Compositio Mathematica **119:** 209–212, 1999.

A Complete Surface in M_6 in Characteristic > 2

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(Received: 7 April 1998; accepted in final form: 25 August 1998)

Abstract. We construct in all characteristics $p > 2$ a complete surface in the moduli space of smooth genus 6 curves. The surface is contained in the locus of curves with automorphisms.

Mathematics Subject Classification (1991): 14H10.

Key words: moduli of curves.

We consider the following question: 'What is the number of essential parameters on which a complete family of smooth curves of genus g depends?' or equivalently, 'What is the maximal dimension of a complete subvariety of M_g, the moduli space of smooth curves of genus g?' In [3] Diaz provided an upper bound for the dimension of such a subvariety: for $g \geqslant 2$ this dimension is at most $g - 2$. The moduli space M_g itself is irreducible, quasi-projective of dimension $3g - 3$ ($g \geqslant 2$). Diaz proved his result in characteristic 0, but his bound also holds in characteristic > 0 (see [6]).

In order to see how good Diaz's bound is, one has to construct complete subvarieties of M_g. This turns out to be a difficult problem, in any characteristic. Only in genus 3 Diaz's bound is sharp, since it is known that M_g contains complete curves if g is at least 3 (see [4]). In higher genera almost nothing is known. The best result we know is a construction of complete subvarieties of arbitrary dimension $d \geqslant 1$ in M_g with $g \geqslant 2^{d+1}$. This construction gives a complete surface in M_8. For $g = 4, 5, 6$ and 7 the existence of a complete surface in M_g is an open question.

Starting from a complete curve in M_3, we construct a complete surface in M_6. However, this construction only works in characteristic $\neq 0, 2$. Our result is:

THEOREM 1. *In any characteristic $p > 2$ the moduli space M_6 of smooth genus 6 curves contains a complete surface.*

To construct in characteristic 0 a complete surface in M_6 seems more difficult. This is more or less similar to the fact that the moduli space A_g of principally polarized abelian varieties of dimension g contains in characteristic $p > 0$ complete subvarieties of rather high dimension [7]. The corresponding situation in characteristic 0 is completely unknown.

Our construction depends heavily on a theorem of Keel. The starting point is a complete family $C \to B$ of smooth genus 3 curves. The idea is to construct a family of double covers ramified in two *distinct* points over the fibers of $C \to B$. To parametrize pairs of distinct branch points, we consider $C \times_B C$. On $C \times_B C$ we want to contract Δ to a point. To achieve this, we define on $C \times_B C$ a nef and big line bundle L with the property that $L|_\Delta$ is trivial. To show that the global sections of a power of L do not have a base locus we use a theorem of Keel which is only valid in positive characteristic. Keel's theorem relies on a lemma, which roughly states that if $L|_{\Delta_k}$ is trivial, then $L^{\otimes p}|_{\Delta_{kp}}$ is trivial, see [5, Lemma 1.7]. Here Δ_i is the ith order neighborhood of Δ, the subscheme of $C \times_B C$ defined by I^{k+1}, where I is the ideal sheaf of Δ. This is where the Frobenius map is used.

The line bundle L exists in all characteristics, but we do not know how to prove its eventual freeness in characterictic 0. Instead of using Keel's theorem, we tried to prove by direct methods that L is free on $C \times_B C$. To prove this, we need that L is trivial on Δ_i for every $i > 0$. Unfortunately, we don't know how to establish this.

1. The Construction

Let $C \to B$ be a family of smooth genus 3 curves over a complete one-dimensional base B, having the property that the induced map $B \to M_3$ has finite fibres. We consider the fibre product $C \times_B C$. Let $\Delta \subset C \times_B C$ be the relative diagonal and $\pi_1, \pi_2 \colon C \times_B C \to C$ the projections on the first and second coordinate. On $C \times_B C$ consider the line bundle L associated to the divisor $(\pi_1^* + \pi_2^*)(K_{C/B}) + 2\Delta$. In characteristic $p > 0$ we can prove that a sufficiently high power of L is free:

THEOREM 2. *Let L be the line bundle L associated to the divisor $(\pi_1^* + \pi_2^*)(K_{C/B}) + 2\Delta$. Then L satisfies on $C \times_B C$:*

(i) *the restriction of L to Δ is trivial;*
(ii) *L is big and nef on $C \times_B C$ and big on any subvariety not containing Δ;*
(iii) *in characteristic $p > 0$ a sufficiently high multiple of L is free and defines a birational morphism of $C \times_B C$ to a projective threefold. Under this morphism, Δ is contracted to a point.*

Proof. (i) Let $\Delta \colon C \to C \times_B C$ be the diagonal map $c \mapsto (c, c)$. Then according to the adjunction formula $\Delta^*(K_{C\times_B C} + \Delta) \cong K_C$. Now $\Delta^*(K_{C\times_B C}) \cong K_{C/B} + K_C$, so it follows that $\Delta^*(\Delta) \cong -K_{C/B}$. Hence $\Delta^*(L) \cong \mathcal{O}_C$ and the restriction of L to Δ is trivial.

(ii) Let X be a subvariety of $C \times_B C$. If X has dimension 1, then

$$\big(\pi_1^*(K_{C/B}) + \pi_2^*(K_{C/B})\big) \cdot X = K_{C/B} \cdot (\pi_{1,*} + \pi_{2,*})(X) > 0,$$

since $K_{C/B}$ is ample on C (see [1]) and since $\pi_{1,*}(X)$ and $\pi_{2,*}(X)$ cannot both be zero-dimensional. If X has dimension $s > 1$, then using a similar argument one proves that $\big(\pi_1^*(K_{C/B}) + \pi_2^*(K_{C/B})\big)^s \cdot X > 0$. Hence by the Nakai–Moishezon criterion $\pi_1^*(K_{C/B}) + \pi_2^*(K_{C/B})$ is ample on $C \times_B C$. Since $L = \pi_1^*(K_{C/B}) +$

https://doi.org/10.1023/A:1001776408994 Published online by Cambridge University Press

$\pi_2^*(K_{C/B}) + 2\Delta$, L is the sum of an ample and an effective divisor, hence big by one of the equivalent criteria for bigness. By (i) L is nef.

Moreover, if we restrict L to any positive dimensional subvariety X not containing Δ, then $L|_X$ is the sum of the restriction of an ample divisor plus an effective divisor, hence $L|_X$ also big.

(iii) To show that L is eventually free, we use a result of Keel which states that in characteristic $p > 0$ a nef and big line bundle L is eventually free iff $L|_{E(L)}$ is eventually free [5, Theorem 1.2]. Here $E(L)$ is the exceptional locus of L; this is the union of all subvarieties along which L is not big. By (i) L restricted to Δ is free. By (ii) L is nef and big. Together (i) and (ii) imply $E(L) = \Delta$.

Proof of Theorem 1. From Theorem 2 we conclude that for some n the global sections of $L^{\otimes n}$ yield a morphism $\phi: C \times_B C \to \mathbf{P}^N$ which contracts the diagonal to a point. In $\mathbf{P}^N$ choose a hyperplane not meeting the image of Δ. Then $T = \phi^*(H) \subset C \times_B C \setminus \Delta$ is a surface which parametrizes pairs of *distinct* points on the fibers of the family $C \to B$. Now by standard arguments one constructs a complete family $X \to S$, each fiber being a double cover of a fiber of $C \to B$ ramified in the two distinct points determined by $t \in T$ [8, Sect. 1]. Locally we take square roots, so we have to exclude the case that the characteristic is 2. Since $C \to B$ is a family of smooth genus 3 curves, $X \to S$ is a family of smooth genus 6 curves. The base S is a finite cover of T. It is needed to overcome the monodromy arising from the fact that for one pair of distinct branch points one can choose a finite number of distinct coverings. The base S maps into the locus of curves in M_6 having non-trivial automorphisms. We claim that this image is two-dimensional. To prove this, note that the structural map from S to M_6 factors as $S \to \mathcal{R}_{3,2} \to M_6$, where $\mathcal{R}_{3,2}$ parametrizes double coverings of genus 3 curves ramified in two distinct points. The image of S in $\mathcal{R}_{3,2}$ is clearly two-dimensional: S maps to M_3 with one-dimensional fibers and one-dimensional image. Moreover, the map $\mathcal{R}_{3,2} \to M_6$ is quasi-finite, since the image parametrizes smooth genus 6 curves with an involution with a genus 3 quotient and a genus 6 curve admits only finitely many involutions.

2. Remark

Consider the *difference* map $C \times_B C \to \mathrm{Jac}(C/B)$, $(x, y) \mapsto [x - y]$. This map contracts the diagonal $\Delta \subset C \times_B C$ to a curve. Any hypersurface in $\mathrm{Jac}(C/B)$ not meeting this curve pulls back to a complete two-dimensional subvariety T in $C \times_B C$ not meeting the diagonal. Starting from such a subvariety one can, as in the proof of Theorem 1, construct a family of smooth genus 6 curves. This would give a different construction of a complete two-dimensional family of smooth genus 6 curves. But such a hypersurface is hard to find, as the following result of E. Colombo and P. Pirola [2] shows: Let $\pi: \mathcal{A} \to B$ be a family of Abelian varieties of relative dimension g over a smooth complete curve B, with zero section $e: B \to \mathcal{A}$ and not isogenous to a family $\mathcal{A}_1 \times_B \mathcal{A}_2$ with $\mathcal{A}_1$ isotrivial. Let Z be an effective relative ample divisor on $\mathcal{A}$ and C a curve on $\mathcal{A}$. Then $C \cap Z \neq \emptyset$.

3. Characteristic 0

In characteristic 0 there is one point at which our construction may fail: the line bundle L associated to the divisor $(\pi_1^* + \pi_2^*)(K_{C/B}) + 2\Delta$ may not be eventually free. However, in the case B is a point, Keel proves that in all characteristics the line bundle L is eventually free [5, Theorem 3.0]. One can try to mimic his proof for the case in which B is a curve. The hard part is to show that for every $k > 0$ the restriction of L to the kth order neighborhood of Δ inside $C \times_B C$ is trivial. However, we are unable to prove this.

References

1. Arakelov, S.: Families of algebraic curves with fixed degeneracies, *Izv. Akad. Nauk SSSR Ser. Mat*. **35** (1971), 1269–1293.
2. Colombo, E. and Pirola, P.: Relative ample divisors on complete families of abelian varieties, *J. Pure Appl. Algebra*, to appear.
3. Diaz, S.: A bound on the dimensions of complete subvarieties of $\mathcal{M}_g$, *Duke Math. J*. **51**(2) (1984), 405–408.
4. Diaz, S.: Complete subvarieties of the moduli space of smooth curves, In: S. J. Bloch (ed.), *Algebraic Geometry (Bowdoin 1985)*, Proc. Symp. Pure Math. 1, Amer. Math. Soc., Providence, 1987, pp. 77–81.
5. Keel, S.: Base point freeness for nef and big line bundles in positive characteritic, with applications to M_g and to 3-fold mmp, alg-geom/9703003, 1997.
6. Looijenga, E.: On the tautological ring of M_g, *Invent. Math*. **121** (1995), 411–419.
7. Oort, F.: Subvarieties of moduli spaces, *Invent. Math*. **24** (1974), 95–117.
8. Zaal, C.: Explicit complete curves in the moduli space of curves of genus three, *Geom. Dedicata* **56** (1995), 185–196.