0) \to (T_g, z_0)$ be any self-homeomorphism of T_g. Then τ and η can be used to construct a closed compact 3-manifold which we will denote by $X_g U_\tau X'_g$ by making the identification:

(7) $$\tau(z) = \eta(z)$$

for every point $z \in T_g$. Our 3-manifold is, of course, given as a Heegaard splitting of genus g; moreover, *every* 3-manifold which admits a decomposition as a Heegaard splitting of genus g can be obtained in this way, by allowing τ to range over the full group of homeomorphisms of T_g, or (eliminating obvious duplications) by allowing the induced automorphism τ_* to range over Aut $\pi_1 T_g$/Inn $\pi_1 T_g$ $= H(T_g)$.

Suppose that the action of τ_* is given by:

$$(8)\qquad \begin{aligned}(a_i)\tau_* &= A_i(a_1, \cdots, a_g, b_1, \cdots, b_g)\\ (b_i)\tau_* &= B_i(a_1, \cdots, a_g, b_1, \cdots, b_g),\ i = 1, \cdots, g\end{aligned}$$

Observe that the action of τ_* induces a homomorphism Φ'_τ which maps $\pi_1 T_g$ to $\pi_1 X'_g$, and which is defined by

$$(9)\qquad \begin{aligned}(A_i(a_1, \cdots a_g, b_1, \cdots, b_g))\Phi'_\tau &= 1\\ (B_i(a_1, \cdots, a_g, b_1, \cdots, b_g))\Phi'_\tau &= \hat{b}_i^{-1},\ i = 1, \cdots, g\end{aligned}$$

If we apply Van Kampen's Theorem to the 3-manifold $X_g U_\tau X'_g$, noting that $X_g \cap X'_g = T_g = T'_g$, we can obtain a presentation for the fundamental group of $X_g U_\tau X'_g$:

$$(10)\qquad \pi_1(X_g U_\tau X_g) = \left\{ \begin{matrix} \hat{a}_1, \cdots, \hat{a}_g \\ \\ \hat{b}_1, \cdots \hat{b}_g \end{matrix} \right\} : \left\{ \begin{matrix} (a_i)\tau_*\Phi = (a_i)\eta_*\Phi',\ i = 1, \cdots, g \\ \\ (b_i)\tau_*\Phi = (b_i)\eta_*\Phi',\ i = 1, \cdots, g \end{matrix} \right\}$$

Using (8), this reduces to:

$$(11)\qquad \pi_1(X_g U_\tau X'_g) = \langle \hat{a}_1, \cdots, \hat{a}_g; A_i(\hat{a}_1, \cdots, \hat{a}_g, 1, \cdots, 1)\ \ 1 \leqq i \leqq g \rangle$$

Thus we have established

THEOREM 1. *Let $\tau_* \in Aut\ \pi_1 T_g$. Let the action of τ_* be given by equation* (8). *Then the fundamental group of the three manifold $X_g U_\tau X'_g$ admits the presentation* (11).

3. Characterization of the homology 3-spheres

Let Sp($2g, Z$) denote the group of $2g \times 2g$ symplectic matrices with integral entries [5]. A natural homomorphism, which we denote by ψ, exists from Aut $\pi_1 T_g$/Inn $\pi_1 T_g$ onto Sp($2g, Z$): If $\tau_* \in$ Aut $\pi_1 T_g$ is any representative of an element $[\tau_*]$ in Aut $\pi_1 T_g$/Inn $\pi_1 T_g$, and if the action of τ_* is given by (8), and if:

$$(12)\qquad \begin{aligned} A_i(a_1, \cdots, a_g, b_1, \cdots, b_g) &= \prod_{k=1}^{g} a_k^{\tau_{i,k}} b_k^{\tau_{i,g+k}} \bmod [\pi_1 T_g, \pi_1 T_g] \\ B_i(a_1, \cdots, a_g, b_1, \cdots, b_g) &= \prod_{k=1}^{g} a_k^{\tau_{g+i,k}} b_k^{\tau_{g+i,g+k}} \bmod [\pi_1 T_g, \pi_1 T_g] \end{aligned}$$

Then the homomorphism ψ : Aut $\pi_1 T_g$/Inn $\pi_1 T_g \to$ Sp($2g, Z$) is defined by

$$(13)\qquad ([\tau_*])\psi = (\tau_{rs})$$

That is, the image of $[\tau_*]$ under ψ is the matrix whose entries are the exponents of a_k and b_k occuring in (12). Let K denote the kernel of ψ.

Let $N(x_1, \cdots, x_r)$ be the smallest normal subgroup of $\pi_1 T_g$ containing the elements $x_1, \cdots, x_r$ of $\pi_1 T_g$. Define subgroups A, B of Aut $\pi_1 T_g$/Inn $\pi_1 T_g$ by:

$$A = \{[\tau_*]/(N(a_1, \cdots, a_g))\tau_* \subseteq N(a_1, \cdots, a_g)\} \tag{14}$$
$$B = \{[\tau_*]/(N(b_1, \cdots, b_g))\tau_* \subseteq N(b_1, \cdots, b_g)\}$$

We assert:

THEOREM 2: *$(X_g U_\tau X'_g)$ is a homology 3-sphere if and only if $[\tau_*] \in AKB$*

PROOF. If $\tau_* \in \text{Aut}\, \pi_1 T_g$, then $([\tau_*])\psi$ will be a matrix in the group $\text{Sp}(2g, Z)$, that is a $2g \times 2g$ matrix of the form

$$([\tau_*])\psi = (\tau_{ij}) = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \tag{15}$$

where Y_1, Y_2, Y_3, Y_4 are $g \times g$ matrices of integers satisfying the symplectic conditions [5]:

(16)
$$\begin{array}{ll} (16.1)\ Y_1 Y_2^t = Y_2 Y_1^t & (16.4)\ Y_2^t Y_4 = Y_4^t Y_2 \\ (16.2)\ Y_1^t Y_3 = Y_3^t Y_1 & (16.5)\ Y_1 Y_4^t - Y_2 Y_3^t = I \\ (16.3)\ Y_3 Y_4^t = Y_4 Y_3^t & (16.6)\ Y_1^t Y_4 - Y_3^t Y_2 = I \end{array}$$

where the symbol Y^t denotes the transpose of the matrix Y. Using the presentation given in equation (11) for $\pi_1(X_g U_\tau X'_g)$, and the definition of the integers τ_{ij} in equation (12), we note that the first homology group of $X_g U_\tau X'_g$ will be trivial if and only if the $g \times g$ matrix Y_1 has determinant ± 1. Noting that the matrix

$$\begin{bmatrix} Y_1 & 0 \\ 0 & (Y_1^t)^{-1} \end{bmatrix}$$

is symplectic if determinant $Y_1 = \pm 1$ it then follows that

$$([\tau_*])\psi = \begin{bmatrix} Y_1 & 0 \\ 0 & (Y_1^t)^{-1} \end{bmatrix} \begin{bmatrix} I & \tilde{Y}_2 \\ \tilde{Y}_3 & \tilde{Y}_4 \end{bmatrix} \tag{17}$$

where the matrix on the far right is symplectic, and therefore satisfies the symplectic conditions (16). Equations (16.1) and (16.2) imply that $\tilde{Y}_2$ and $\tilde{Y}_3$ are symmetric matrices. Equation (16.6) then reduces to:

$$\tilde{Y}_4 = I + \tilde{Y}_3 \tilde{Y}_2 \tag{18}$$

Therefore

$$([\tau_*])\psi = \begin{bmatrix} I & 0 \\ (Y_1^t)^{-1} \tilde{Y}_3 Y_1^{-1} & I \end{bmatrix} \begin{bmatrix} Y_1 & Y_1 \tilde{Y}_2 \\ 0 & (Y_1^t)^{-1} \end{bmatrix} \tag{19}$$

In (19), the matrix on the left is in the subgroup $(A)\psi \subseteq \text{Sp}(2g, Z)$, while the matrix on the right is in $(B)\psi$. It then follows that τ_* must have represented an element in $((A)\psi(B)\psi)\psi^{-1} = AKB$

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Conversely, suppose that $[\tau_*] \in AKB$. Then

$$([\tau_*])\psi = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix} \tag{20}$$

Since both matrices in equation (20) are in $\mathrm{Sp}(2g, Z)$, condition (16.5) implies that $P_1 P_4^t = Q_1 Q_4^t = I$, therefore det P_1, det $Q_1 = \pm 1$. Therefore:

$$([\tau_*])\psi = \begin{bmatrix} P_1 Q_1 & * \\ * & * \end{bmatrix} \tag{21}$$

where det $P_1 Q_1 = \pm 1$. This implies that the group $\pi_1(X_g U_\tau X_g')$, when abelianized is trivial, hence $(X_g U_\tau X_g')$ is a homology 3-sphere.

4. The Poincaré conjecture as a problem about $H(T_g)$

We begin by establishing a result which allows us to set up a correspondence between Heegaard splittings of S^3 and the complex $AB \subset H(T_g)$.

THEOREM 3. *The 3-manifold $X_g U_\tau X_g'$ is topologically equivalent to a 3-sphere if and only if $[\tau_*] \in AB$.*

PROOF. The necessity of the condition of Theorem 3 will be shown to follow from a result due to Waldhausen:

LEMMA 1 [*Waldhausen* 16]. *If S^3 admits two Heegaard splittings of the same genus, say $\tilde{X}_g U_{\tilde{\tau}} \tilde{X}_g'$ and $X_g U_\tau X_g'$, then there is a homeomorphism h mapping $S^3 \to S^3$ such that $\tilde{T}_g h = T_g$.*

The conventions adopted in Section 2 ensure that if τ is taken to be the identity map, then $X_g U_{id} X_g'$ will be S^3, and we use this as a "standard" Heegaard splitting of S^3. Suppose that $X_g U_\tau X_g'$ is also S^3. Then by Lemma 1 there is a homeomorphism h mapping $X_g U_{id} X_g' \to X_g U_\tau X_g'$ such that the restrictions b_1 and b_2 of h to T_g and T_g' respectively satisfy $T_g h_1 = T_g$, $T_g' h_2 = T_g'$. Since h must preserve the boundary identification, we have

$$b_1 \tau \eta = \eta b_2, \text{ or } \tau = b_1^{-1} \eta b_2 \eta^{-1} \tag{22}$$

This gives the required product representation, since b_1^{-1} induces an automphism α_*, with $[\alpha_*] \in A$ and $\eta b_2 \eta^{-1}$ induces an automorphism β_* with $[\beta_*] \in B$.

To see that the condition of Theorem 3 is sufficient, suppose that $[\tau_*] = [\alpha_*][\beta_*]$, where $[\alpha_*] \in A$ and $[\beta_*] \in B$. By the handlebody theorem [see, for example, 15] we can select representatives α and β for $[\alpha_*]$ and $[\beta_*]$ respectively in such a way that α^{-1} is induced by a homeomorphism $h_1 : X_g \to X_g$, while $\eta^{-1}\beta\eta$ is induced by a homeomorphism $h_2 : X'_g \to X'_g$. Using h_1 and h_2 we define a homeomorphism $h : X_g U_{id} X'_g \to X_g U_\tau X'_g$ by the rules $h | X_g = h_1$ and $h | X'_g = h_2$. It is easily checked that h is well-defined on $X_g \cap X'_g$, hence $X_g U_\tau X'_g$ is homeomorphic to S^3. This completes the proof of Theorem 3.

It is now possible to reformulate the Poincaré conjecture as a problem about the group $H(T_g)$.

COROLLARY 1. *The 3-dimensional Poincaré conjecture is true if and only if the following conjecture about the group $H(T_g)$ is true. Let $[\tau_*] \in AKB \subset H(T_g)$. Let the action of τ_* be given by equation* (8). *Let $\pi_1(X_g U_\tau X'_g)$ be the abstract group presented by equation* (11). *Then $\pi_1(X_g U_\tau X'_g) = 1$ only if $[\tau_*] \in AB$.*

PROOF. By Theorem 2, a necessary and sufficient condition for $X_g U_\tau X'_g$ to be a homology 3-sphere is that $[\tau_*] \in AKB$. By Theorem 3 a necessary and sufficient condition for $X_g U_\tau X'_g$ to be a topological 3-sphere is that $[\tau_*] \in AB$. Hence the Poincaré conjecture is true if and only if the subclass of AKB which corresponds, in our representation, to all homotopy 3-spheres is precisely AB. This proves Corollary 1.

5. Some algebraic questions

To try to understand the remaining questions involved in Corollary 1, we propose a series of problems whose solutions might lead to a resolution of the Poincaré conjecture. Each of these can be specialized to the case $g = 2$, which is the first real case of interest, since the Poincaré conjecture is known to be true for $g = 0$ and 1.

PROBLEM 1. Among all groups which have presentations of the type given in equation (11), characterize those which define the trivial group. This seems to be an extremely difficult problem, however by Corollary 1, we may restrict ourselves to elements $[\tau_*] \in AKB$, and we expect our answer to be $[\tau_*] \in AB$.

PROBLEM 2. If a direct attack on Problem 1 fails, one might hope to make further progress by following an indirect path and attempting to amass further data about the group $H(T_g)$. Generators and defining relations are known for $H(T_2)$, which is a Z_2 – central extension of the homeotopy group of a 6-punctured sphere [see 2]. The latter group [see 3, 7, 8] is, in turn, closely related to Artin's braid group [see 7]. The results in [2] generalize to a proper subgroup of $H(T_g)$ if $g > 2$, and the problem of characterizing $H(T_g)$ by generators and defining relations is an open question for $g > 2$. This problem is important not only for its applications to the study of 3-manifolds, but also for its potential applications to Riemann surface theory [e.g. see 9] and for our understanding of the automorphism group of a free group.

PROBLEM 3. Characterize the subgroup K of $H(T_g)$, $g \geqq 2$. Is K finitely generated? Finitely presented? Very little is known, even for $g = 2$. We conjecture that for $g = 2$ the group K is a free group of infinite rank.

PROBLEM 4. Study the subgroups A and B of $H(T_g)$, $g \geqq 2$. Generators are known for A if $g = 2$ [see 4]. Since A and B are conjugate subgroups, only one of these groups need be investigated. Of particular interest in connection with Corollary 1 would be double coset representatives for A and B in the complex AKB.

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