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# Hadamard designs

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In this paper it is shown that an Hadamard design with each letter repeated once and only once can exist for 2, 4 and 8 letters only.

L.D. Baumert and Marshall Hall, Jr have found a design with four letters each repeated three times. Their design and the design on four letters each repeated once, found by J. Williamson, is the totality previously published.

An *Hadamard design* is a square array of letters which commute in pairs, and to which signs are attached, so that the scalar product of any two distinct rows considered as vectors is zero.

There are three main types of Hadamard designs: those of Williamson and this paper where each letter is repeated once only in each row; those of Baumert-Hall where each letter is repeated more than once but each distinct letter is repeated the same number of times; and those in which the letters occur an unequal number of times.

This last category has been extensively studied and used (see Marshall Hall, Jr [2] and Jennifer Wallis [3]) in constructing Hadamard matrices. The second category has been used by L.D. Baumert and Marshall Hall, Jr [1] to construct an Hadamard matrix of order 156 . The Baumert-Hall design is

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$$\begin{array}{rrrrcrrrrcrrrr}
A & A & A & B & & -B & C & -C & -D & & B & C & -D & -D \\
A & -A & B & -A & & -B & -D & D & -C & & -B & -D & -C & -C \\
A & -B & -A & A & & -D & D & -B & B & & -C & -D & C & -C \\
B & A & -A & -A & & D & D & D & C & & C & -B & -B & -C \\
\\
B & -D & D & D & & A & A & A & C & & -C & B & -C & B \\
B & C & -D & D & & A & -A & C & -A & & -D & C & B & -B \\
D & -C & B & -B & & A & -C & -A & A & & B & C & D & -D \\
-C & -D & -C & -D & & C & A & -A & -A & & -D & B & -B & -B \\
\\
D & -C & -B & -B & & -B & C & C & -D & & A & A & A & D \\
-D & -B & C & C & & C & B & B & -D & & A & -A & D & -A \\
C & -B & -C & C & & D & -B & -D & -B & & A & -D & -A & A \\
-C & -D & -D & C & & -C & -B & B & B & & D & A & -A & -A .
\end{array}$$

No other example of this type has been published.

Of the first mentioned category, Williamson [4] gave the following example

$$\begin{array}{rrrr}
A & B & C & D \\
-B & A & -D & C \\
-C & D & A & -B \\
-D & -C & B & A .
\end{array}$$

We now study further this type of Hadamard design. To simplify our discussion, we introduce the following definition:

An Hadamard design on $n$ letters or an *$n$-letter design* is an Hadamard design with $n$ distinct letters where each letter occurs once only in each row and column.

We have already seen that there is a 4-letter design. We would like to know if there are $n$-letter designs with the number of distinct letters, $n$, greater than four. So we shall try to construct such a design.

Suppose there is an Hadamard design on the letters $A, B, C, D, \ldots$, whose first two rows are

$$\begin{array}{ccccc}
\pm A & \pm B & \pm C & \pm D & \ldots \\
\pm B & \pm A & \pm D & \pm C & \ldots
\end{array}$$

(clearly any design can be put in this form by suitable relabelling; the signs of the elements are not important at the moment). There will be a

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row whose first element is $\pm C$ ; reorder the rows so as to make this the third row. $\pm C$ is in the $(3, 1)$ position, so $\pm A$ must occur in the $(3, 3)$ position. On the other hand the $C$ element of row 3 occurs under the $\pm B$ element of row 2 , so there must be a $\pm B$ in the $(3, 4)$ position, and similarly the $(3, 2)$ element must be $\pm D$ . So the first three rows are

$$\begin{array}{ccccc} \pm A & \pm B & \pm C & \pm D & \ldots \\ \pm B & \pm A & \pm D & \pm C & \ldots \\ \pm C & \pm D & \pm A & \pm B & \ldots . \end{array}$$

In the same way, if we move the row whose first element is $\pm D$ into the position of row 4 , we find the design can be put in the form

$$\begin{array}{ccccc} \pm A & \pm B & \pm C & \pm D & \ldots \\ \pm B & \pm A & \pm D & \pm C & \ldots \\ \pm C & \pm D & \pm A & \pm B & \ldots \\ \pm D & \pm C & \pm B & \pm A & \ldots \\ . & . & . & . & \\ . & . & . & . & \\ . & . & . & . & \end{array}$$

by interchanging columns.

Write down the 4-letter design and repeat one letter of this design in the $(5, 5)$ position, then the letters in the $(t, 5)$ positions $1 \leqq t \leqq 4$ must be all different from each other and from the first mentioned four letters.

So $n = 8$ is the next smallest possible order for an Hadamard design on $n$ letters.

By repeating the above method, assuming the existence of an 8-letter design, we see that this implies $n = 16$ is the next possible order for an Hadamard design on $n$ letters. Continuing we see

LEMMA 1. *An $n$-letter Hadamard design can exist only if $n = 2^s$ , $s$ a positive integer.*

Obviously we may rearrange the rows and columns of an $n$-letter design until a single letter occurs in the diagonal positions. Then, since multiplying through a row by $-1$ does not alter the condition of

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the scalar product giving zero, we may always choose this diagonal element to have positive sign. Now we note there are eight possible ways of combining two letters into a 2-letter design:

$$\begin{matrix} X & Y \\ -Y & X \end{matrix} \qquad \begin{matrix} X & Y \\ Y & -X \end{matrix} \qquad \begin{matrix} X & -Y \\ Y & X \end{matrix} \qquad \begin{matrix} X & -Y \\ -Y & -X \end{matrix}$$

$$\begin{matrix} -X & Y \\ -Y & -X \end{matrix} \qquad \begin{matrix} -X & Y \\ Y & X \end{matrix} \qquad \begin{matrix} -X & -Y \\ Y & -X \end{matrix} \qquad \begin{matrix} -X & -Y \\ -Y & X \end{matrix} ,$$

but of these only two

$$\begin{matrix} X & Y \\ -Y & X \end{matrix} \qquad \begin{matrix} X & Y \\ Y & -X \end{matrix}$$

are distinct as the other six may be obtained from these two by substituting $\mp X$ for $\pm X$ and $\mp Y$ for $\pm Y$. We note

LEMMA 2. *If any letter of a 2-letter design is replaced by its negative throughout the design then the new design is a 2-letter design.*

Since any $n$-letter design is an array of 2-letter designs interwoven we have:

LEMMA 3. *If any letter of an $n$-letter design is replaced throughout the design by its negative then the new design is an $n$-letter design.*

Another obvious property is

LEMMA 4. *If any row or column of an $n$-letter design is multiplied throughout by $-1$ then the new design is an $n$-letter design.*

Every element of the first row of an $n$-letter design may be chosen with positive sign, for if $-X$ occurs then we may use Lemma 3 to obtain a new design with the required property.

Also we may choose the $(4t+1, 4t+1)$, $(4t+1, 4t+2)$, $(4t+1, 4t+3)$, $(4t+1, 4t+4)$ elements with positive sign, $t = 1, 2, \ldots$ . For suppose one of these, the $(4t+1, s)$ element had negative sign. Then using Lemma 4 we may multiply the $s$th column by $-1$ and so the $(s, s)$ element and the $(1, s)$ element, both formerly with positive sign, acquire a negative sign, but our element $(4t+1, s)$ has the required sign. Lemma 4 is now used to multiply the $s$th row by $-1$ so the

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diagonal element again has a positive sign, and finally Lemma 3 is used to replace the $(1, s)$ element by its negative throughout.

Finally, it is clear that the rows and columns may be interchanged so that the $(4t+1, 4t+k)$ element equals the $(1, k)$ element for $k = 1, 2, 3, 4$ .

So we have shown

LEMMA 5. *Any n-letter design is equivalent under the operations*

*(a) interchange of rows or columns,*

*(b) multiplying rows and columns by* $-1$ ,

*(c) replacing any letter by its negative throughout the design,*

*to an n-letter design with the properties*

*(i) every element of the first row has positive sign,*

*(ii) the diagonal element is constant and has positive sign,*

*(iii) the elements* $(4t+1, 4t+k) = (1, k)$ *for* $k = 1, 2, 3, 4$ , $0 \leqq t \leqq \frac{n-4}{4}$ ,

*(iv) the elements* $(4t+1, 4t+k)$ , $k = 1, 2, 3, 4$ , $0 \leqq t \leqq \frac{n-4}{4}$ *occur with positive sign.*

We will call an $n$-letter design satisfying conditions *(i)*, *(ii)*, *(iii)*, *(iv)* of Lemma 5 a *normalized n-letter design*.

Constructing the normalized 4-letter design on the letters $A, B, C, D$ we have the following first row and diagonal

$$\begin{matrix} A & B & C & D \\ & A & & \\ & & A & \\ & & & A \end{matrix} .$$

Then depending on whether we make the $(2, 3)$ element $+D$ or $-D$ we see

LEMMA 6. *There are two normalized 4-letter designs on the letters* $A, B, C, D$ ; *they are*

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$$\begin{array}{rrrr} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{array} \quad \text{and} \quad \begin{array}{rrrr} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{array} .$$

But these are equivalent under properties *(b)* and *(c)* of Lemma 5.

We now use this first-mentioned 4-letter design to construct an 8-letter design on the letters $A, B, C, D, E, F, G, H$ . Then using Lemma 5 we must have

$$\begin{array}{rrrrrrrr} A & B & C & D & E & F & G & H \\ -B & A & D & -C & & & & \\ -C & -D & A & B & & & & \\ -D & C & -B & A & & & & \\ & & & & A & B & C & D \\ & & & & & A & & \\ & & & & & & A & \\ & & & & & & & A \end{array} ,$$

in a normalized 8-letter design. Then immediately by considering the $(1, k)$ , $(1, k+4)$ and $(5, k+4)$ elements for $k = 1, 2, 3, 4$ , we see the $(5, k)$ elements must be $-E, -F, -G, -H$ respectively.

Now considering the $(2, k)$ , $(5, k)$ and $(5, k+4)$ elements we see the $(2, k+4)$ elements must be $F, -E, -H, G$ respectively. Similarly the $(3, k+4)$ elements must be $G, H, -E, -F$ and the $(4, k+4)$ elements must be $H, -G, F, -E$ . So we have

$$\begin{array}{rrrrrrrr} A & B & C & D & E & F & G & H \\ -B & A & D & -C & F & -E & -H & G \\ -C & -D & A & B & G & H & -E & -F \\ -D & C & -B & A & H & -G & F & -E \\ -E & -F & -G & -H & A & B & C & D \\ & & & & & A & & \\ & & & & & & A & \\ & & & & & & & A \end{array} .$$

Consideration of the $(k, 6)$ , $(6, 6)$ and $(k, k)$ elements shows the $(6, k)$ elements must be $-F, E, -H, G$ . Similarly the $(7, k)$ elements

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must be $-G, H, E, -F$ and the $(8, k)$ elements $-H, -G, F, E$. The $(5, 5)$, $(5, m)$ and $(m, m)$ elements for $m = 6, 7, 8$ show us the $(m, 5)$ elements must be $-B, -C, -D$. So now we have

$$\begin{array}{cccc@{\qquad}cccc}
A & B & C & D & E & F & G & H \\
-B & A & D & -C & F & -E & -H & G \\
-C & -D & A & B & G & H & -E & -F \\
-D & C & -B & A & H & -G & F & -E \\[6pt]
-E & -F & -G & -H & A & B & C & D \\
-F & E & -H & G & -B & A & & \\
-G & H & E & -F & -C & & A & \\
-H & -G & F & E & -D & & & A .
\end{array}$$

Finally the $(2, s)$, $(6, s)$ and $(2, 4+s)$ elements for $s = 3, 4$ show the $(6, 4+s)$ elements must be $-D, C$. The $(4, 1)$, $(7, 1)$ and $(4, 6)$ elements show the $(7, 6)$ element must be $D$; the $(4, 3)$, $(7, 3)$ and $(4, 8)$ elements show the $(7, 8)$ element must be $-B$; and the $(4, t)$, $(8, t)$ and $(4, 4+t)$ elements for $t = 2, 3$ show the $(8, 4+t)$ elements must be $-C, B$. So we have the 8-letter design

$$\begin{array}{cccc@{\qquad}cccc}
A & B & C & D & E & F & G & H \\
-B & A & D & -C & F & -E & -H & G \\
-C & -D & A & B & G & H & -E & -F \\
-D & C & -B & A & H & -G & F & -E \\[6pt]
-E & -F & -G & -H & A & B & C & D \\
-F & E & -H & G & -B & A & -D & C \\
-G & H & E & -F & -C & D & A & -B \\
-H & -G & F & E & -D & -C & B & A .
\end{array}$$

We note that using one of the normalized 4-letter designs to form a normalized 8-letter design forced the other normalized 4-letter design into the 8-letter design. So, in a normalized 8-letter design both normalized 4-letter designs must appear.

LEMMA 7. *As there are two normalized 4-letter designs which can be used as the upper left hand block in the construction, there are two normalized 8-letter designs. They are equivalent under Lemma 5. If the normalized 8-letter design is written in the form*

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$$\begin{matrix} X_1 & Y_1 \\ Y_2 & X_2 \end{matrix}$$

*where* $X_1$ *is one normalized 4-letter design then* $X_2$ *must be the other normalized 4-letter design.*

Obviously if we start with four letters within an $n$-letter design in normalized form and put them into the upper left hand block using our equivalence relations, then they must form a 4-letter design within themselves.

In proving Lemma 7, we have shown that if $A, B, C, D$ form a normalized 4-letter design and $A, B, C, D, E, F, G, H, I, J, \ldots$ are the letters of an $n$-letter design then we may write them in the form:

$$(1) \qquad \begin{matrix} A & B & C & D & E & F & G & H & I & J & \ldots \\ & A & & & & & & & & & \\ & & A & & & & & & & & \\ & & & A & & & & & & & \\ & & & & A & B & C & D & & & \\ & & & & & A & & & & & \\ & & & & & & A & & & & \\ & & & & & & & A & & & \\ & & & & & & & & A & B & \ldots \end{matrix}$$

but then the letters $A, B, C, D, E, F, G, H$ in the top left hand corner of (1) form within themselves an 8-letter normalized design. Now there are only two normalized 4-letter designs, call them $X_1$ and $X_2$ respectively. If we denote the 4-letter designs in (1) on $E, F, G, H$ by $Y_1$ and $Y_2$ we have (1) is

$$(2) \qquad \begin{matrix} X_1 & Y_1 & Z_1 & \ldots \\ Y_2 & X_2 & U_1 & \ldots \\ Z_2 & U_2 & X_i & \ldots \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{matrix}$$

Now because (2) is a normalized $n$-letter design, all the letters $Z_r$

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are distinct from the letters in $X_j$ and $Y_m$ . Similarly the letters in $U_s$ are distinct from the letters in $Z_r$, $X_j$, $Y_m$ .

We may rewrite (2) as

$$
(3) \qquad \begin{array}{cccc}
X_1 & Z_1 & Y_1 & \dots \\
Z_2 & X_i & U_2 & \dots \\
Y_2 & U_1 & X_2 & \dots \\
\cdot & & & \\
\cdot & & & \\
\cdot & & &
\end{array}
$$

using the admissible equivalence relation of interchanging rows and columns. We see (3) is still normalized. Hence by the above reasoning the block

$$
\begin{array}{cc}
X_1 & Z_2 \\
Z_2 & X_i
\end{array}
$$

must form an 8-letter normalized design and so $X_i = X_2$ .

Now rewrite (2) as

$$
(4) \qquad \begin{array}{cccc}
X_2 & U_1 & Y_2 & \dots \\
U_2 & X_i & Z_2 & \dots \\
Y_1 & Z_1 & X_1 & \dots \\
\cdot & & & \\
\cdot & & & \\
\cdot & & &
\end{array}
$$

by interchanging rows and columns. (4) may not be normalized as some of the letters of the first row of $U_1$, $Y_2$, ... may have negative signs, but using the equivalence relation which permits the replacing of any letter by its negative throughout the design we may normalize (4) to

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$$(5)\qquad \begin{matrix} X_2 & U_1{}^* & Y_2{}^* & \cdots \\ U_2{}^* & X_i & Z_2 & \cdots \\ Y_1{}^* & Z_1 & X_1 & \cdots \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{matrix} \quad .$$

We note that (5) is now normalized and that $X_2$, $X_i$ were unaffected by the change.

Then by the above reasoning the block

$$\begin{matrix} X_2 & U_1{}^* \\ U_2{}^* & X_i \end{matrix}$$

must form an 8-letter design so $X_i = X_1$ . But this is a contradiction, so we have

THEOREM 8. *There is no* $n$*-letter design with* $n > 8$ .

COROLLARY 9. *There are* $n$*-letter designs for* $n = 2, 4,$ *and* 8, *and for no other* $n$ .

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