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A NOVEL ANALYTICAL APPROACH FOR PRICING DISCRETELY SAMPLED GAMMA SWAPS IN THE HESTON MODEL

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Abstract

The main purpose of this paper is to present a novel analytical approach for pricing discretely sampled gamma swaps, defined in terms of weighted variance swaps of the underlying asset, based on Heston's two-factor stochastic volatility model. The closed-form formula obtained in this paper is in a much simpler form than those proposed in the literature, which substantially reduces the computational burden and can be implemented efficiently. The solution procedure presented in this paper can be adopted to derive closed-form solutions for pricing various types of weighted variance swaps, such as self-quantoed variance and entropy swaps. Most interestingly, we discuss the validity of the current solutions in the parameter space, and provide market practitioners with some remarks for trading these types of weighted variance swaps.

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1. Introduction

Volatility derivatives have been playing an increasingly important role in the banking and finance industry in recent years. Volatility measures the standard deviation of the returns of an underlying asset. Therefore, volatility derivatives have been the most commonly used measurement for the assessment of risk. In the last two decades, these have been introduced to provide investors with the opportunity to take a direct position, not on the underlying asset itself, but on its volatility. Therefore, investors can use volatility derivatives to trade the spread between the realized and implied volatility levels, or hedge against the risk of volatility to their portfolios.

Variance and volatility swaps are considered to be the first and most fundamental financial products. Nowadays, they are the most popular volatility-based derivatives, for their effective provision of volatility exposure. Variance swaps have been actively

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traded in over-the-counter (OTC) markets since the global financial crisis. More specifically, variance swaps are financial derivatives that are considered to be forward contracts on annualized realized variance. In today's financial markets, variance swaps on stock indices are highly liquid, and are widely used by investors as an easy way to trade future realized variance against the current implied variance. Moreover, OTC variance swaps can be linked to other types of underlying assets, such as commodities or currencies. Hence, they can be very useful in hedging volatility risk exposure, or in taking a position on the future realized volatility of the underlying asset. An explosive increase in the trading volume of variance swaps has been witnessed in recent years (see http://cfe.cboe.com/education/finaleuromoneyvarpaper.pdf). As a result of this increase, many researchers have proposed various types of valuation approaches for pricing variance swaps with the realized variance, defined in terms of either continuous sampling or discrete sampling.

Apart from variance swaps, there is another type of the third generation volatility derivatives, known as *a gamma swap*, which is considered to be a weighted variance swap. Gamma swaps have been used in financial markets to protect investors from the impact of volatility or variance spikes when the underlying price falls close to zero. According to a handbook published by Banque Nationale de Paris Paribas (BNP Paribas) [6], OTC gamma swaps, written on the €-Stoxx 50, Nikkei 225, and Standard & Poor's 500 (S&P 500) indices, have been traded since March 2001. In general, gamma swaps obviate the need for embedding a cap, which otherwise is usually embedded in volatility or variance swaps to protect the swap seller from crash risk. The payoff of a gamma swap is identical to that of a variance swap, except that the daily squared return of the underlying asset is weighted by the spot price divided by the initial spot price. In other words, when the underlying price in the current period crashes (approaches zero), the squared return is extremely large, but the weighted return approaches zero; an investor can buy a gamma swap and sell a variance swap to limit the losses that could arise from a decreasing current price, and tap into the potential of a rising current price. For a similar reason, when introducing gamma swaps, but with different weights, self-quantoed variance and entropy swaps are considered to be interesting volatility derivatives for investors when the market is influenced by the leverage effect that refers to the generally negative correlation between an asset return and its changes in volatility. Unlike variance and volatility swaps, there is relatively little work on pricing gamma swaps, such as in [11, 12]. In this study, we focus our attention on developing an analytical approach to price gamma swaps, based on the Heston two-factor stochastic volatility model [5]. Moreover, we aim to extend the approach to pricing self-quantoed variance and entropy swaps.

This study demonstrates that the analytical approach presented by Rujivan and Zhu [10] can be adopted to derive a closed-form formula for the fair strike price of gamma swaps, with the realized variance defined in terms of weighted variance swaps of the underlying price based on Heston's two-factor stochastic volatility model [5]. The main contribution of this paper is threefold. Firstly, our closed-form formula presented in this paper is in a much simpler form than the one proposed by Zheng and Kwok [12]. For example, Zheng and Kwok's formula [12] contains a second order differential operator that needs to be worked out in order to obtain the values of the fair strike prices of gamma swaps, while Zheng and Kwok's procedure [12] can be completely avoided by using our closed-form formula. Consequently, our closed-form formula substantially reduces the computational burden and can be implemented efficiently. Moreover, we show that our current approach can be easily extended to the case of pricing self-quantoed variance and entropy swaps introduced by Crosby [4]. Secondly, with the simplest form of the solution for pricing gamma swaps presented in this paper, we include a discussion on the validity of our solutions in a subspace of the original parameter space of the Heston model. Therefore, market practitioners need to be cautious, making sure that their model parameters, extracted from market data, are in the right parameter subspace when any of these analytical pricing formulae are used to calculate the fair strike price of a discretely sampled gamma swap. Thirdly, we derive two propositions, which may have some practical implications when market practitioners need to compare between the fair strike prices of variance swaps and weighted variance swaps discussed in this paper.

The remainder of the paper is organized as follows. In Section 2, we review and simplify the results from pricing variance swaps under the Heston model proposed by Rujivan and Zhu [10]. In Section 3, we derive a closed-form formula for fair strike prices of gamma swaps by adopting Rujivan and Zhu's approach [9]. An interesting discussion on the validity of our solution is provided in Section 4. We further derive closed-form formulae for fair strike prices of self-quantoed variance and entropy swaps in Section 5. Some remarks on the fair strike prices of the weighted variance swaps presented in this paper comprise two propositions in Section 6. In Section 7, some numerical examples are given, demonstrating the practical implications of the two propositions. Finally, a brief conclusion is given in Section 8.

2. Pricing variance swaps under the Heston model

In the present section, we briefly review the Heston stochastic volatility model [5] which we adopt to describe the dynamics of the underlying asset, and give the assumptions and notation used in this paper. Next, we provide some descriptions of variance swaps, and simplify the results obtained by using Rujivan and Zhu's approach [10] for the pricing of variance swaps. In particular, we shall apply the results in Section 6 for deriving a proposition used to compare between the fair strike prices of variance swaps and weighted variance swaps discussed in this paper.

Following Rujivan and Zhu [9, 10], we begin by considering a probability space $(\Omega, \mathcal{F}, Q)$, where Q is a risk-neutral probability measure. The dynamics of the underlying price S_t is described by the diffusion processes with a stochastic instantaneous variance v_t as

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{v_t}S_t\,d\tilde{B}_t^S, \\ dv_t = \kappa^*(\theta^* - v_t)\,dt + \sigma_V\sqrt{v_t}\,d\tilde{B}_t^V, \end{cases} \tag{2.1}$$

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where κ^* and θ^* are the risk-neutral parameters and σ_V is the so-called volatility of volatility. The parameter, $r = r_0 - d_0$, has r_0 as a risk-free interest rate and d_0 as a constant dividend yield. The Wiener processes with respect to the risk-neutral probability measure Q, $d\widetilde{B}_t^S$ and $d\widetilde{B}_t^V$, are assumed to be correlated with a constant correlation coefficient ρ, that is, $(d\widetilde{B}_t^S, d\widetilde{B}_t^V) = \rho\, dt$. For the rest of this paper, our analysis will be based on the probability space $(\Omega, \mathcal{F}, Q)$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$. The conditional expectation with respect to $\mathcal{F}_t$ is denoted by $E_t^Q = E^Q[\cdot \mid \mathcal{F}_t]$.

In order to ensure that the Heston model (2.1) is a proper stochastic volatility model with the variance process reverting to a positive mean level, we make the following assumptions.

Assumption 2.1. *All parameters $r_0, d_0, \kappa^*, \theta^*, \sigma_V$, and an initial instantaneous variance v_0 are strictly positive.*

In addition, the stochastic volatility process is the so-called square-root process. Hence, to ensure that the variance is always positive, a further assumption, known as the Feller condition, is also needed (see the articles by Cox et al. and Heston [3, 5]).

Assumption 2.2. *The parameters κ^*, θ^*, and σ_V satisfy the inequality $2\kappa^*\theta^* \geq \sigma_V^2$.*

Due to Assumptions 2.1 and 2.2, we define a parameter space for the Heston model (2.1) as follows:

$$\Theta = \{p = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) \in (\mathbb{R}^+)^5 \times [-1, 1] \mid 2\kappa^*\theta^* \geq \sigma_V^2\}. \tag{2.2}$$

The parameter space Θ will be referred to in Sections 4–7. Furthermore, for any given set $\mathcal{D}$ and a real-valued function $f : \mathcal{D} \times \Theta \to \mathbb{R}$, the value of f at $(x, p) \in \mathcal{D} \times \Theta$ is denoted by $f(x; p)$ or simply $f(x)$ when the parameter p is fixed.

Variance swaps are forward contracts on the future realized variance of the returns of the specified underlying asset. For a given maturity $T > 0$, the value of a variance swap can be written as $V_T = (\sigma_R^2 - K_{\text{var}}) \times L$, where σ_R^2 is the annualized realized variance over the contract life $[0, T]$, K_{var} is the annualized delivery price (or *strike price* in what follows) for the variance swap, and L is the notional amount of the swap in dollars per annualized volatility point squared.

Rujivan and Zhu [10] have proposed a simple closed-form formula for pricing variance swaps with the log-return realized variance

$$\sigma_{R,d}^2(0, N, T) = \frac{AF}{N} \sum_{i=1}^{N} \ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big) \times 100^2 = \frac{1}{T} \sum_{i=1}^{N} \ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big) \times 100^2,$$

where S_{t_i} is the closing price of the underlying asset at the ith observation time t_i and there are altogether N observations. Here AF is the annualized factor converting this expression to an annualized variance. In the following proposition, we simplify the results obtained by Rujivan and Zhu [10] for pricing variance swaps with the realized variance $\sigma_{R,d}^2(0, N, T)$, which is closely related to the realized variance defined for gamma swaps.

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PROPOSITION 2.3. *Suppose that S_t follows the dynamics described in (2.1). Let $X_t = \ln S_t$. Then $K_{\text{var}} = E_0^Q[\sigma_{R,d}^2]$ can be written in terms of a quadratic form of the initial instantaneous variance v_0 as*

$$K_{\text{var}}(T,\Delta t,v_0) = \frac{100^2}{T}\bigg[\bigg\{\sum_{i=1}^{N}\tilde{A}_0(\Delta t,t_{i-1})\bigg\} + \bigg\{\sum_{i=1}^{N}\tilde{A}_1(\Delta t,t_{i-1})\bigg\}v_0 + \bigg\{\sum_{i=1}^{N}\tilde{A}_2(\Delta t,t_{i-1})\bigg\}v_0^2\bigg], \tag{2.3}$$

where

$$\begin{aligned}
\tilde{A}_0(\Delta t,t_{i-1}) &= \tilde{F}_0(\Delta t) + \theta^*(1-e^{-\kappa^* t_{i-1}})\tilde{F}_1(\Delta t) \\
&\quad + \theta^*\Big(\theta^* + \frac{\sigma_V^2}{2\kappa^*}\Big)(1-e^{-\kappa^* t_{i-1}})^2\tilde{F}_2(\Delta t), \\
\tilde{A}_1(\Delta t,t_{i-1}) &= e^{-\kappa^* t_{i-1}}\tilde{F}_1(\Delta t) + \frac{(2\kappa^*\theta^* + \sigma_V^2)}{\kappa^*}(1-e^{-\kappa^* t_{i-1}})e^{-\kappa^* t_{i-1}}\tilde{F}_2(\Delta t), \\
\tilde{A}_2(\Delta t,t_{i-1}) &= e^{-2\kappa^* t_{i-1}}\tilde{F}_2(\Delta t)
\end{aligned}$$

and

$$\tilde{F}_0(\Delta t) = \tilde{\alpha}_1 + \tilde{\alpha}_2\Delta t + \tilde{\alpha}_3(\Delta t)^2 + \tilde{\alpha}_4 e^{-\kappa^*\Delta t} + \tilde{\alpha}_5\Delta t e^{-\kappa^*\Delta t} + \tilde{\alpha}_6 e^{-2\kappa^*\Delta t}, \tag{2.4}$$

$$\tilde{F}_1(\Delta t) = \tilde{\beta}_1 + \tilde{\beta}_2\Delta t + \tilde{\beta}_3 e^{-\kappa^*\Delta t} + \tilde{\beta}_4\Delta t e^{-\kappa^*\Delta t} + \tilde{\beta}_5 e^{-2\kappa^*\Delta t}, \tag{2.5}$$

$$\tilde{F}_2(\Delta t) = \Big(\frac{e^{-\kappa^*\Delta t}-1}{2\kappa^*}\Big)^2$$

with $\Delta t = t_i - t_{i-1}$. The constants in (2.4) and (2.5) are given by

$$\begin{cases}
\tilde{\alpha}_1 = \dfrac{\theta^*(2\kappa^*\theta^* - 8(\kappa^*)^2 + 16\kappa^*\rho\sigma_V - 5\sigma_V^2)}{8(\kappa^*)^3}, & \tilde{\alpha}_2 = \dfrac{\theta^*(2(2r-\theta^*)\kappa^* + 4\kappa^*(\kappa^*-\rho\sigma_V) + \sigma_V^2)}{4(\kappa^*)^2}, \\[2ex]
\tilde{\alpha}_3 = \dfrac{(2r-\theta^*)^2}{4}, & \tilde{\alpha}_4 = \dfrac{\theta^*(2(\kappa^*)^2 - \kappa^*(\theta^* + 4\rho\sigma_V) + \sigma_V^2)}{2(\kappa^*)^3}, \\[2ex]
\tilde{\alpha}_5 = \dfrac{\theta^*((\theta^* - 2r - 2\rho\sigma_V)\kappa^* + \sigma_V^2)}{2(\kappa^*)^2}, & \tilde{\alpha}_6 = \dfrac{\theta^*(2\theta^*\kappa^* + \sigma_V^2)}{8(\kappa^*)^3}, \\[2ex]
\tilde{\beta}_1 = \dfrac{\{\sigma_V^2 - 2\kappa^*\theta^* + 4\kappa^*(\kappa^*-\rho\sigma_V)\}}{4(\kappa^*)^3}, & \tilde{\beta}_2 = \dfrac{(\theta^*-2r)}{2\kappa^*}, \\[2ex]
\tilde{\beta}_3 = \dfrac{(\theta^* + \rho\sigma_V - \kappa^*)}{(\kappa^*)^2}, & \tilde{\beta}_4 = \dfrac{\{(2r-\theta^*+2\rho\sigma_V)\kappa^* - \sigma_V^2\}}{2(\kappa^*)^2}, \\[2ex]
\tilde{\beta}_5 = -\dfrac{(2\theta^*\kappa^* + \sigma_V^2)}{4(\kappa^*)^3}.
\end{cases}$$

3. Pricing gamma swaps under the Heston model

In this section, we apply Rujivan and Zhu's approach [10] to derive a closed-form formula for pricing discretely sampled gamma swaps.

3.1. Gamma swaps Gamma swaps are forward contracts on the future realized variance, defined in terms of a weighted variance swap as

$$\sigma_\Gamma^2 = \frac{AF}{N}\sum_{i=1}^{N} w_i \ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\times 100^2 = \frac{100^2}{T}\sum_{i=1}^{N}\frac{S_{t_i}}{S_{t_0}}\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big),$$

where the weight w_i is chosen to be $S_{t_i}/S_{t_0}, i = 1, 2, \ldots, N$. As explained by Zheng and Kwok [12], one reason for choosing the weight to be the underlying level is to provide the embedded damping of the large downside variance, when the underlying price crashes close to zero. When the underlying price in the current period crashes (approaches zero), that is, when the squared return is extremely large, but the weighted return approaches zero, an investor can buy a gamma swap and sell a variance swap to limit the losses that could arise from a decreasing current price, and to tap into the potential of a rising current price.

In a risk-neutral world, the value of a gamma swap at time t is the expected present value of the future payoff, $V_t = E_t^Q[e^{-r(T-t)}(\sigma_\Gamma^2 - K_\Gamma)L]$. This should be zero at the beginning of the contract, since there is no cost to enter into a swap. Therefore, the fair strike price of a gamma swap can be defined as $K_\Gamma = E_0^Q[\sigma_\Gamma^2]$, after initially setting the value of $V_t = 0$. The gamma swap valuation problem is, therefore, reduced to calculating the expectation value of the future realized variance in the risk-neutral world.

3.2. A novel analytical approach for pricing gamma swaps Following Rujivan and Zhu's approach [10], we begin with taking the expectation of σ_Γ^2 as follows:

$$K_\Gamma = E_0^Q[\sigma_\Gamma^2] = \frac{100^2}{T}\sum_{i=1}^{N}\frac{1}{S_{t_0}}E_0^Q[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2],$$

where $X_t = \ln S_t$ for all $t \geqslant 0$. Therefore, the problem of pricing gamma swaps is reduced to calculating N conditional expectations in the form of

$$\frac{1}{S_0}E_0^Q[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2] \tag{3.1}$$

for some fixed equal time period Δt and N different tenors $t_i = i\Delta t$ $(i = 0, 1, \ldots, N)$. In the rest of this subsection, we will focus on calculating the expectation of this expression. In the process of calculating this expectation, i is fixed, where both t_i and t_{i-1} are regarded as known constants.

Using the facts that $\mathcal{F}_0 \subset \mathcal{F}_{t_{i-1}}$ and $S_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$-measurable, we apply the tower property [2] to the conditional expectation (3.1), which yields a double conditional expectation

$$E_0^Q[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2] = E_0^Q[E_{t_{i-1}}^Q[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2]]. \tag{3.2}$$

The conditional expectation with respect to $\mathcal{F}_{t_{i-1}}$ on the right-hand side (RHS) of equation (3.2), that is, $E_{t_{i-1}}^Q[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2]$, can be computed by using the following proposition.

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Proposition 3.1. *Suppose that S_t follows the dynamics described in (2.1) and $\omega = \rho\sigma_V - \kappa^* \neq 0$. Let $X_t = \ln S_t$; then*

$$E^Q_{t_{i-1}}[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2] = e^{X_{t_{i-1}}}\{F_0(\Delta t) + F_1(\Delta t)v_{t_{i-1}} + F_2(\Delta t)v^2_{t_{i-1}}\}, \tag{3.3}$$

$$F_0(\Delta t) = (\alpha_1 + \alpha_2\Delta t + \alpha_3(\Delta t)^2 + \alpha_4 e^{\omega\Delta t} + \alpha_5\Delta t e^{\omega\Delta t} + \alpha_6 e^{2\omega\Delta t})e^{r\Delta t}, \tag{3.4}$$

$$F_1(\Delta t) = (\beta_1 + \beta_2\Delta t + \beta_3 e^{\omega\Delta t} + \beta_4\Delta t e^{\omega\Delta t} + \beta_5 e^{2\omega\Delta t})e^{r\Delta t}, \tag{3.5}$$

$$F_2(\Delta t) = \left(\frac{e^{\omega\Delta t} - 1}{2\omega}\right)^2 e^{r\Delta t}, \tag{3.6}$$

where $\Delta t = t_i - t_{i-1}$ and the constants in (3.4) and (3.5) are given by

$$\begin{cases} \alpha_1 = \dfrac{\kappa^*\theta^*(2\kappa^*\theta^* - 8(\kappa^*)^2 + (8\rho^2 - 5)\sigma_V^2)}{8\omega^4}, & \alpha_2 = \dfrac{\kappa^*\theta^*(\sigma_V^2 - 4\kappa^*\omega + 4r\rho\sigma_V - (4r + 2\theta^*)\kappa^*)}{-4\omega^3}, \\ \alpha_3 = \dfrac{(\kappa^*\theta^* - 2r\omega)^2}{4\omega^2}, & \alpha_4 = \dfrac{\kappa^*\theta^*(2(\kappa^*)^2 + (1 - 2\rho^2)\sigma_V^2 - \kappa^*\theta^*)}{2\omega^4}, \\ \alpha_5 = \dfrac{\kappa^*\theta^*\{\kappa^*\theta^* - 2r\omega + \sigma_V(\sigma_V - 2\rho\omega)\}}{-2\omega^3}, & \alpha_6 = \dfrac{\kappa^*\theta^*(2\kappa^*\theta^* + \sigma_V^2)}{8\omega^4}, \\ \beta_1 = \dfrac{(\sigma_V^2 - 2\kappa^*\theta^* - 4\kappa^*\omega)}{-4\omega^3}, & \beta_2 = \dfrac{\kappa^*\theta^* - 2r\omega}{2\omega^2}, \\ \beta_3 = \dfrac{\kappa^*(\theta^* + \omega)}{-\omega^3}, & \beta_4 = \dfrac{2r\omega - \sigma_V(\sigma_V - 2\rho\omega) - \kappa^*\theta^*}{2\omega^2}, \\ \beta_5 = \dfrac{2\kappa^*\theta^* + \sigma_V^2}{4\omega^3}. \end{cases} \tag{3.7}$$

The proof of this proposition is given in Appendix A.

Utilizing Proposition 3.1, a closed-form formula for computing $E^Q_0[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2]$ is obtained as shown in the next proposition.

Proposition 3.2. *Suppose that $\omega = \rho\sigma_V - \kappa^* \neq 0$. The conditional expectation in (3.1) can be written in terms of a quadratic form of the initial instantaneous variance v_0 as*

$$\frac{1}{S_0}E^Q_0[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2] = g_i(\Delta t, v_0) = \bar{A}_0(\Delta t, t_{i-1}) + \bar{A}_1(\Delta t, t_{i-1})v_0 + \bar{A}_2(\Delta t, t_{i-1})v_0^2 \tag{3.8}$$

for all $i = 1, 2, \ldots, N$, $v_0 > 0$, where $\Delta t = t_i - t_{i-1}$. Set $\eta = \kappa^*\theta^*(2\kappa^*\theta^* + \sigma_V^2)/2\omega^2$ and $\gamma = (2\kappa^*\theta^* + \sigma_V^2)/\omega$. Then

$$\bar{A}_0(\Delta t, t_{i-1}) = e^{rt_{i-1}}F_0(\Delta t) + f_1(t_{i-1})F_1(\Delta t) + f_2(t_{i-1})F_2(\Delta t), \tag{3.9}$$

$$\bar{A}_1(\Delta t, t_{i-1}) = e^{(r+\omega)t_{i-1}}F_1(\Delta t) + \{\gamma e^{(r+2\omega)t_{i-1}}(1 - e^{-\omega t_{i-1}})\}F_2(\Delta t), \tag{3.10}$$

$$\bar{A}_2(\Delta t, t_{i-1}) = e^{(r+2\omega)t_{i-1}}F_2(\Delta t), \tag{3.11}$$

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$$f_1(t_{i-1}) = \frac{\kappa^*\theta^*}{\omega} e^{(r+\omega)t_{i-1}}(1 - e^{-\omega t_{i-1}}),$$
$$f_2(t_{i-1}) = \eta e^{(r+2\omega)t_{i-1}}(1 - e^{-\omega t_{i-1}})^2.$$

The proof of this proposition is provided in Appendix B.

Now, with the conditional expectation expressed in (3.8), we can directly adopt Proposition 3.2 to obtain the pricing formula for the realized variance σ_Γ^2, which can be written in a quadratic form of the initial instantaneous variance v_0 as

$$K_\Gamma(\Delta t, v_0) = \frac{100^2}{T}\bigg\{\bigg(\sum_{i=1}^{N} \bar{A}_0(\Delta t, t_{i-1})\bigg) + \bigg(\sum_{i=1}^{N} \bar{A}_1(\Delta t, t_{i-1})\bigg)v_0 + \bigg(\sum_{i=1}^{N} \bar{A}_2(\Delta t, t_{i-1})\bigg)v_0^2\bigg\}. \tag{3.12}$$

Note that our formula (3.12) for pricing gamma swaps is in a much simpler form than the one derived by Zheng and Kwok [12]. They are not in exactly the same form. It is easy to verify that the latter can be derived from the former. Using Mathematica, we can show that Zheng and Kwok's formula [12] reduces to our formula after some algebraic manipulations. However, the current formula is obtained using a much simpler approach, and it is also in a much simpler form, which can be exploited to explore some properties of the solution, which are discussed in the next section.

There is a major advantage of using formula (3.12) for computing fair strike prices of gamma swaps, which should be noted here. Indeed, Zheng and Kwok's solution [12] for pricing gamma swaps is given in an *implicit* form, in the sense that some differentiation operators remain in their formula (3.1), which involves the calculation of second order derivatives of the parameter functions. On the other hand, our solution, written in a quadratic form of v_0, as shown in (3.12), is the simplest, and can be readily computed. Although the derivatives can be calculated with the aid of a symbolic package, such as Maple or Mathematica, it is still much better to have pricing formulae for variance and gamma swaps that need no further differentiations, like the ones presented in (2.3) and (3.12). Clearly, this distinguishing feature of our solutions, with reduced computational time and effort, makes our formulae an exciting improvement over the formulae presented by Zheng and Kwok [12].

A crucial point should be noted here about formula (3.12), when $\omega = \rho\sigma_V - \kappa^* = 0$. In this case, formula (3.12) can no longer be used to compute K_Γ, since the functions and constants defined in (3.4)–(3.7) can have infinite values. This is also the case for Zheng and Kwok's formula [12] for pricing gamma swaps, as shown in their formulae (3.1) and (3.5). Although they did not explicitly mention this particular point, that is, some restrictions need to be imposed in the parameter space, we can easily see from their formulae (3.1) and (3.5) that there is a subspace in Θ in which their solution is valid in the sense of guaranteeing a finite fair strike price of a gamma swap in some cases.

For the sake of completeness in our pricing formula, we provide a closed-form formula for pricing gamma swaps in this particular case, which is described in the following proposition.

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PROPOSITION 3.3. *Suppose that S_t follows the dynamics described in* (2.1) *and* $\omega = 0$. *Let* $X_t = \ln S_t$. *Then*

$$\begin{aligned}
E^{Q}_{t_{i-1}}[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2] &= e^{X_{t_{i-1}}}\{F^*_0(\Delta t) + F^*_1(\Delta t)v_{t_{i-1}} + F^*_2(\Delta t)v^2_{t_{i-1}}\},\\
F^*_0(\Delta t) &= \tfrac{1}{48}e^{r\Delta t}(\Delta t)^2\{48r^2 + 24r\kappa^*\theta^*\Delta t + \kappa^*\theta^*(24 + 8\rho\sigma_V\Delta t\\
&\quad + (3\kappa^*\theta^* + \sigma_V^2)(\Delta t)^2)\},\\
F^*_1(\Delta t) &= \tfrac{1}{12}e^{r\Delta t}\Delta t\{12 + \Delta t(12r + 6\rho\sigma_V + (3\kappa^*\theta^* + \sigma_V^2)\Delta t)\},\\
F^*_2(\Delta t) &= \tfrac{1}{4}e^{r\Delta t}(\Delta t)^2.
\end{aligned}$$

Moreover, the conditional expectation in (3.1) *can be written in terms of a quadratic form of the initial instantaneous variance* v_0 *as*

$$\frac{1}{S_0}E^{Q}_{0}[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2] = \bar{A}^*_0(\Delta t, t_{i-1}) + \bar{A}^*_1(\Delta t, t_{i-1})v_0 + \bar{A}^*_2(\Delta t, t_{i-1})v_0^2$$

for all $i = 1, 2, \ldots, N$, $v_0 > 0$, *where* $\Delta t = t_i - t_{i-1}$. *Set* $\eta^* = (1/2)\kappa^*\theta^*(2\kappa^*\theta^* + \sigma_V^2)$ *and* $\gamma^* = 2\kappa^*\theta^* + \sigma_V^2$. *Then*

$$\begin{aligned}
\bar{A}^*_0(\Delta t, t_{i-1}) &= e^{rt_{i-1}}\{F^*_0(\Delta t) + \kappa^*\theta^* t_{i-1}F^*_1(\Delta t) + \eta^* t^2_{i-1}F^*_2(\Delta t)\},\\
\bar{A}^*_1(\Delta t, t_{i-1}) &= e^{rt_{i-1}}\{F^*_1(\Delta t) + \gamma^* t_{i-1}F^*_2(\Delta t)\},\\
\bar{A}^*_2(\Delta t, t_{i-1}) &= e^{rt_{i-1}}F^*_2(\Delta t),
\end{aligned}$$

and

$$K^*_\Gamma(\Delta t, v_0) = \frac{100^2}{T}\bigg\{\bigg(\sum_{i=1}^{N}\bar{A}^*_0(\Delta t, t_{i-1})\bigg) + \bigg(\sum_{i=1}^{N}\bar{A}^*_1(\Delta t, t_{i-1})\bigg)v_0 + \bigg(\sum_{i=1}^{N}\bar{A}^*_2(\Delta t, t_{i-1})\bigg)v_0^2\bigg\},$$

where $K^*_\Gamma(\Delta t, v_0)$ *denotes the fair strike price of a gamma swap in this particular case.*

The proof of this proposition is analogous to the proof of Propositions 3.1 and 3.2 and, therefore, omitted here.

4. Validity of the closed-form formula

The construction of the simple pricing formula for gamma swaps presents some very interesting discussions in terms of the validity of the solution in the parameter space and the determination of the required parameters, which will be discussed in this section. The purpose of such an investigation is to ensure that one of the fundamental assumptions that the fair strike price of a gamma swap should be of a finite and positive value for a given set of parameters determined from market data, that is, $0 \le K_\Gamma < \infty$, is indeed satisfied.

Case $\omega = \rho\sigma_V - \kappa^* \neq 0$. We consider a subspace of the parameter space Θ defined in (2.2) as follows:

$$\Theta_1 = \{p = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) \in \Theta \mid \rho\sigma_V - \kappa^* \neq 0\}.$$

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In this case, the finiteness of K_Γ can be readily established. This is because, from (3.4)–(3.7), one can verify that the functions $F_k(\Delta t; p), k = 0, 1, 2$, are finite for all $\Delta t \geqslant 0$ and $p = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) \in \Theta_1$. Thus, from equation (3.8), $E_0^Q[(S_{t_i}/S_0)\ln^2(S_{t_i}/S_{t_{i-1}})] < \infty$ for all $i = 1, 2, \ldots, N$ and it immediately follows from (3.12) that $K_\Gamma < \infty$.

On the other hand, the strict positivity of K_Γ can only be ensured by a sufficient condition, as shown in the following proposition.

Proposition 4.1. *Let $p = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) \in \Theta_1$ be a parameter vector of the Heston model* (2.1)*. Set*

$$\Delta t_p^* = \min[\{\tau > 0 | F_0(\tau; p) F_1(\tau; p) = 0\} \cup \{\infty\}]. \tag{4.1}$$

Then, the following assertions are true:

(1) *Δt_p^* is either strictly positive or infinite, depending on p;*
(2) *for any $\Delta t \in (0, \Delta t_p^*)$ with $T = N\Delta t$ for some positive integer N,*

  (2.1) *$0 < K_\Gamma(T, \Delta t, v_0; p) < \infty$ for all $v_0 > 0$ and*
  (2.2) *$K_\Gamma(T, \Delta t, v_0; p)$ strictly increases with respect to v_0 on $(0, \infty)$.*

The proof of this proposition can be found in Appendix C.

Case $\omega = \rho\sigma_V - \kappa^* = 0$. Set $\Theta_2 = \Theta - \Theta_1$. One can follow the method, presented in Appendix C, in order to obtain the finiteness and positivity of K_Γ^* in a similar fashion as claimed in Proposition 4.1.

An important issue that should be addressed for the pricing formula in (3.12) is that we have imposed conditions in terms of the sampling frequency and the market-extracted model parameters in order to obtain a financially meaningful value of the fair strike price of a gamma swap. In other words, the subspace of Θ, derived in Proposition 4.1, in which the pricing formula is valid to the payoff function of a contract, provides evidence that the parameters extracted from market data are contract dependent, when a stochastic volatility model is adopted to price a derivative contract.

5. Applications of the novel analytical approach

We demonstrate some applications of our approach for deriving closed-form formulae for the fair strike prices of self-quantoed variance and entropy swaps in the following subsections. The results obtained in this section are based on the assumption that $\omega = \rho\sigma_V - \kappa^* \neq 0$. For the case of $\omega = 0$, similar results can be obtained and thus they are omitted here for ease of exposition.

5.1. Pricing self-quantoed variance swaps According to Crosby [4], when the weight w_i is chosen to be $S_{t_N}/S_0, i = 1, 2, \ldots, N$, the future realized variance, defined by

$$\sigma_S^2 = \frac{AF}{N} \sum_{i=1}^{N} w_i \ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \times 100^2 = \frac{100^2}{T} \sum_{i=1}^{N} \frac{S_{t_N}}{S_0} \ln^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right),$$

is a floating leg payoff at time T of the so-called self-quantoed variance swap.

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To find the analytical fair strike price of a self-quantoed variance swap, we adopt the technique used in Section 3.2 to compute the expectation of σ_S^2 as follows:

$$K_S = E_0^Q[\sigma_S^2] = \frac{100^2}{T}\sum_{i=1}^{N}\frac{1}{S_0}E_0^Q\Big[S_{t_N}\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big] = \frac{100^2}{T}\sum_{i=1}^{N}\frac{1}{S_0}E_0^Q[(X_{t_i}-X_{t_{i-1}})^2 e^{X_{t_N}}]. \tag{5.1}$$

Applying the tower property [2] to the conditional expectation on the RHS of (5.1) yields

$$\frac{1}{S_0}E_0^Q[(X_{t_i}-X_{t_{i-1}})^2 e^{X_{t_N}}] = \frac{1}{S_0}E_0^Q[E_{t_{i-1}}^Q[(X_{t_i}-X_{t_{i-1}})^2 E_{t_i}^Q[e^{X_{t_N}}]]] = w_i^s g_i(\Delta t, v_0), \tag{5.2}$$

in which $E_{t_i}^Q[e^{X_{t_N}}] = E_{t_i}^Q[S_{t_N}] = e^{X_{t_i}+r(t_N-t_i)}$ and

$$w_i^s = e^{r(t_N-t_i)} = e^{(r_0-d)(t_N-t_i)}$$

for all $i = 1, 2, \ldots, N$. Since Proposition 3.2 provides a formula for $g_i(\Delta t, v_0)$, we immediately obtain the pricing formula for self-quantoed variance swaps as

$$K_S(\Delta t, v_0) = \frac{100^2}{T}\Big\{\Big(\sum_{i=1}^{N}\bar{A}_0(\Delta t, t_{i-1})w_i^s\Big) + \Big(\sum_{i=1}^{N}\bar{A}_1(\Delta t, t_{i-1})w_i^s\Big)v_0 + \Big(\sum_{i=1}^{N}\bar{A}_2(\Delta t, t_{i-1})w_i^s\Big)v_0^2\Big\}.$$

5.2. Pricing entropy swaps As introduced by Crosby [4], the future realized variance defined by

$$\sigma_E^2 = \frac{AF}{N}\sum_{i=1}^{N} w_i \ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\times 100^2 = \frac{100^2}{T}\sum_{i=1}^{N}\frac{S_{t_i}}{S_{t_{i-1}}}\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)$$

is a floating leg payoff at time T of the so-called entropy swap, where the weight w_i is chosen to be $S_{t_i}/S_{t_{i-1}}$, $i = 1, 2, \ldots, N$.

Using the technique presented in Section 3.2 to compute the expectation of σ_E^2 gives

$$K_E = E_0^Q[\sigma_E^2] = \frac{100^2}{T}\sum_{i=1}^{N}E_0^Q\Big[\frac{S_{t_i}}{S_{t_{i-1}}}\ln^2\Big(\frac{S_{t_i}}{S_{t_{i-1}}}\Big)\Big] = \frac{100^2}{T}\sum_{i=1}^{N}E_0^Q[e^{-X_{t_{i-1}}}e^{X_{t_i}}(X_{t_i}-X_{t_{i-1}})^2]. \tag{5.3}$$

Applying the tower property [2] to the conditional expectation on the RHS of (5.3) and using Proposition 3.1 yield

$$\begin{aligned} E_0^Q[e^{-X_{t_{i-1}}}(X_{t_i}-X_{t_{i-1}})^2 e^{X_{t_i}}] &= E_0^Q[e^{-X_{t_{i-1}}}E_{t_{i-1}}^Q[e^{X_{t_i}}(X_{t_i}-X_{t_{i-1}})^2]] \\ &= E_0^Q[F_0(\Delta t) + F_1(\Delta t)v_{t_{i-1}} + F_2(\Delta t)v_{t_{i-1}}^2] \\ &= F_0(\Delta t) + F_1(\Delta t)E_0^Q[v_{t_{i-1}}] + F_2(\Delta t)E_0^Q[v_{t_{i-1}}^2]. \end{aligned} \tag{5.4}$$

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The closed-form formulae for $E_0^Q[v_{t_{i-1}}]$ and $E_0^Q[v_{t_{i-1}}^2]$ were worked out by Broadie and Jain [1], which can be expressed as linear and quadratic functions of v_0, respectively. Therefore, the fair strike prices of entropy swaps can also be written in terms of a quadratic form of the initial instantaneous variance v_0 as

$$K_E(\Delta t, v_0) = \frac{100^2}{T}\bigg\{\bigg(\sum_{i=1}^{N}\bar{B}_0(\Delta t, t_{i-1})\bigg) + \bigg(\sum_{i=1}^{N}\bar{B}_1(\Delta t, t_{i-1})\bigg)v_0 + \bigg(\sum_{i=1}^{N}\bar{B}_2(\Delta t, t_{i-1})\bigg)v_0^2\bigg\},$$

where

$$\begin{aligned}\bar{B}_0(\Delta t, t_{i-1}) &= F_0(\Delta t) + (\theta^*(1 - e^{-\kappa^* t_{i-1}}))F_1(\Delta t)\\ &\quad + \bigg(\frac{\theta^*(2\kappa^*\theta^* + \sigma_V^2)}{2\kappa^*}(1 - e^{-\kappa^* t_{i-1}})^2\bigg)F_2(\Delta t),\end{aligned} \tag{5.5}$$

$$\bar{B}_1(\Delta t, t_{i-1}) = e^{-\kappa^* t_{i-1}}F_1(\Delta t) + \bigg(\frac{(2\kappa^*\theta^* + \sigma_V^2)}{\kappa^*}(1 - e^{-\kappa^* t_{i-1}})e^{-\kappa^* t_{i-1}}\bigg)F_2(\Delta t), \tag{5.6}$$

$$\bar{B}_2(\Delta t, t_{i-1}) = e^{-2\kappa^* t_{i-1}}F_2(\Delta t). \tag{5.7}$$

6. Remarks for trading weighted variance swaps

In the following proposition, we provide some remarks which will be useful for market practitioners who want to trade weighted variance swaps discussed in this paper.

6.1. Comparison between the fair strike prices of weighted variance swaps

PROPOSITION 6.1. *Let $p = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) \in \Theta_1$ and $r = r_0 - d_0$. Then, for any $v_0 > 0$, we make the following assertions.*

(1) *If $r > 0$ and $r + 2\rho\sigma_V > 0$, then there exists $T_p^* > 0$ such that for any $\Delta t \in (0, T_p^*)$,*

$$K_E(\Delta t, v_0; p) < K_\Gamma(\Delta t, v_0; p) < K_S(\Delta t, v_0; p). \tag{6.1}$$

(2) *If $r < 0$ and $r + 2\rho\sigma_V < 0$, then there exists $T_p^* > 0$ such that for any $\Delta t \in (0, T_p^*)$,*

$$K_S(\Delta t, v_0; p) < K_\Gamma(\Delta t, v_0; p) < K_E(\Delta t, v_0; p).$$

(3) *If $r = 0$ and $\rho = 0$, then, for any $\Delta t \geq 0$,*

$$K_S(\Delta t, v_0; p) = K_\Gamma(\Delta t, v_0; p) = K_E(\Delta t, v_0; p).$$

The proof of Proposition 6.1 is given in Appendix D.

Next, we discuss some implications of Proposition 6.1 with regard to the three assertions by focusing on the parameters r and ρ. Assertion (1) implies that if $r > 0$, then entropy swaps are more prone to be cheaper than the self-quantoed variance and gamma swaps, unless $r + 2\rho\sigma_V \leq 0$. It is clear that market practitioners should buy entropy swaps rather than gamma or self-quantoed variance swaps in order to hedge against volatility risk when $r > 0$ and $\rho \geq 0$. However, the correlation coefficient between the stock price and its variance is generally negative due to the leverage effect and hence the condition $r + 2\rho\sigma_V > 0$ may not be fulfilled in some cases of ρ and σ_V. Gamma swaps can be either cheaper or more expensive than entropy swaps under the leverage effect, which will be demonstrated in Section 7.

We next consider assertion (2). If $r < 0$, then market practitioners should buy a self-quantoed variance swap, unless $r + 2\rho\sigma_V \geq 0$. It should be noted that the conclusion is always true under the leverage effect, since the condition $r + 2\rho\sigma_V < 0$ always holds. In Section 7, we provide some numerical results in Example 7.1, illustrating the implications of Proposition 6.1 with detailed discussions. Assertion (3) implies that pricing self-quantoed variance and entropy swaps reduces to pricing gamma swaps when the two underlying Wiener processes in the Heston model (2.1) are independent and the risk-free interest rate r_0 equals the constant dividend yield d_0. We shall discuss this point further in the next subsection.

6.2. Comparison between the fair strike prices of variance and weighted variance swaps When the two underlying Wiener processes in the Heston model (2.1) are independent, that is, $\rho = 0$, the following proposition will be useful for market practitioners who need to compare between the fair strike prices of variance and weighted variance swaps discussed in this paper.

PROPOSITION 6.2. *Let $p = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) \in \Theta_1, r = r_0 - d_0$, and set $\rho = 0$. Then, for any $v_0 > 0$, we have the following assertions.*

(1) *If $r > 0$, then there exists $\hat{T}_p^* > 0$ such that for any $\Delta t \in (0, \hat{T}_p^*)$,*

$$K_{\mathrm{var}}(\Delta t, v_0; p) < K_E(\Delta t, v_0; p) < K_\Gamma(\Delta t, v_0; p) < K_S(\Delta t, v_0; p). \tag{6.2}$$

(2) *If $r < 0$, then there exists $\hat{T}_p^* > 0$ such that for any $\Delta t \in (0, \hat{T}_p^*)$,*

$$K_S(\Delta t, v_0; p) < K_\Gamma(\Delta t, v_0; p) < K_E(\Delta t, v_0; p) < K_{\mathrm{var}}(\Delta t, v_0; p). \tag{6.3}$$

(3) *If $r = 0$, then, for any $\Delta t \geq 0$,*

$$K_E(\Delta t, v_0; p) = K_\Gamma(\Delta t, v_0; p) = K_S(\Delta t, v_0; p) = K_{\mathrm{var}}(\Delta t, v_0; p).$$

The proof of Proposition 6.2 can be found in Appendix E.

Now we discuss some implications of Proposition 6.2. Suppose that $\rho = 0$. Assertion (1) suggests that market practitioners should buy variance swaps rather than weighted variance swaps, in order to hedge against volatility risk when the underlying price tends to rise up, that is, $r > 0$. On the other hand, when the underlying price tends to fall down, assertion (2) implies that market practitioners should buy weighted variance swaps. In addition, from assertion (3), the values of weighted variance swaps can be approximated by the value of the variance swap when the level of the underlying price does not change much, that is, $r \approx 0$ or $r_0 \approx d_0$. Hence, we conclude that the differences between the values of standard variance swaps and the weighted variance swaps are due to (i) the difference between the risk-free interest rate r_0 and the constant dividend yield d_0 and (ii) the correlation coefficient ρ. In other words, gamma and self-quantoed variance and entropy swaps have the same attractive property of being

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exposed to correlation between the underlying price and its variance. Similar results have been proposed by Overhaus et al. [8] under the assumption that the realized variances of weighted variance swaps are approximated by continuously sampled realized variances (see the book by Overhaus et al. [8, pp. 68–69]).

Under the leverage effect ($\rho < 0$), or inverse leverage effect ($\rho > 0$), assertions (1)–(2) of Proposition 6.2 may not be true in general. In Section 7, we demonstrate in Example 7.2 that the condition $\rho = 0$ in Proposition 6.2 is only one of the sufficient conditions needed to obtain the assertions. Moreover, the numerical results show that the fair strike prices of the gamma swaps decline when ρ becomes more negative. A similar result has been proposed by Zheng and Kwok [12] when the underlying price and its variance follow the Heston model with jumps. They explained the result such that when the leverage effect becomes stronger and the volatility of the underlying asset price runs high, the gamma swap assigns lower weights to the sampled values of higher realized variance due to the decline in the underlying asset price. Very interestingly, under the Heston model (2.1), our numerical results show that the fair strike prices of the gamma swaps exhibit more sensitivity to ρ when the volatility of variance σ_V becomes positively larger.

As mentioned in Section 1, regarding the reason for introducing gamma swaps, when the underlying price tends to fall down, that is, when $r < 0$, an investor can buy a gamma swap and sell a variance swap to limit the losses. Assertion (2) of Proposition 6.2 allows the investor to perform this procedure properly, since gamma swaps are cheaper than variance swaps, as shown in the inequality (6.3). Most interestingly, the empirical data of the 6-month gamma and variance swap strike prices on the €-Stoxx 50 index, shown in the BNP Paribas handbook [6, p. 25]), supports the assertion that the gamma swap strike prices were slightly lower than the variance swap strike prices during August 2001–February 2005.

7. Numerical examples and discussions

In this section, some numerical examples are presented to illustrate the implications of Propositions 6.1–6.2, discussed in the previous section. In Example 7.1, we show that when the sufficient conditions in assertion (1) of Proposition 6.1 are satisfied, the inequality (6.1) holds for $v_0 > 0$. However, we also show that the inequality (6.1) can no longer be true if the sufficient conditions are violated. In Example 7.2, we demonstrate that the condition $\rho = 0$ in Proposition 6.2 is only a sufficient condition for obtaining the inequality (6.2). Furthermore, the numerical results show that the fair strike prices of the gamma swaps exhibit more sensitivity to ρ when σ_V becomes positively larger. In our numerical examples, we assume the number of trading days in 1 year to be 252, that is, $T = 1$, $N = 252$, and $\Delta t = T/N = 1/252$.

Example 7.1. This example considers the fair strike prices discussed in Proposition 6.1 when $r > 0$. First, we set the parameter vector to be $p_1 = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) = (0.20, 0.10, 11.35, 0.022, 0.0618, -0.64)$, which is used by Zhu and Lian [13], but σ_V is decreased by 10 times. Note that $r = r_0 - d_0 = 0.10$ and $p_1 \in \Theta_1$ and the condition

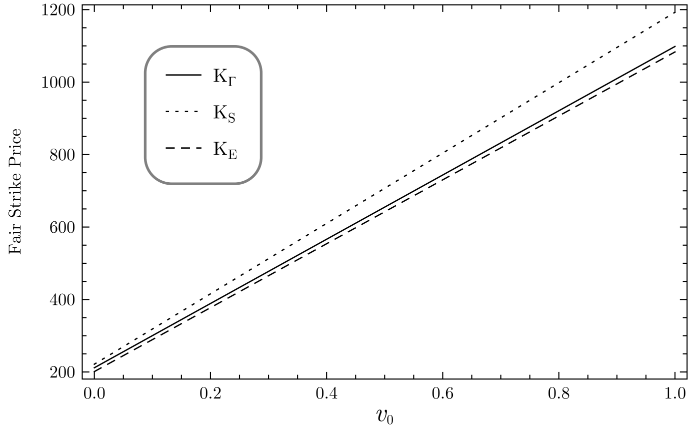

FIGURE 1. Variations of $K_\Gamma(\Delta t, v_0; p_1)$, $K_S(\Delta t, v_0; p_1)$, and $K_E(\Delta t, v_0; p_1)$ against $v_0 \in (0, 1)$.

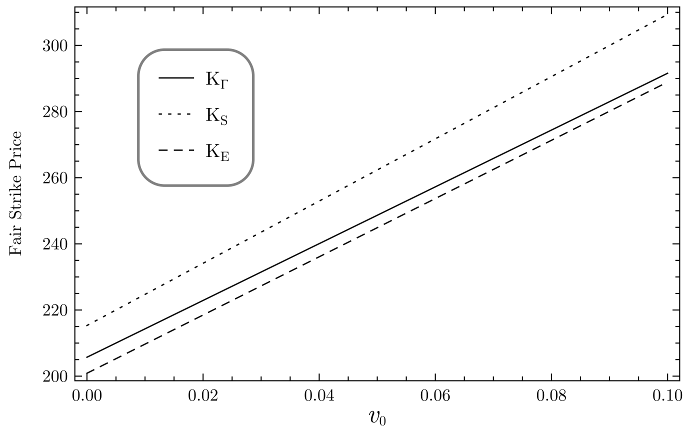

FIGURE 2. Variations of $K_\Gamma(\Delta t, v_0; p_2)$, $K_S(\Delta t, v_0; p_2)$, and $K_E(\Delta t, v_0; p_2)$ against $v_0 \in (0, 0.1)$.

$r + 2\rho\sigma_V > 0$ holds. As displayed in Figure 1, $K_\Gamma(\Delta t, v_0; p_1)$, $K_S(\Delta t, v_0; p_1)$, and $K_E(\Delta t, v_0; p_1)$ are plotted against $v_0 \in (0, 1)$ such that the fair strike prices satisfy the inequality (6.1). Next, we set the parameter vector to be $p_2 = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) = (0.20, 0.10, 11.35, 0.022, 0.618, -0.64)$. We have $p_2 \in \Theta_1$, but $r + 2\rho\sigma_V < 0$. Figure 2 shows that $K_\Gamma(\Delta t, v_0; p_2)$, $K_S(\Delta t, v_0; p_2)$, and $K_E(\Delta t, v_0; p_2)$ satisfy the inequality (6.1) for all $v_0 \in (0, 0.1)$. As displayed by Figure 3, however, the fair strike prices do not satisfy the inequality (6.1) because $K_\Gamma(\Delta t, v_0; p_2) < K_E(\Delta t, v_0; p_2) < K_S(\Delta t, v_0; p_2)$ for all $v_0 \in (0.5, 0.8)$. Furthermore, we set the parameter vector to be $p_3 = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) = (0.03, 0.01, 11.35, 0.022, 0.618, -0.64)$, where r is reduced to 0.02. From Figure 4, we have $K_\Gamma(\Delta t, v_0; p_3) < K_S(\Delta t, v_0; p_3) < K_E(\Delta t, v_0; p_3)$ for all

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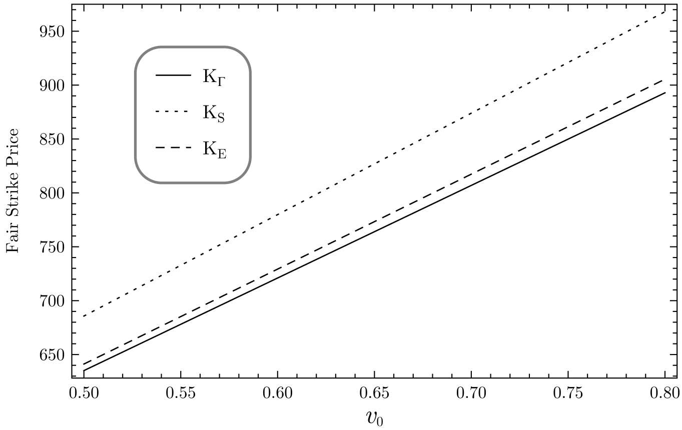

FIGURE 3. Variations of $K_\Gamma(\Delta t, v_0; p_2)$, $K_S(\Delta t, v_0; p_2)$, and $K_E(\Delta t, v_0; p_2)$ against $v_0 \in (0.5, 0.8)$.

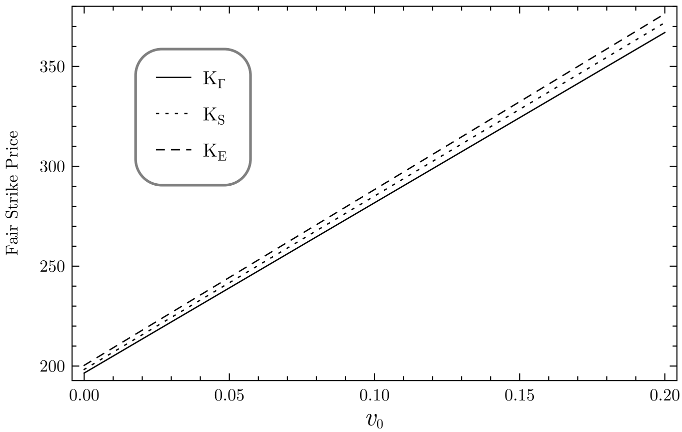

FIGURE 4. Variations of $K_\Gamma(\Delta t, v_0; p_3)$, $K_S(\Delta t, v_0; p_3)$, and $K_E(\Delta t, v_0; p_3)$ against $v_0 \in (0, 0.2)$.

$v_0 \in (0, 0.2)$. The results obtained by using p_2 and p_3 demonstrate that the inequality (6.1) can no longer be true if condition $r + 2\rho\sigma_V > 0$ is violated.

EXAMPLE 7.2. We consider the fair strike prices discussed in Proposition 6.2 by focusing on the parameter ρ. We choose $p_4(\rho) := (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) = (0.10, 0.20, 11.35, 0.022, 0.0618, \rho)$ for any $\rho \in [-1, 1]$ such that $p_4(\rho) \in \Theta_1$, where $r = r_0 - d_0 = -0.10$ is set to be negative. In other words, we have assumed that the underlying price tends to fall down in the current period. As displayed in Figure 5, $K_{\text{var}}(\Delta t, v_0; p_4(\rho))$, $K_\Gamma(\Delta t, v_0; p_4(\rho))$, $K_S(\Delta t, v_0; p_4(\rho))$, and $K_E(\Delta t, v_0; p_4(\rho))$ are plotted against $\rho \in [-1, 1]$, where we set $v_0 = 0.5$. One can see from Figure 5 that the fair strike prices satisfy the inequality (6.3) for all $\rho \in [-1, 1]$. In particular, $K_\Gamma(\Delta t, v_0; p_4(\rho)) < K_{\text{var}}(\Delta t, v_0; p_4(\rho))$. As pointed out in Section 6.2, this result agrees with the empirical

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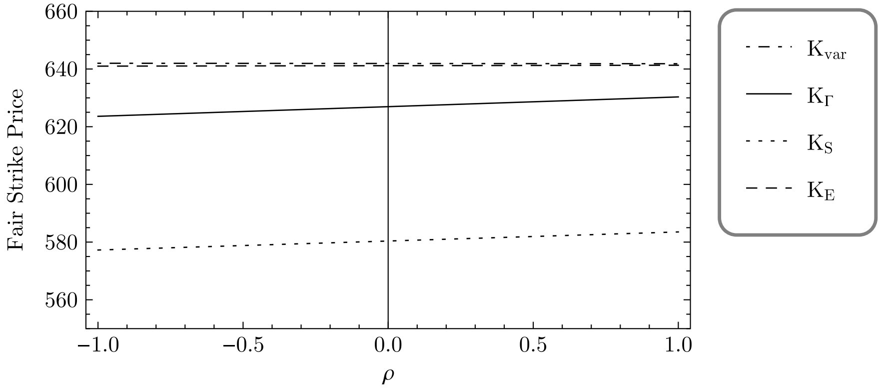

FIGURE 5. Variations of $K_{\text{var}}(p_4(\rho))$, $K_\Gamma(p_4(\rho))$, $K_S(p_4(\rho))$, and $K_E(p_4(\rho))$ against $\rho \in [-1, 1]$ with $\sigma_V = 0.0618$.

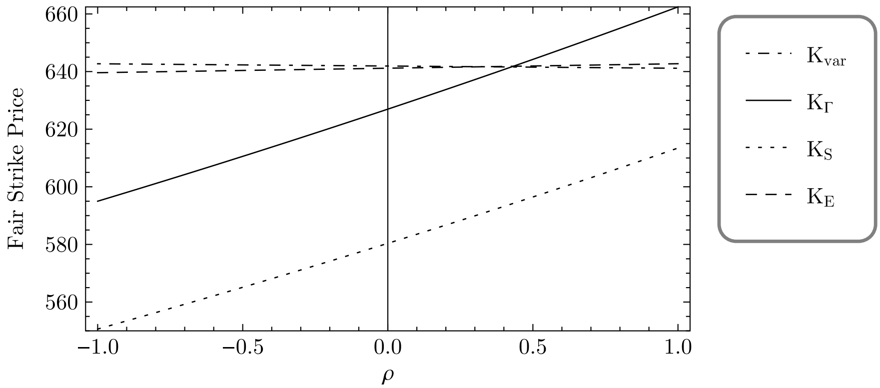

FIGURE 6. Variations of $K_{\text{var}}(p_5(\rho))$, $K_\Gamma(p_5(\rho))$, $K_S(p_5(\rho))$, and $K_E(p_5(\rho))$ against $\rho \in [-1, 1]$ with $\sigma_V = 0.618$.

data of the 6-month gamma and variance swap strike prices on the €-Stoxx 50 index, shown in the BNP Paribas handbook [6], such that the gamma swap strike price is lower than the variance swap strike price when the underlying price tends to fall down. Finally, we change the parameter vector to be $p_5(\rho) = (r_0, d_0, \kappa^*, \theta^*, \sigma_V, \rho) = (0.10, 0.20, 11.35, 0.022, 0.618, \rho)$ for any $\rho \in [-1, 1]$ such that $p_5(\rho) \in \Theta_1$. As displayed in Figure 6, $K_{\text{var}}(\Delta t, v_0; p_5(\rho))$, $K_\Gamma(\Delta t, v_0; p_5(\rho))$, $K_S(\Delta t, v_0; p_5(\rho))$, and $K_E(\Delta t, v_0; p_5(\rho))$ satisfy the inequality (6.3) only for all $\rho \in [-1, 0.4)$. The numerical results obtained by using $p_4(\rho)$ and $p_5(\rho)$ demonstrate that the condition $\rho = 0$ is only a sufficient condition to obtain assertion (2) of Proposition 6.2. In addition, Figures 5 and 6 illustrate that the fair strike prices of the gamma swaps exhibit more sensitivity to ρ when σ_V becomes positively larger, as previously discussed in Section 6.2.

8. Conclusions

In this paper, a simple closed-form formula for pricing discretely sampled gamma swaps based on Heston's two-factor stochastic volatility model [5] is presented. The closed-form formula for the fair strike prices of gamma swaps is in a much simpler form than that presented earlier in the literature [12]. Furthermore, we have provided examples of restrictions on model parameters, that is, subspaces of the parameter space, that need to be imposed in order for the derived formula to lead to a financially meaningful fair strike price. Another contribution of the paper is that we have demonstrated that the closed-form formulae for the fair strike prices of self-quantoed variance and entropy swaps can be readily obtained by adopting this approach. Finally, we have provided some remarks in Propositions 6.1–6.2, and carried out some numerical tests for market practitioners who need to compare between the fair strike prices of variance and weighted variance swaps discussed in the paper.

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Appendix A

Proof of Proposition 3.1. We first apply Itô's lemma [7] to the transformation $X_t = \ln S_t$. This gives

$$\begin{cases} dX_t = (r - \frac{1}{2}v_t)\,dt + \sqrt{v_t}\,d\tilde{B}_t^S, \\ dv_t = \kappa^*(\theta^* - v_t)\,dt + \sigma_V\sqrt{v_t}\,d\tilde{B}_t^V. \end{cases} \tag{A.1}$$

Consider the two-dimensional Itô process (X_t, v_t). Let $c \in \mathbb{R}$. We set

$$U_i(x, v, t) = E^Q_{t_{i-1}}[e^{X_t}(X_t - c)^2] = E^Q[e^{X_t}(X_t - c)^2 | (X_{t_{i-1}} = x, v_{t_{i-1}} = v)] \tag{A.2}$$

for all $(x, v, t) \in D_i = \mathbb{R} \times \mathbb{R}^+ \times [t_{i-1}, t_i)$. Applying the risk-neutral pricing and Feynman–Kac formula for (A.1) and (A.2), U_i satisfies the following partial differential equation (PDE):

$$\frac{\partial U_i}{\partial t} + \frac{1}{2}v\frac{\partial^2 U_i}{\partial x^2} + \frac{1}{2}\sigma_V^2 v\frac{\partial^2 U_i}{\partial v^2} + \rho\sigma_V v\frac{\partial^2 U_i}{\partial x \partial v} + \left(r - \frac{1}{2}v\right)\frac{\partial U_i}{\partial x} + \kappa^*(\theta^* - v)\frac{\partial U_i}{\partial v} = 0 \tag{A.3}$$

for all $(x, v, t) \in D_i$, subject to the terminal condition

$$U_i(x, v, t_i) = e^x(x - c)^2 = e^x(x^2 - 2cx + c^2) \tag{A.4}$$

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with $(x, v) \in \mathbb{R} \times \mathbb{R}^+$. Let $\tau = t_i - t$. We solve the PDE (A.3), subject to (A.4), by assuming that its solution can be written in the form

$$U_i(x, v, t) = e^x\{H_0(t_i - t) + H_1(t_i - t)v + H_2(t_i - t)v^2 + H_3(t_i - t)x + H_4(t_i - t)x^2 + H_5(t_i - t)xv\} \tag{A.5}$$

for all $(x, v, t) \in D_i$. Following the method proposed by Rujivan and Zhu [9] and setting $\omega = \rho\sigma_V - \kappa^*$, we can show that $H_k(\tau)$, $k = 0, 1, \ldots, 5$, satisfy the following system of linear ordinary differential equations (ODEs):

$$\begin{cases} \dfrac{dH_4}{d\tau} = rH_4, \dfrac{dH_5}{d\tau} = (r + \omega)H_5 + H_4, \\ \dfrac{dH_2}{d\tau} = (r + 2\omega)H_2 + \dfrac{1}{2}H_5, \dfrac{dH_3}{d\tau} = r(H_3 + 2H_4) + \kappa^*\theta^* H_5, \\ \dfrac{dH_1}{d\tau} = (r + \omega)H_1 + \dfrac{1}{2}H_3 + H_4 + (2\kappa^*\theta^* + \sigma_V^2)H_2 + (r + \rho\sigma_V)H_5, \\ \dfrac{dH_0}{d\tau} = r(H_0 + H_3)\kappa^*\theta^* H_1, \end{cases} \tag{A.6}$$

for all $\tau > 0$, subject to the initial conditions

$$H_4(0) = 1, \quad H_5(0) = 0, \quad H_2(0) = 0, \quad H_3(0) = -2c, \quad H_1(0) = 0, \quad H_0(0) = c^2. \tag{A.7}$$

By assuming that $\omega \neq 0$ and using the symbolic package DSolve in Mathematica for solving the initial value problem (A.6) subject to (A.7), the solutions can be expressed as

$$\begin{cases} H_4(\tau) = e^{r\tau}, H_5(\tau) = (e^{(r+\omega)\tau} - e^{r\tau})\omega^{-1}, \\ H_2(\tau) = F_2(\tau), H_3(\tau) = G(\tau) - 2ce^{r\tau}, \\ H_1(\tau) = F_1(\tau) - cH_5(\tau), \\ H_0(\tau) = F_0(\tau) + c^2 e^{r\tau} - cG(\tau) \end{cases} \tag{A.8}$$

for all $\tau \geqslant 0$, where $G(\tau) = [\kappa^*\theta^* e^{(r+\omega)\tau} + e^{r\tau}\{2r\omega^2\tau - \kappa^*\theta^*(\omega\tau + 1)\}]\omega^{-2}$ and $F_k(\tau)$, $k = 0, 1, 2$, are given in (3.4)–(3.6), respectively. In particular, by setting $x = c$ in (A.5) and using (3.4)–(3.6), we obtain

$$U_i(c, v, t) = e^c\{F_0(t_i - t) + F_1(t_i - t)v + F_2(t_i - t)v^2\} \tag{A.9}$$

for all $t \in [t_{i-1}, t_i)$ and $v \in \mathbb{R}^+$. Next, we consider the left-hand side (LHS) of (3.3). Since $X_{t_{i-1}}$ and $v_{t_{i-1}}$ are $\mathcal{F}_{t_{i-1}}$-measurable, we can set $x = X_{t_{i-1}}, v = v_{t_{i-1}}, t = t_{i-1}$ in (A.5) and $c = X_{t_{i-1}}$ in (A.7). Then we use (A.9) to obtain the RHS of equation (3.3). The proof of Proposition 3.1 is now complete. □

Appendix B

Proof of Proposition 3.2. Utilizing Proposition 3.1, the conditional expectation on the RHS of equation (3.2) can be written as

$$E_0^Q[E_{t_{i-1}}^Q[e^{X_{t_i}}(X_{t_i} - X_{t_{i-1}})^2]] = (E_0^Q[e^{X_{t_{i-1}}}])F_0(\Delta t) + (E_0^Q[e^{X_{t_{i-1}}}v_{t_{i-1}}])F_1(\Delta t) + (E_0^Q[e^{X_{t_{i-1}}}v_{t_{i-1}}^2])F_2(\Delta t), \tag{B.1}$$

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where $\Delta t = t_i - t_{i-1}$. It is easy to show that

$$E_0^Q[e^{X_{t_{i-1}}}] = E_0^Q[S_{t_{i-1}}] = S_0 e^{rt_{i-1}}. \tag{B.2}$$

Next, we set

$$\begin{cases} U_0^{(1)}(x, v, t) = E_0^Q[e^{X_t} v_t] = E^Q[e^{X_t} v_t \mid (X_0 = x, v_0 = v)], \\ U_0^{(2)}(x, v, t) = E_0^Q[e^{X_t} v_t^2] = E^Q[e^{X_t} v_t^2 \mid (X_0 = x, v_0 = v)] \end{cases} \tag{B.3}$$

for $(x, v, t) \in D_0 = \mathbb{R} \times \mathbb{R}^+ \times [0, t_{i-1})$. Applying the risk-neutral pricing and Feynman–Kac formula for (A.1) and (B.3), $U_0^{(1)}$ and $U_0^{(2)}$ satisfy the PDE (A.3) for all $(x, v, t) \in D_0$, subject to the terminal conditions

$$U_0^{(1)}(x, v, t_{i-1}) = e^x v \quad \text{and} \quad U_0^{(2)}(x, v, t_{i-1}) = e^x v^2$$

for all $(x, v) \in \mathbb{R} \times \mathbb{R}^+$. We assume that

$$\begin{cases} U_0^{(1)}(x, v, t) = e^x(H_0^{(1)}(t_{i-1} - t) + H_1^{(1)}(t_{i-1} - t)v), \\ U_0^{(2)}(x, v, t) = e^x(H_0^{(2)}(t_{i-1} - t) + H_1^{(2)}(t_{i-1} - t)v + H_2^{(2)}(t_{i-1} - t)v^2) \end{cases} \tag{B.4}$$

for all $(x, v, t) \in D_0$. Let $\tau = t_{i-1} - t$ and $\omega = \rho\sigma_V - \kappa^*$. Following the method used in Appendix A, we obtain the following system of linear ODEs:

$$\begin{cases} \dfrac{dH_1^{(1)}}{d\tau} = (r + \omega)H_1^{(1)}, \quad \dfrac{dH_0^{(1)}}{d\tau} = rH_0^{(1)} + \kappa^*\theta^* H_1^{(1)}, \\ \dfrac{dH_2^{(2)}}{d\tau} = (r + 2\omega)H_2^{(2)}, \quad \dfrac{dH_1^{(2)}}{d\tau} = (r + \omega)H_1^{(2)} + (2\kappa^*\theta^* + \sigma_V^2)H_2^{(2)}, \\ \dfrac{dH_0^{(2)}}{d\tau} = rH_0^{(2)} + \kappa^*\theta^* H_1^{(2)} \end{cases} \tag{B.5}$$

for all $\tau > 0$, subject to the initial conditions

$$H_1^{(1)}(0) = 1, \quad H_0^{(1)}(0) = 0, \quad H_2^{(2)}(0) = 1, \quad H_1^{(2)}(0) = 0, \quad H_0^{(2)}(0) = 0. \tag{B.6}$$

Assuming that $\omega \neq 0$ and using DSolve for solving the initial value problem (B.5) subject to (B.6) gives

$$\begin{cases} H_1^{(1)}(\tau) = e^{(r+\omega)\tau}, \quad H_0^{(1)}(\tau) = \dfrac{\kappa^*\theta^*}{\omega}(e^{(r+\omega)\tau} - e^{r\tau}), \\ H_2^{(2)}(\tau) = e^{(r+2\omega)\tau}, \quad H_1^{(2)}(\tau) = \gamma(e^{(r+2\omega)\tau} - e^{(r+\omega)\tau}), \\ H_0^{(2)}(\tau) = \eta e^{(r-2\kappa^*)\tau}(e^{\kappa^*\tau} - e^{\rho\sigma_V\tau})^2, \end{cases}$$

where $\gamma = (2\kappa^*\theta^* + \sigma_V^2)/\omega$ and $\eta = [\kappa^*\theta^*(2\kappa^*\theta^* + \sigma_V^2)]/2\omega^2$. Setting $x = X_0, v = v_0$, and $t = 0$ in (B.4) yields

$$\begin{cases} E_0^Q[e^{X_{t_{i-1}}} v_{t_{i-1}}] = e^{X_0}(H_0^{(1)}(t_{i-1}) + H_1^{(1)}(t_{i-1})v_0), \\ E_0^Q[e^{X_{t_{i-1}}} v_{t_{i-1}}^2] = e^{X_0}(H_0^{(2)}(t_{i-1}) + H_1^{(2)}(t_{i-1})v_0 + H_2^{(2)}(t_{i-1})v_0^2). \end{cases} \tag{B.7}$$

Inserting (B.2) and (B.7) into the RHS of (B.1) and multiplying by $1/S_0$, the result obtained can be simplified to get equation (3.8). This completes the proof of Proposition 3.2. □

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Appendix C

Proof of Proposition 4.1. We will demonstrate that the proposition can be easily obtained by investigating the monotonicity and convexity properties of F_0 and F_1 derived from a system of linear ODEs (A.6) with the initial conditions (A.7). Firstly, we show that Δt_p^* defined in (4.1) is either strictly positive or infinite. From (A.7) and (A.8), we have

$$F_0(0) = 0 \quad \text{and} \quad F_1(0) = 0.$$

Using (A.6) and (A.8), we get

$$\begin{aligned}\frac{dF_1}{d\tau} &= \frac{dH_1}{d\tau} + c\frac{dH_5}{d\tau} \\ &= \{k_1 H_1 + \tfrac{1}{2}H_3 + H_4 + k_2 H_2 + k_3 H_5\} + c\{k_1 H_5 + H_4\},\end{aligned} \tag{C.1}$$

where $k_1 = r + \omega, k_2 = 2\kappa^*\theta^* + \sigma_V^2$, and $k_3 = r + \rho\sigma_V$. Applying (A.7) to (C.1), we obtain

$$\left.\frac{dF_1}{d\tau}\right|_{\tau=0} = \{-c + 1\} + c = 1 > 0. \tag{C.2}$$

Since $F_1(0) = 0$, using (C.2) we can show that F_1 is strictly positive and increasing on $(0, \Delta t_p^{F_1}]$ for some $\Delta t_p^{F_1} > 0$. Next, we compute $dF_0/d\tau$ at $\tau = 0$. From (A.6) and (A.8), we have

$$\frac{dF_0}{d\tau} = \frac{dH_0}{d\tau} + c\frac{dG}{d\tau} - rc^2 e^{r\tau} = \{r(H_0 + H_3) + \kappa^*\theta^* H_1\} + c\frac{dG}{d\tau} - rc^2 e^{r\tau}. \tag{C.3}$$

Applying (A.7) to (C.3) yields

$$\left.\frac{dF_0}{d\tau}\right|_{\tau=0} = \{r(c^2 - 2c)\} + c\{2r\} - rc^2 = 0. \tag{C.4}$$

Next, consider $d^2F_0/d\tau^2$ at $\tau = 0$. From (C.3) and using (A.6), we arrive at

$$\begin{aligned}\frac{d^2F_0}{d\tau^2} &= \left\{r\left(\frac{dH_0}{d\tau} + \frac{dH_3}{d\tau}\right) + \kappa^*\theta^*\frac{dH_1}{d\tau}\right\} + c\frac{d^2G}{d\tau^2} - r^2c^2e^{r\tau} \\ &= [r\{(r(H_0 + H_3) + \kappa^*\theta^* H_1) + (r(H_3 + 2H_4) + \kappa^*\theta^* H_5)\} \\ &\quad + \kappa^*\theta^*(k_1H_1 + \tfrac{1}{2}H_3 + H_4 + k_2H_2 + k_3H_5)] + c\frac{d^2G}{d\tau^2} - r^2c^2e^{r\tau}.\end{aligned} \tag{C.5}$$

Applying (A.7) to (C.5), we obtain

$$\begin{aligned}\left.\frac{d^2F_0}{d\tau^2}\right|_{\tau=0} &= \{r(\{r\{c^2 - 2c\}\} + \{r\{-2c + 2\}\} + \kappa^*\theta^*\{-c + 1\}\} + c\{4r^2 + \kappa^*\theta^*\} - r^2c^2 \\ &= 2r^2 + \kappa^*\theta^* > 0.\end{aligned} \tag{C.6}$$

From (C.4) and (C.6), it follows that F_0 has a local minimum at $\tau = 0$. Since $F_0(0) = 0$, there exists $\Delta t_p^{F_0} > 0$ such that F_0 is strictly positive and increasing on $(0, \Delta t_p^{F_0}]$. Consequently, F_0 and F_1 are strictly positive and increasing on $(0, \min(\Delta t_p^{F_0}, \Delta t_p^{F_1}))$.

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Clearly, a positive root of F_0 or F_1 may or may not exist depending on p. Suppose that either F_0 or F_1 has a positive root. Since Δt_p^* is the smallest positive root of the product function $F_0(\tau; p)F_1(\tau; p)$, $\Delta t_p^* > \min(\Delta t_p^{F_0}, \Delta t_p^{F_1}) > 0$. On the other hand, if neither F_0 nor F_1 has a positive root, this implies that $\Delta t_p^* = \infty$.

Secondly, we consider $K_\Gamma(T, \Delta t, v_0; p)$ as in (3.12). Since $F_i(\Delta t; p), i = 0, 1, 2$, are finite and strictly positive for all $\Delta t \in (0, \Delta t_p^*)$ and $p \in \Theta_1$, one can verify that the coefficients of $F_i, i = 0, 1, 2$, as given in (3.9)–(3.11), are always finite and strictly positive, unless $\omega = 0$. Using these results, we obtain assertion (2.1). Next, one can verify that assertion (2.2) is a consequence of assertion (2.1), since for any given $\Delta t \in (0, \Delta t_p^*)$ and $t_i = i\Delta t \in [0, T], i = 0, 1, \ldots, N$,

$$\frac{\partial K_\Gamma}{\partial v_0} = \frac{100^2}{T}\bigg\{\bigg(\sum_{i=1}^{N} \bar{A}_1(\Delta t, t_{i-1}; p)\bigg) + 2\bigg(\sum_{i=1}^{N} \bar{A}_2(\Delta t, t_{i-1}; p)\bigg)v_0\bigg\} > 0 \quad \text{for all } v_0 > 0.$$

Now the proof of Proposition 4.1 is complete. □

Appendix D

Proof of Proposition 6.1. From (3.9)–(3.11) and (5.5)–(5.7), for any $i \in \{1, 2, \ldots, N\}$, we set

$$\begin{aligned}
\bar{A}_0(\Delta t, t_{i-1}) - \bar{B}_0(\Delta t, t_{i-1}) &= d_i^{(0)}(\Delta t)F_0(\Delta t) + d_i^{(1)}(\Delta t)F_1(\Delta t) + d_i^{(2)}(\Delta t)F_2(\Delta t),\\
\bar{A}_1(\Delta t, t_{i-1}) - \bar{B}_1(\Delta t, t_{i-1}) &= d_i^{(3)}(\Delta t)F_1(\Delta t) + d_i^{(4)}(\Delta t)F_2(\Delta t),\\
\bar{A}_2(\Delta t, t_{i-1}) - \bar{B}_2(\Delta t, t_{i-1}) &= d_i^{(5)}(\Delta t)F_2(\Delta t),
\end{aligned}$$

where

$$d_i^{(0)}(\Delta t) = e^{r(i-1)\Delta t} - 1, \tag{D.1}$$

$$d_i^{(1)}(\Delta t) = \frac{\kappa^*\theta^*}{\omega} e^{(r+\omega)(i-1)\Delta t}(1 - e^{-\omega(i-1)\Delta t}) - \theta^*(1 - e^{-\kappa^*(i-1)\Delta t}), \tag{D.2}$$

$$\begin{aligned}
d_i^{(2)}(\Delta t) = {}& \frac{\kappa^*\theta^*(2\kappa^*\theta^* + \sigma_V^2)}{2\omega^2} e^{(r+2\omega)(i-1)\Delta t}(1 - e^{-\omega(i-1)\Delta t})^2\\
&- \frac{\theta^*(2\kappa^*\theta^* + \sigma_V^2)}{2\kappa^*}(1 - e^{-\kappa^*(i-1)\Delta t})^2,
\end{aligned} \tag{D.3}$$

$$d_i^{(3)}(\Delta t) = e^{(r+\omega)(i-1)\Delta t} - e^{-\kappa^*(i-1)\Delta t}, \tag{D.4}$$

$$\begin{aligned}
d_i^{(4)}(\Delta t) = {}& \frac{(2\kappa^*\theta^* + \sigma_V^2)}{\omega} e^{(r+2\omega)(i-1)\Delta t}(1 - e^{-\omega(i-1)\Delta t})\\
&- \frac{(2\kappa^*\theta^* + \sigma_V^2)}{\kappa^*}(1 - e^{-\kappa^*(i-1)\Delta t})e^{-\kappa^*(i-1)\Delta t},
\end{aligned} \tag{D.5}$$

$$d_i^{(5)}(\Delta t) = e^{(r+2\omega)(i-1)\Delta t} - e^{-2\kappa^*(i-1)\Delta t}, \tag{D.6}$$

and $\omega = \rho\sigma_V - \kappa^*$. From (D.1)–(D.6), we have $d_i^{(j)}(0) = 0$ for all $j = 1, 2, \ldots, 5$ and $i = 1, 2, \ldots, N$. Next, we investigate the monotonicity and convexity properties of

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$d_i^{(j)}$s defined in (D.1)–(D.6) with respect to $\tau = \Delta t$. The following derivatives can be obtained by using MATHEMATICA:

$$\frac{d}{d\tau} d_i^{(0)}(\tau)\Big|_{\tau=0} = (i-1)r, \tag{D.7}$$

$$\frac{d}{d\tau} d_i^{(1)}(\tau)\Big|_{\tau=0} = 0, \quad \frac{d^2}{d\tau^2} d_i^{(1)}(\tau)\Big|_{\tau=0} = 2(i-1)^2\kappa^*\theta^*\Big(r + \frac{1}{2}\rho\sigma_V\Big), \tag{D.8}$$

$$\frac{d}{d\tau} d_i^{(2)}(\tau)\Big|_{\tau=0} = \frac{d^2}{d\tau^2} d_i^{(2)}(\tau)\Big|_{\tau=0} = 0,$$

$$\frac{d^3}{d\tau^3} d_i^{(2)}(\tau)\Big|_{\tau=0} = 3(i-1)^3\kappa^*\theta^*(2\kappa^*\theta + \sigma_V^2)(r + \rho\sigma_V), \tag{D.9}$$

$$\frac{d}{d\tau} d_i^{(3)}(\tau)\Big|_{\tau=0} = (i-1)(r + \rho\sigma_V), \tag{D.10}$$

$$\frac{d}{d\tau} d_i^{(4)}(\tau)\Big|_{\tau=0} = 0, \quad \frac{d^2}{d\tau^2} d_i^{(4)}(\tau)\Big|_{\tau=0} = 2(i-1)^2(2\kappa^*\theta^* + \sigma_V^2)\Big(r + \frac{3}{2}\rho\sigma_V\Big), \tag{D.11}$$

$$\frac{d}{d\tau} d_i^{(5)}(\tau)\Big|_{\tau=0} = (i-1)(r + 2\rho\sigma_V) \tag{D.12}$$

for all $i = 1, 2, \ldots, N$.

Suppose that $r > 0$ and $r + 2\rho\sigma_V > 0$. From Proposition 4.1 and (D.7)–(D.12), one can show that there exists $T_p^* > 0$ such that

$$F_k(\Delta t) > 0, \tag{D.13}$$

$$d_i^{(j)}(\Delta t) > 0 \tag{D.14}$$

for all $\Delta t \in (0, T_p^*)$, $k = 0, 1, 2$, $j = 1, 2, \ldots, 5$, and $i = 2, 3, \ldots, N$. The obtained results (D.13)–(D.14) and $r > 0$ are sufficient conditions to show that

$$K_E(\Delta t, v_0; p) < K_\Gamma(\Delta t, v_0; p) < K_S(\Delta t, v_0; p)$$

for all $\Delta t \in (0, T_p^*)$. Hence, we obtain assertion (1).

On the other hand, let $r < 0$ and $r + 2\rho\sigma_V < 0$. By using Proposition 4.1 and (D.13)–(D.14) once again, one can easily derive (D.13) and

$$d_i^{(j)}(\Delta t) < 0 \tag{D.15}$$

for all $\Delta t \in (0, T_p^*)$, $k = 0, 1, 2$, $j = 1, 2, \ldots, 5$, and $i = 2, 3, \ldots, N$. The obtained results (D.13) and (D.15) and $r < 0$ imply that

$$K_E(\Delta t, v_0; p) > K_\Gamma(\Delta t, v_0; p) > K_S(\Delta t, v_0; p)$$

for all $\Delta t \in (0, T_p^*)$ and we immediately obtain assertion (2). Finally, assertion (3) is trivial, due to the fact that

$$d_i^{(j)}(\Delta t)|_{r=0,\rho=0} = 0$$

for all $\Delta t \in (0, T_p^*)$, $j = 1, 2, \ldots, 5$, and $i = 1, 2, \ldots, N$. This completes the proof of Proposition 6.1. □

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Appendix E

Proof of Proposition 6.2. Suppose that S_t follows the dynamics described in (2.1). Let $X_t = \ln S_t$. From Rujivan and Zhu [10, Proposition 2.6], we have

$$E_0^Q[(X_{t_i} - X_{t_{i-1}})^2] = \tilde{A}_0(\Delta t, t_{i-1}) + \tilde{A}_1(\Delta t, t_{i-1})v_0 + \tilde{A}_2(\Delta t, t_{i-1})v_0^2 \tag{E.1}$$

for all $i = 1, 2, \ldots, N$ and $v_0 > 0$, where $\Delta t = t_i - t_{i-1}$. Using (E.1), (3.8), (5.2), and (5.4) by setting $\rho = 0$, we obtain the following estimates:

$$E_0^Q\Big[\frac{S_{t_i}}{S_0}(X_{t_i} - X_{t_{i-1}})^2\Big] - e^{rt_i}E_0^Q[(X_{t_i} - X_{t_{i-1}})^2] = h(\Delta t, t_i, v_0)e^{(r-\kappa^*)t_i}r, \tag{E.2}$$

$$E_0^Q\Big[\frac{S_{t_N}}{S_0}(X_{t_i} - X_{t_{i-1}})^2\Big] - e^{rt_N}E_0^Q[(X_{t_i} - X_{t_{i-1}})^2] = h(\Delta t, t_i, v_0)e^{(rt_N-\kappa^* t_i)}r, \tag{E.3}$$

$$E_0^Q\Big[\frac{S_{t_i}}{S_{t_{i-1}}}(X_{t_i} - X_{t_{i-1}})^2\Big] - e^{r\Delta t}E_0^Q[(X_{t_i} - X_{t_{i-1}})^2] = h(\Delta t, t_i, v_0)e^{(r\Delta t-\kappa^* t_i)}r \tag{E.4}$$

for all $i = 1, 2, \ldots, N$, where

$$h(\Delta t, t_i, v_0) = \frac{2\{(v_0 - \theta^*)(e^{\kappa^*\Delta t} - 1) + \kappa^*\theta^* e^{\kappa^* t_i}\Delta t\}\Delta t}{\kappa^*}. \tag{E.5}$$

Next, we investigate the monotonicity and convexity properties of h with respect to $\tau = \Delta t$. Note that

$$h(0, t_i, v_0) = 0, \qquad \frac{d}{d\tau}h(\tau, t_i, v_0)\Big|_{\tau=0} = 0,$$

and

$$\frac{d^2}{d\tau^2}h(\tau, t_i, v_0)\Big|_{\tau=0} = 4\{v_0 + (e^{\kappa^* t_i} - 1)\theta^*\} > 0.$$

The obtained results imply that there exists $\Delta t_p^h > 0$ such that

$$h(\Delta t, t_i, v_0) > 0 \tag{E.6}$$

for all $\Delta t \in (0, \Delta t_p^h)$. Due to Propositions 4.1 and 6.1, we choose $\hat{T}_p^* = \min(\Delta t_p^*, T_p^*, \Delta t_p^h)$. In what follows, we consider $\Delta t \in (0, \hat{T}_p^*)$.

Case $r > 0$: From (E.4) and (E.6), one can verify that

$$E_0^Q\Big[\frac{S_{t_i}}{S_{t_{i-1}}}(X_{t_i} - X_{t_{i-1}})^2\Big] - E_0^Q[(X_{t_i} - X_{t_{i-1}})^2] > 0 \tag{E.7}$$

for all $i = 1, 2, \ldots, N$. The inequality (E.7) implies that

$$K_{\mathrm{var}}(\Delta t, v_0; p) < K_E(\Delta t, v_0; p) \tag{E.8}$$

for all $\Delta t \in (0, \hat{T}_p^*)$. Using (E.8) and assertion (1) of Proposition 6.1 with $\rho = 0$, we immediately obtain assertion (1) of Proposition 6.2.

Case $r < 0$: From (E.4) and (E.6), one can verify that

$$E_0^Q\left[\frac{S_{t_i}}{S_{t_{i-1}}}(X_{t_i} - X_{t_{i-1}})^2\right] - E_0^Q[(X_{t_i} - X_{t_{i-1}})^2] < 0 \tag{E.9}$$

for all $i = 1, 2, \ldots, N$. The inequality (E.9) implies that

$$K_{\text{var}}(\Delta t, v_0; p) > K_E(\Delta t, v_0; p) \tag{E.10}$$

for all $\Delta t \in (0, \hat{T}_p^*)$. Using (E.10) and assertion (2) of Proposition 6.1 with $\rho = 0$, we then obtain assertion (2) of Proposition 6.2.

Case $r = 0$: It is obvious from (E.2)–(E.5) that assertion (3) of Proposition 6.2 is true.

The proof of Proposition 6.2 is now complete. □

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