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REMARKS ON COMPLETENESS IN SPACES OF LINEAR OPERATORS

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Whereas a locally convex Hausdorff space X inherits any completeness properties that the space of continuous linear operators, $L(X)$, in X, may have (for the topology of pointwise convergence in X), this is not so in the converse situation and is the problem discussed here. The barrelledness of X in its Mackey topology plays an important role: if $L(X)$ is quasicomplete, then X is barrelled for its Mackey topology. Consequently, for Mackey spaces X it turns out that $L(X)$ is quasicomplete if and only if X is quasicomplete and barrelled: this is false if sequential completeness is substituted for quasicompleteness. Furthermore, there exist non-barrelled spaces X for which X and $L(X)$ are quasicomplete (sequentially complete). Hence, although barrelledness is a sufficient condition for completeness of $L(X)$ in various senses, it is certainly not necessary.

Let X be a Hausdorff topological vector space, always assumed to be locally convex, and $L(X)$ be the space of continuous linear operators in X equipped with the strong operator topology, that is, the topology of pointwise convergence in X.

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The following procedure is a familiar one: a given sequence (or net) of elements $\{T_n\}$ in $L(X)$ has the property that for each $x \in X$ the sequence (or net) $\{T_n\}$ is Cauchy in X and, hence, modulo certain completeness conditions in X, there is an element Tx, of X, such that $T_n x \to Tx$ in X. The assignment

$$T : x \to Tx , x \in X ,$$

is then a linear operation in X and the problem is to determine the continuity of T.

One possible solution to this problem is to determine whether the space $L(X)$, which is a locally convex space in its own right, is complete in an appropriate sense. Unfortunately, even though the underlying space X may have very strong completeness properties, it turns out, as seen by the series of examples below, that these may not be inherited by the space $L(X)$. Of course, in the case when X is a Banach space the situation is well understood. It is not clear, however, what the situation is in the setting of (not so familiar) non-normable spaces. The purpose of this simple note is to clarify this point.

Whether or not X is barrelled in its Mackey topology plays an important role. For, if the space $L(X)$ is quasicomplete, then necessarily X is barrelled for its Mackey topology (see Proposition 1). Consequently, for Mackey spaces (that is spaces X which have their Mackey topology) it turns out that $L(X)$ is quasicomplete if, and only if, X is quasicomplete and barrelled (see Corollary 1.1). Examples are given which show that the quasicompleteness of X and $L(X)$ in this statement cannot be replaced by sequential completeness. Furthermore, if we do not require the underlying space to have its Mackey topology, then it is shown that there are classes of non-barrelled spaces X for which both X and $L(X)$ are quasicomplete (sequentially complete). Accordingly, although barrelledness is a sufficient condition for the completeness of $L(X)$ in various senses, it is certainly not a necessary condition.

The space of continuous linear functionals on X will be denoted by X'. If we wish to indicate the topology ρ of a space X, possibly different from its original topology, we will denote the space by X_ρ.

In particular, σ or more precisely $\sigma(X,X')$ and τ (or $\tau(X,X')$) will denote the weak topology and the Mackey topology on X, respectively. Recall that X is quasicomplete (sequentially complete) if every bounded Cauchy net (Cauchy sequence) converges to some element of X. The adjoint of an operator $T \in L(X)$, which is an element of $L(X'_\sigma)$, is denoted by T'.

The collection of all seminorms q_x of the form

$$q_x : T \to q(Tx) \ , \quad T \in L(X) \ ,$$

where q is any continuous seminorm in X and $x \in X$ is arbitrary determines the topology of $L(X)$. Since the correspondence $\sum_i x_i \otimes x'_i \to \xi \in (L(X))'$, defined by

$$\xi : T \to \sum_i \langle Tx_i, x'_i \rangle \ , \quad T \in L(X) \ ,$$

is an (algebraic) isomorphism of the tensor product $X \otimes X'$ onto the dual of the locally convex Hausdorff space $L(X)$ it follows that the weak topology, $L(X)_\sigma$, of $L(X)$, is precisely the weak operator topology ([5], p. 139, Corollary 4).

The first observation which can be made is that the space X is complete (quasicomplete, sequentially complete) whenever $L(X)$ is complete (respectively quasicomplete, sequentially complete). For, if $\{x_\alpha\}$ is a net of elements in X, then for any non-zero element $x' \in X'$ the net of operators $\{T_\alpha\}$ in $L(X)$ defined by

$$T_\alpha : x \to \langle x, x' \rangle x_\alpha, \quad x \in X \ ,$$

for each α, has the property that

$$q_x(T_\alpha - T_\beta) = q(T_\alpha x - T_\beta x) = |\langle x, x' \rangle| q(x_\alpha - x_\beta) \ ,$$

for each α and β and each continuous seminorm q_x in $L(X)$. Accordingly, $\{T_\alpha\}$ is an $L(X)$-Cauchy net (bounded Cauchy net, Cauchy sequence) whenever $\{x_\alpha\}$ is a Cauchy net in X (respectively bounded Cauchy net, Cauchy sequence), in which case it follows from the hypothesis

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on $L(X)$ that there is an operator T in $L(X)$ such that $T_\alpha \to T$ in $L(X)$. If $x \in X$ is any element such that $\lambda = \langle x, x' \rangle$ is non-zero, then the X-convergence of the net $\{x_\alpha = \lambda^{-1} T_\alpha x\}$, to the element $\lambda^{-1} Tx$, of course, follows from the fact that $\lambda^{-1} T_\alpha \to \lambda^{-1} T$ in $L(X)$.

The more important problem in practice is the converse question, namely to determine what completeness properties the space of operators $L(X)$ inherits from those of X. Unfortunately, the state of affairs in this case is not so satisfactory.

It is easy to exhibit complete spaces X for which $L(X)$ fails to be complete. For example, if X is any infinite dimensional Fréchet space, then X'_σ is not complete ([5], p. 195, Exercise 21b) and, hence, $L(X)$ cannot be complete ([4], p. 143, Proposition 2(b)). The situation for sequentially complete or quasicomplete spaces X is no better.

EXAMPLE 1. A sequentially complete space X for which $L(X)$ is not sequentially complete.

Let Y be any barrelled, locally convex space for which there exists an element e in the bidual, but not in Y itself, with the property that there exists a sequence of elements $\{e_n\} \subseteq Y$ such that

$$\lim_{n\to\infty} \langle e_n, \xi \rangle = \langle e, \xi \rangle , \quad \xi \in Y' .$$

In particular then, Y cannot be semi-reflexive. A sufficient condition for the existence of such a sequence $\{e_n\}$ is that Y' be separable for the strong dual topology ([5], p. 143). For instance, if $Y = c_0$ and $e = (1,1,1,\ldots) \in l^\infty \backslash c_0$, then the sequence $\{e_n\} \subseteq c_0$, where e_n is the characteristic function of the set $\{1,2,\ldots,n\}$, for each $n = 1,2,\ldots,$ provides such an example.

If X is the space Y' equipped with its weak-star topology $\sigma(Y', Y)$, then certainly X is sequentially complete ([5], p. 148, Proposition 6.1). To see that the space $L(X)$ is not sequentially complete we observe that if $\xi \in X$ is any non-zero element, then the Cauchy sequence $\{T_n\} \subset L(X)$ given by

$$T_n : x \to \langle x, e_n \rangle \xi , \quad x \in X ,$$

for each $n = 1,2,\ldots$, has the property that it converges pointwise in X to the linear operator T specified by

$$T : x \to \langle x, e\rangle \xi \ , \quad x \in X \ .$$

Since $e \notin Y$ it follows that $T \notin L(X)$ and, hence, $\{T_n\}$ has no limit in $L(X)$.

EXAMPLE 2. A quasicomplete space X for which $L(X)$ is not sequentially complete, hence also not quasicomplete.

The space X of Example 1, being quasicomplete, also applies in this instance.

The previous examples show that without additional assumptions on the space X , very little can be deduced about the completeness properties of $L(X)$ from those of X . Even if the space X is barrelled, in addition to being complete, the discussion prior to Example 1 shows that $L(X)$ may fail to be complete. However, it is a consequence of the Banach-Steinhaus theorem, for example, that barrelledness of X is a sufficient condition for $L(X)$ to be sequentially complete (quasicomplete) whenever X is sequentially complete (quasicomplete). The first part of the following result may be considered as a partial converse to this statement.

PROPOSITION 1. *Let* X *be a locally convex Hausdorff space.*

(i) *If the space* $L(X)$ *is quasicomplete, then also* X'_σ *is quasi-complete and, in particular,* X_τ *is barrelled.*

(ii) *If* $L(X)$ *is sequentially complete, then also* X'_σ *is sequentially complete.*

Proof of (i) is based on the well-known criterion that a Mackey space Y is barrelled if and only if Y'_σ is quasicomplete ([3], p. 305, Proposition 4).

So, let $\{\xi_\alpha\}$ be a bounded Cauchy net in X'_σ . Fix a non-zero vector $z \in X$ and define a net of operators $\{T_\alpha\} \subseteq L(X)$ by

$$T_\alpha : x \to \langle x, \xi_\alpha\rangle z \ , \quad x \in X \ ,$$

for each α . Then $\{T_\alpha\}$ is a bounded Cauchy net in $L(X)$. Accordingly, there is an operator $T \in L(X)$ such that $T_\alpha \to T$ in $L(X)$. Since each operator T_α has its range in the (closed) 1-dimensional subspace of X spanned by z and the limit operator T is continuous, it follows that there exists $\xi \in X'$ such that

$$Tx = \langle x, \xi \rangle z \ , \quad x \in X \ .$$

Pick any element x' of X' for which $\langle z, x' \rangle \neq 0$. Then it follows from the fact that $\langle x, \xi_\alpha \rangle z \to \langle x, \xi \rangle z$ in X (hence, also in X_σ), for each $x \in X$, that

$$\lim_\alpha \langle x, \xi_\alpha \rangle \langle z, x' \rangle = \langle x, \xi \rangle \langle z, x' \rangle \ , \quad x \in X \ .$$

Since $\langle z, x' \rangle \neq 0$, this shows that $\xi_\alpha \to \xi$ in X'.

The proof of (ii) is analogous to that of (i) where now $\{\xi_\alpha\}$ is chosen to be a Cauchy sequence in X'_σ.

COROLLARY 1.1. *Let* X *be a Mackey space. Then* $L(X)$ *is quasicomplete if and only if* X *is barrelled and quasicomplete.*

The following result provides another class of quasicomplete spaces X for which $L(X)$ is not quasicomplete (see Example 2).

COROLLARY 1.2. *Let* Y *be a non-reflexive, barrelled space and* X *denote the space* Y' *equipped with its Mackey topology* $\tau = \tau(Y', Y)$. *Then* X *is quasicomplete but* $L(X)$ *is not quasicomplete. In particular,* X *is not barrelled.*

Proof. Since Y is barrelled the space $X = Y'_\tau$ is certainly quasicomplete ([5], p. 148, Proposition 6.1). If $L(X)$ were quasicomplete, then Proposition 1(i) would imply that X is barrelled for its Mackey topology $\tau(X, X')$, that is, Y' would be barrelled for $\tau(Y', Y)$. Since also $Y = Y_\tau$ is barrelled it would follow that Y is reflexive ([5], p. 145, Theorem 5.7) which is a contradiction. Accordingly, $L(X)$ cannot be quasicomplete and, hence, the space X, being quasicomplete, cannot be barrelled.

In view of Corollary 1.1, Proposition 1(i) and the remarks prior to it, a natural question to ask is whether there exist quasicomplete (sequentially complete) spaces X for which $L(X)$ is also quasicomplete (sequentially complete), but such that X is not barrelled. Of course, in

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the quasicomplete case X cannot then be a Mackey space. We proceed to show that the answer in both cases is positive.

A locally convex Hausdorff space X is said to have the *Schur property* if weakly convergent *sequences* in X are convergent for the given topology in X. Examples of such spaces are $\ell^1(S)$, where S is an arbitrary index set ([1], p. 33, Corollary 2), the dual space to the James Hagler space JH ([2], p. 215) or any Montel space ([3], p. 370, Proposition 3).

EXAMPLE 3. A sequentially complete space X for which $L(X)$ is also sequentially complete, but such that X is not barrelled.

Let Y be any barrelled space with the Schur property which is weakly sequentially complete (for example $\ell^1(S)$ or any Montel space). If X denotes the space Y_σ, then X is sequentially complete by the assumption on Y (it is quasicomplete if and only if Y is semi-reflexive ([5], p. 144)). Let $\{T_n\}$ be a Cauchy sequence in $L(X)$. Then for each element x of X the sequence $\{T_n x\}$ is weakly Cauchy in Y and, hence, there is an element $Tx \in X$ such that $T_n \to Tx$ in X. The Schur property of Y then implies that $T_n x \to Tx$ in Y, for each $x \in X$. Since the sequence $\{T_n\}$ is also contained in the space $L(Y) = L(Y_\tau)$, by Proposition 7.4 on p. 158 of [5], it follows from the Banach-Steinhaus theorem applied in the space Y, that the linear operator

$$T : x \to Tx , \quad x \in Y ,$$

being the pointwise limit in Y of the sequence $\{T_n\}$, belongs to $L(Y)$ and hence, also to $L(X)$. This shows that $L(X)$ is sequentially complete.

If the space X were barrelled it would have its Mackey topology and hence, the topologies $\tau(Y,Y')$ and $\sigma(Y,Y')$ would agree on Y, a phenomenon which rarely happens. For example, if Y is an infinite dimensional Banach space, then certainly $\tau(Y,Y') \neq \sigma(Y,Y')$; see Lemma 1 on p. 39 of [1], for example. Hence, the space X is not barrelled in this case. Similarly if Y is an infinite dimensional Montel space (in which case it is a Mackey space), then although the topologies $\tau(Y,Y')$ and $\sigma(Y,Y')$ agree on the bounded sets of Y they are not in general equal; in such cases X will not be barrelled.

REMARK. It is perhaps interesting to note that there exist infinite dimensional Montel spaces Y for which the topologies $\sigma(Y,Y')$ and $\tau(Y,Y')$ do agree in Y. For example, this is the case if Y is the space of all complex sequences equipped with the topology of co-ordinate-wise convergence.

EXAMPLE 4. A quasicomplete space X for which $L(X)$ is also quasi-complete, but such that X is not barrelled.

Let Y be an infinite dimensional, reflexive Banach space and X denote the space Y'_σ. Then certainly X is quasicomplete. If X were barrelled it would have its Mackey topology $\tau(X,X') = \tau(Y',Y)$ and so $\sigma(Y',Y) = \tau(Y',Y)$. Since Y is reflexive, it follows that $\sigma(Y',Y) = \sigma(Y',Y'')$ and $\tau(Y',Y) = \tau(Y',Y'')$. Accordingly, Y' would be an infinite dimensional Banach space for which the norm topology $\tau(Y',Y'')$ equals the weak topology $\sigma(Y',Y'')$, which is nonsense. Accordingly, X is not barrelled.

Let $\{T_\alpha\}$ be a bounded Cauchy net in $L(X)$. Then $\{T'_\alpha\} \subseteq L(X'_\sigma) = L(Y_\sigma)$. Since $L(Y_\sigma)$ and $L(Y)$ are equal as vector spaces it follows from

$$\sup_\alpha |\langle x, T'_\alpha \xi\rangle| = \sup_\alpha |\langle T_\alpha x, \xi\rangle| < \infty ,$$

for each $\xi \in X' = Y$ and $x \in X = Y'$, and the Principle of Uniform Boundedness that the operator norms of the net $\{T'_\alpha\}$, considered as a subset of $L(Y)$, are bounded, by M, say.

Now, the quasicompleteness of X implies that for each $x \in X$ there is an element Tx of X such that $T_\alpha x \to Tx$, in X, that is,

$$\lim_\alpha \langle T_\alpha x, \xi\rangle = \langle Tx, \xi\rangle , \quad \xi \in Y .$$

To show that $L(X)$ is quasicomplete it suffices to show that the limit operator T belongs to $L(X)$. Fix $x \in X$ and $\xi \in Y$. Then for each α we have

$$\begin{aligned} |\langle Tx, \xi\rangle| &\le |\langle Tx, \xi\rangle - \langle T_\alpha x, \xi\rangle| + |\langle x, T'_\alpha \xi\rangle| \\ &\le |\langle Tx, \xi\rangle - \langle T_\alpha x, \xi\rangle| + M\|x\| \, \|\xi\| . \end{aligned}$$

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Take the limit with respect to α gives

$$|\langle Tx, \xi\rangle| \le M\|x\| \, \|\xi\| \, , \quad \xi \in Y \, , \, x \in X \, .$$

Accordingly, $T : Y' \to Y'$ is continuous for the norm topology in Y' (with $\|T\| \le M$) and, hence, is also continuous for the weak topology $\sigma(Y', Y'')$. But then the reflexivity of Y implies that $\sigma(Y', Y'') = \sigma(Y', Y)$ and so is an element of $L(X)$, as required.

PROPOSITION 2. *Let X be a locally convex Hausdorff space. If $L(X'_\sigma)$ is quasicomplete, then*

(i) *X'_τ is barrelled and, in particular, X is semi-reflexive,*

(ii) *both X_σ and X are quasicomplete, and*

(iii) *$L(X'_\tau)$ is quasicomplete.*

Proof. That X_σ is quasicomplete and X'_τ is barrelled is immediate from Proposition 1(i). Then the barrelledness of X'_τ implies that X is semi-reflexive ([5], p. 144, Proposition 5.5). Accordingly, X is also quasicomplete ([5], p. 144, Corollary 1). Finally, (iii) follows from Proposition 4 on p. 210 of [3] since $L(X'_\tau)$ has a basis of neighbourhoods of zero consisting of $L(X'_\sigma)$-closed sets.

EXAMPLE 5. A quasicomplete, non-Mackey space X for which $L(X)$ is sequentially complete, but not quasicomplete.

Let Y be a weakly sequentially complete, non-reflexive, barrelled space with the Schur property (for example $Y = \ell^1$) . If $X = Y'_\sigma$, then X is quasicomplete, but $L(X) = L(Y'_\sigma)$ is not quasicomplete. Otherwise, Proposition 2(iii) would imply that $L(Y'_\tau)$ is quasicomplete which contradicts Corollary 1.2.

To show that $L(X)$ is sequentially complete, let $\{T_n\}$ be a Cauchy sequence in $L(X)$. Then the dual operators $T'_n : Y_\sigma \to Y_\sigma$, $n = 1, 2, \ldots$, which belong to $L(Y_\sigma)$ are also continuous for the Mackey topology in Y , that is, $\{T'_n\} \subseteq L(Y)$. Furthermore, it follows from the identities

$$\langle x, T'_n\xi\rangle = \langle T_n x, \xi\rangle \, , \quad \xi \in Y = X' \, , \, x \in X \, ,$$

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for each $n = 1,2,\ldots$, that $\{T_n'\xi\}$ is Cauchy in Y_σ for each $\xi \in Y$. Then, for each $\xi \in Y$, the weak sequential completeness of Y guarantees the existence of $S\xi \in Y$ such that $T_n'\xi \to S\xi$ in Y_σ and, hence, $T_n'\xi \to S\xi$ in Y by the Schur property. Then the Banach-Steinhaus theorem applied in the space Y implies that the linear operator

$$S : \xi \to S\xi \ , \quad \xi \in Y \ ,$$

belongs to $L(Y)$ and so, $S' \in L(Y_\sigma') = L(X)$. Furthermore, $T_n' \to S$ in $L(Y)$ implies that $T_n \to S'$ in $L(X)$. Accordingly, $\{T_n\}$ has a limit in $L(X)$.

It is clear from Corollary 1.1 that it is not possible to exhibit a non-barrelled, quasicomplete Mackey space X for which $L(X)$ is also quasicomplete. However, we do have the following situation, which should be compared with Example 3 (where the space X is not a Mackey space).

EXAMPLE 6. A non-barrelled, sequentially complete Mackey space X for which $L(X)$ is also sequentially complete.

Let Y be any space satisfying the hypotheses of Example 5. Then $X = Y_\tau'$ is certainly sequentially complete. In fact, X is quasicomplete by Corollary 1.2. Furthermore, it also follows from Corollary 1.2 that $L(X)$ is not quasicomplete and X is not barrelled. Finally, the sequential completeness of $L(X) = L(Y_\tau')$ follows from that of $L(Y_\sigma')$ by an argument as in the proof of Proposition 2(iii).

Summarizing Corollary 1.2 and Examples 5 and 6 we have

PROPOSITION 3. *Let* X *be a non-reflexive, barrelled space. Then*

(i) X' *is quasicomplete for any locally convex topology consistent with the duality of* X *and* X',

(ii) *neither of the spaces* X_σ' *or* X_τ' *is barrelled, and*

(iii) *neither of the spaces* $L(X_\sigma')$ *or* $L(X_\tau')$ *is quasicomplete.*

If X *is weakly sequentially complete and has the Schur property, in addition to satisfying the above hypotheses, then both of the spaces* $L(X_\sigma')$ *and* $L(X_\tau')$ *are sequentially complete.*

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