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EXISTENCE AND BLOW-UP OF SOLUTIONS FOR A NON-LOCAL FILTRATION AND POROUS MEDIUM PROBLEM

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Abstract We consider a non-local filtration equation of the form

$$u_t = \Delta K(u) + \lambda f(u)\biggl(\int_{\Omega} f(u)\,\mathrm{d}x\biggr)^{-p}$$

and a porous medium equation, in this case $K(u) = u^m$, with some boundary and initial data u_0, where $0 < p < 1$ and $f, f', f'' > 0$. We prove blow-up of solutions for sufficiently large values of the parameter $\lambda > 0$ and for any $u_0 > 0$, or for sufficiently large values of $u_0 > 0$ and for any $\lambda > 0$.

Keywords: existence; blow-up; non-local parabolic problems; porous medium; filtration equation

2000 *Mathematics subject classification:* Primary 35K55; 35K65
Secondary 35K60; 74H35

1. Introduction

In this work we prove existence, uniqueness and blow-up of solutions of the following non-local initial boundary-value problem:

$$u_t = \Delta K(u) + \frac{\lambda f(u)}{(\int_{\Omega} f(u)\,\mathrm{d}x)^p}, \quad x \in \Omega, \quad t > 0, \tag{1.1 a}$$

$$\mathcal{B}(u) := \frac{\partial K(u)}{\partial \hat{n}} + \beta(x)K(u) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.1 b}$$

$$u(x, 0) = u_0(x) > 0, \quad x \in \Omega, \tag{1.1 c}$$

where $\hat{n}$ is the outward-pointing unit normal vector field and Ω is a bounded domain of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. We impose positive initial data in order to avoid degeneration of solutions of (1.1). Also, it is sufficient for the classical solution $u_0 \in L^{\infty}(\Omega)$. We introduce boundary conditions of the form $\mathcal{B}(u)$. These types of conditions are a consequence of Fourier's law for diffusion and conservation of mass, or heat conduction and conservation of energy. The usual type of boundary condition of the form $\partial u/\partial\hat{n} + \beta(x)u = 0$ seems not to have any physical significance. Here $0 \leqslant \beta = \beta(x) \leqslant \infty$

is $C^{1+\alpha}(\partial\Omega)$, $\alpha > 0$, whenever it is bounded ($\beta \equiv 0$, $\beta \equiv \infty$ and $0 < \beta < \infty$ mean Neumann, Dirichlet and Robin boundary conditions, respectively; we may also have mixed boundary conditions). λ and p are positive parameters with $p \in (0, 1)$ and f satisfies

$$f(s) > 0, \quad f'(s) > 0, \quad f''(s) > 0 \quad \text{for } s \geqslant 0, \tag{1.2 a}$$

$$\int_b^\infty \frac{\mathrm{d}s}{f^{1-p}(s)} < \infty, \tag{1.2 b}$$

for some $b \geqslant 0$, e.g. $f(s) = (1+s)^{1+k}$ for $k > p/(1-p)$ or $f(s) = \mathrm{e}^s$. The function $K = K(s) \in C^3(\mathbb{R}_+)$ satisfies either

$$K(s), K'(s), K''(s) > 0 \quad \text{for } s \geqslant 0 \tag{1.3 a}$$

or

$$K(s), K'(s), K''(s) > 0 \quad \text{for } s > 0 \quad \text{with } K(0) = K'(0) = K''(0) = 0; \tag{1.3 b}$$

examples of such functions are $K(s) = \mathrm{e}^s$ or $K(s) = s^m$, respectively (see also [**10**], [**14**, Chapter VI] and [**18**]). Condition (1.3), together with positive initial data and the use of comparison, implies positive classical solutions (see § 2).

Problem (1.1) is the so-called non-local filtration (or generalized porous medium) problem. If $K(u) = u^m$, $m > 1$, $m \in \mathbb{R}$, then (1.1) becomes the non-local porous medium problem:

$$u_t = \Delta u^m + \frac{\lambda f(u)}{(\int_\Omega f(u)\,\mathrm{d}x)^p}, \quad x \in \Omega, \quad t > 0, \tag{1.4 a}$$

$$\mathcal{B}(u) = \frac{\partial u^m}{\partial \hat{n}} + \beta(x) u^m = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.4 b}$$

$$u(x, 0) = u_0(x) > 0, \qquad x \in \Omega. \tag{1.4 c}$$

Our motivation to address (1.4), concerning the conduction term Δu^m (or $\nabla \cdot u^{m-1} \nabla u$), comes from [**19**]. In [**19**], the plasma-heating equation

$$u_t = (u^3 u_x)_x + \lambda f(u) \biggl(\int_{-1}^1 f(u)\,\mathrm{d}x \biggr)^{-2}$$

is used; more precisely, the conduction term $(u^4)_{xx}$ or $(u^3 u_x)_x$ is introduced, where the term u^3 accounts for heat transport dominated by thermal radiation by assuming the Stefan–Boltzmann law for emission of thermal radiation. Actually, (1.4 a) is a generalization of the plasma-heating equation. Problem (1.1) can also be considered as a generalization of (1.4).

On the other hand, if u represents the fluid density of a compressible fluid through a porous medium, then Darcy's law can lead to the equation

$$u_t = \nabla \cdot \sigma(u) \nabla u, \quad \sigma(u) = K'(u), \tag{1.5}$$

which is again the filtration equation. For the latter equation, specializing to an isothermal perfect gas, we can obtain the porous medium equation:

$$u_t = \nabla \cdot (u^m \nabla u), \quad m > 0 \quad \text{or} \quad u_t = \Delta u^m, \quad m > 1. \tag{1.6}$$

Equation (1.6) also models the thickness u of a viscous drop spreading under gravity over a horizontal surface as well as the horizontal spreading of highly fissured volcanos (for more details see [**17**, p. 256]).

Concerning the non-local reaction term of problem (1.1), this comes from modelling ohmic heating phenomena, as well as phenomena which occur in shear bands of metals which are being deformed at high strain rates [**4**–**6**], in the theory of gravitational equilibrium of polytropic stars [**12**], in the investigation of the fully turbulent behaviour of flows, using invariant measures for the Euler equation [**7**] and in modelling aggregation of cells via interaction with a chemical substance (chemotaxis) [**21**]. It is worth mentioning that Galaktionov and Levine [**8**] and, later on, Afanas'eva and Tedeev [**1**] treated, among other problems, a non-local porous medium problem with critical Fujita exponent and proved, for the Cauchy problem, global existence and blow-up of solutions.

This paper is organized as follows: in § 2 we prove existence and uniqueness; in § 3 we prove blow-up of solutions of the porous medium problem for sufficiently large $\lambda > 0$ and for any positive initial data; in § 4, we show blow-up for sufficiently large initial data; finally, in § 5, we prove blow-up for the filtration problem. Note that for the proof of blow-up for the Dirichlet and Robin problems we require Ω to be convex.

2. Existence and uniqueness via a lower–upper solution pair

In contrast to the decreasing case where a maximum principle holds, when $f(s)$ is an increasing function, the existence of an upper solution and a lower solution in the classical sense does not guarantee the existence of a solution of problem (1.1) lying between them. In order to use similar comparison arguments, we introduce the concept of the lower–upper solution pair.

At this point, we just outline some steps of the procedure of the proof by using comparison methods [**2**, **6**, **10**, **13**]. More precisely, we introduce a system of two iteration schemes and, using a lower–upper solution pair, we get two monotonic sequences of functions which are solutions to the system. Then, taking a weak form (integral formulation) of the system and using the monotone convergence theorem as well as regularity arguments, we obtain that the limits of the two sequences, $\underline{u}$, $\bar{u}$ ($\underline{u} \leqslant \bar{u}$), are classical solutions to the system. Finally, by the maximum principle we prove that $\underline{u} \geqslant \bar{u}$, which implies that the system coincides with (1.1 a) and gives us existence and uniqueness.

In order to obtain existence we introduce the concept of lower–upper solution pairs.

Definition 2.1. Let two functions be $z = z(x,t)$, $v = v(x,t) \in C^{2+\alpha,1+\alpha/2}(\Omega_T;\mathbb{R}) \cap C^{\alpha,0}(\bar{\Omega}_T;\mathbb{R})$, $0 < \alpha < 1$, $\Omega_T = \Omega \times (0,T)$. Then (z,v) is called a lower–upper solution pair for problem (1.1) if it satisfies $z(x,t) \leqslant v(x,t)$ for $(x,t) \in \bar{\Omega}_T$ and

$$\begin{aligned} S(z;v) \leqslant 0 \leqslant S(v;z), &\qquad x \in \Omega, \quad 0<t<T,\\ \mathcal{B}(z) \leqslant 0 \leqslant \mathcal{B}(v), &\qquad x \in \partial\Omega,\ 0<t<T,\\ 0 \leqslant z(x,0) \leqslant u_0(x) \leqslant v(x,0), &\qquad x \in \Omega. \end{aligned}$$

Here the operator S is defined by

$$S(z;v) \equiv z_t - \Delta K(z) - \lambda f(z)\biggl(\int_\Omega f(v)\,\mathrm{d}x\biggr)^{-p}.$$

If all the above inequalities are strict, then (z,v) is a strict lower–upper solution pair [**3**,**6**].

Lemma 2.2. *Let (z,v) be a lower–upper solution pair of (1.1). Then $z \leqslant u \leqslant v$.*

Proof. We prove the lemma in two steps.

Step 1. Let (z,v) be a strict lower–upper solution pair of (1.1). We shall show that $z<u<v$. First, we give the proof for the pair (u,v). Let $d(x,t)=v(x,t)-u(x,t)$ [**2**]. We assume that the conclusion is false; that is, there exists a first $\bar{t}$ such that $d(\bar{x},\bar{t})=0$ for some $\bar{x} \in \Omega$. Also, we have that $d(x,t)>0$ for $(x,t) \in \bar{\Omega} \times (0,\bar{t})$, and $d_t(\bar{x},\bar{t}) \leqslant 0$. Moreover, $d(\bar{x},\bar{t})$ attains its minimum at $x=\bar{x}$, so $\nabla d(\bar{x},\bar{t}) = \nabla v(\bar{x},\bar{t}) - \nabla u(\bar{x},\bar{t}) = 0$ and $\Delta d(\bar{x},\bar{t}) \geqslant 0$. Thus, at $(\bar{x},\bar{t})$, we have

$$\begin{aligned} 0 &\geqslant d_t(\bar{x},\bar{t})\\ &= v_t(\bar{x},\bar{t}) - u_t(\bar{x},\bar{t})\\ &= K'(v)\Delta v + K''(v)|\nabla v|^2 - K'(u)\Delta u - K''(u)|\nabla u|^2 + \mathrm{NLTs}\\ &= K'(u)\Delta d(\bar{x},\bar{t}) + K''(v)(|\nabla v|^2 - |\nabla u|^2)\\ &\qquad + \lambda\biggl(\frac{f(v(\bar{x},\bar{t}))}{(\int_\Omega f(u(x,\bar{t}))\,\mathrm{d}x)^p} - \frac{f(u(\bar{x},\bar{t}))}{(\int_\Omega f(v(x,\bar{t}))\,\mathrm{d}x)^p}\biggr)\\ &\geqslant f(u(\bar{x},\bar{t}))\frac{(\int_\Omega f(v(x,\bar{t}))\,\mathrm{d}x)^p - (\int_\Omega f(u(x,\bar{t}))\,\mathrm{d}x)^p}{(\int_\Omega f(v(x,\bar{t}))\,\mathrm{d}x)^p(\int_\Omega f(u(x,\bar{t}))\,\mathrm{d}x)^p}\\ &> 0, \end{aligned}$$

where 'NLTs' is the difference in the non-local terms, the term with the Laplacian is non-negative, the term with the gradient is zero and the difference in the non-local terms is strictly positive due to the monotonicity of f. Hence, $0 \geqslant d_t(\bar{x},\bar{t}) > 0$, which is a contradiction.

Step 2. Let us now assume that (u,v) is a lower–upper solution pair of (1.1). We shall show that $z \leqslant u \leqslant v$. Since f is a convex function, it is Lipschitz continuous and also one-side Lipschitz continuous, i.e. $f(a+b)-f(b) \leqslant La$, where L is a positive constant and $0<a<R$ for some R. Let $v^\varepsilon = v + \varepsilon \mathrm{e}^{\sigma t} > v$ for some $\varepsilon>0$ (we define z^ε similarly).

Actually, we use $0 < \varepsilon \ll 1$, $\varepsilon \mathrm{e}^{\sigma t} < \varepsilon \mathrm{e}^{\sigma T} = R$ and L fixed. Then we have [**2**]

$$\begin{aligned}
S(v^{\varepsilon}; u) &= v^{\varepsilon}_t - \Delta K(v^{\varepsilon}) - \lambda \frac{f(v^{\varepsilon})}{(\int_{\Omega} f(u)\,\mathrm{d}x)^p} \\
&= v_t + \varepsilon\sigma \mathrm{e}^{\sigma t} - K''(v^{\varepsilon})|\nabla v|^2 - K'(v^{\varepsilon})\Delta v - \lambda \frac{f(v^{\varepsilon})}{(\int_{\Omega} f(u)\,\mathrm{d}x)^p} \\
&\geqslant v_t - \Delta K(v) - \lambda \frac{f(v) + L\varepsilon \mathrm{e}^{\sigma t}}{(\int_{\Omega} f(u)\,\mathrm{d}x)^p} + \varepsilon\sigma \mathrm{e}^{\sigma t} \\
&\qquad + (K''(v) - K''(v^{\varepsilon}))|\nabla(v)|^2 + (K'(v) - K'(v^{\varepsilon}))\Delta v \\
&= S(v; u) + \varepsilon \mathrm{e}^{\sigma t}\biggl[\sigma - \lambda \frac{L}{(\int_{\Omega} f(u)\,\mathrm{d}x)^p} - K'''(v)|\nabla v|^2 - K''(v)\Delta v\biggr] + O(\varepsilon^2) \\
&> S(v; u) \geqslant S(u; v).
\end{aligned}$$

The last inequality is due to the fact that σ can be taken to be sufficiently large ($\sigma \gg 1$), while R and L are fixed. The function v is bounded in $C^{2,1}(\bar{\Omega}_T)$. From the first step we derive $v > u$ and $v^{\varepsilon} > u$ for $\varepsilon \ll 1$. Now we have $v^{\varepsilon} = v + \varepsilon \mathrm{e}^{\sigma t} > u$ for any $0 < \varepsilon \ll 1$ and, taking $\varepsilon \to 0$, we get $v \geqslant u$.

The other pair, (z, u), is treated similarly. This completes the proof. □

Now, we show that at least such a lower–upper pair (z, v) exists.

Lemma 2.3. *Let $Z = Z(t)$, $V = V(t) \leqslant b < \infty$ satisfy the inequalities*

$$Z_t - \underline{\varLambda} f(Z) \leqslant 0 \leqslant V_t - \bar{\varLambda} f(V), \quad 0 < t < \hat{T} < \infty, \tag{2.1 a}$$

$$Z(0) = Z_0 = 0 < u_0(x) \leqslant V(0) = V_0, \tag{2.1 b}$$

where $\bar{\varLambda} = \lambda / f^p(0)|\Omega|^p > \underline{\varLambda} = \lambda / f^p(b)|\Omega|^p$ for some $b \gg 1$. Then $0 \leqslant Z(t) \leqslant V(t)$ and (Z, V) is a lower–upper solution pair of problem (1.1) with

$$0 \leqslant Z(t) < u(x, t) \leqslant V(t). \tag{2.2}$$

Proof. It is obvious that the time $\hat{T}$ depends on b, i.e. $\hat{T} = \hat{T}(b)$, with $b = V(\hat{T})$, $\hat{T} < T_{\max}$, where $T_{\max}$ is the maximal existence time of V, ($V(t) \to \infty$ as $t \to T_{\max}-$). The fact that $0 \leqslant Z \leqslant V$ comes from direct integration of (2.1 a), i.e.

$$\int_0^{Z(t)} \frac{\mathrm{d}s}{f(s)} \leqslant \underline{\varLambda} t < \bar{\varLambda} t \leqslant \int_{V_0}^{V(t)} \frac{\mathrm{d}s}{f(s)} \leqslant \int_0^{V(t)} \frac{\mathrm{d}s}{f(s)} < \int_0^{\infty} \frac{\mathrm{d}s}{f(s)} < \infty, \tag{2.3}$$

the last inequality of (2.3) is a consequence of (1.2 b). Moreover, due to the choice of $\underline{\varLambda}$, $\bar{\varLambda}$, we get that (Z, V) is a lower–upper solution pair of problem (1.1).

Now by using the maximum principle we get (2.2). Actually, using similar methods to those in Lemma 2.2, we have $D = D(x, t) = u(x, t) - Z(t)$, with $D(x, 0) > 0$. Assuming now that $D \leqslant 0$, there exists $\bar{t} > 0$ such that $D(\bar{x}, \bar{t}) = 0$ for some $\bar{x} \in \Omega$, while $D > 0$ for $(x, t) \in \bar{\Omega} \times [0, \bar{t})$ and $D_t(\bar{x}, \bar{t}) \leqslant 0$. Then at $(\bar{x}, \bar{t})$ we have $\nabla D = 0$, $\Delta D \geqslant 0$ and

$$0 \geqslant D_t(\bar{x}, \bar{t}) > K''(u)|\nabla D|^2 + K'(u)\Delta D + \underline{\varLambda}[f(u(\bar{x}, \bar{t})) - f(Z(\bar{t}))] = K'(u)\Delta D \geqslant 0,$$

which is a contradiction, i.e. $0 \geqslant D_t > 0$. Similarly, we obtain that $V(t) \geqslant u(x,t)$. This completes the proof. □

Now let (z, v) be a lower–upper solution pair and an iterative scheme starting with $\bar{u}_0 = v$, $\underline{u}_0 = z$ and proceeding according to

$$\underline{I}_n = I(\underline{u}_n; \bar{u}_n) := \underline{u}_{nt} - \Delta(K(\underline{u}_n)) - \frac{\lambda f(\underline{u}_{n-1})}{(\int_\Omega f(\bar{u}_{n-1})\,\mathrm{d}x)^p} = 0, \quad x \in \Omega,\ t > 0, \tag{2.4}$$

$$\bar{I}_n = I(\bar{u}_n; \underline{u}_n) := \bar{u}_{nt} - \Delta(K(\bar{u}_n)) - \frac{\lambda f(\bar{u}_{n-1})}{(\int_\Omega f(\underline{u}_{n-1})\,\mathrm{d}x)^p} = 0, \quad x \in \Omega,\ t > 0, \tag{2.5}$$

and

$$\mathcal{B}(\underline{u}_n) = \mathcal{B}(\bar{u}_n) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2.6}$$

$$\underline{u}_n(x, 0) = \bar{u}_n(x, 0) = u_0(x), \quad x \in \Omega, \tag{2.7}$$

for $n = 1, 2, \dots$.

Remark 2.4. We may also define the iterative scheme (2.4) and (2.5) by

$$S(\underline{u}_n; \bar{u}_{n-1}) = S(\bar{u}_n; \underline{u}_{n-1}) = 0,$$

respectively. But in that case we use a generalized maximum principle instead of the simple one we use here.

Now, we prove that if we have a lower–upper solution pair, we can construct a monotonic sequence which converges to the solution of problem (1.1).

Proposition 2.5. *Let (z, v) be a lower–upper solution pair of (1.1 a) and $\underline{u}_n, \bar{u}_n$ satisfy (2.4), (2.6), (2.7) and (2.5)–(2.7), respectively, for $n = 1, 2, 3, \dots$, with $\underline{u}_0 = Z$ and $\bar{u}_0 = V$. Then we have*

$$\underline{u}_0 < \underline{u}_1 < \cdots < \underline{u}_{n-1} < \underline{u}_n < \cdots < \bar{u}_n < \bar{u}_{n-1} < \cdots < \bar{u}_1 < \bar{u}_0.$$

Proof. We prove the proposition by induction. Firstly, we show that $\underline{u}_{n-1} < \underline{u}_n$ and $\bar{u}_n < \bar{u}_{n-1}$. Indeed,

$$\underline{I}_1 = \underline{I}_1(\underline{u}_1; \bar{u}_1) = \underline{u}_{1t} - \Delta K(\underline{u}_1) - \frac{\lambda f(\underline{u}_0)}{(\int_\Omega f(\bar{u}_0)\,\mathrm{d}x)^p} = 0 \geqslant \underline{u}_{0t} - \Delta K(\underline{u}_0) - \frac{\lambda f(\underline{u}_0)}{(\int_\Omega f(\bar{u}_0)\,\mathrm{d}x)^p}$$

and by using the maximum principle for the filtration problem we get $\underline{u}_1 \geqslant \underline{u}_0$. Similarly, $\bar{u}_1 \leqslant \bar{u}_0$. Now, for the n-step, we have

$$\begin{aligned}\underline{I}_n = \underline{u}_{nt} - \Delta K(\underline{u}_n) - \frac{\lambda f(\underline{u}_{n-1})}{(\int_\Omega f(\bar{u}_{n-1})\,\mathrm{d}x)^p} = 0 \\ = \underline{u}_{(n-1)t} - \Delta K(\underline{u}_{n-1}) - \frac{\lambda f(\underline{u}_{n-2})}{(\int_\Omega f(\bar{u}_{n-2})\,\mathrm{d}x)^p}.\end{aligned}$$

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The latter implies

$$[\underline{u}_{nt} - \Delta K(\underline{u}_n)] - [\underline{u}_{(n-1)t} - \Delta K(\underline{u}_{n-1})] = \frac{\lambda f(\underline{u}_{n-1})}{(\int_\Omega f(\bar{u}_{n-1})\,\mathrm{d}x)^p} - \frac{\lambda f(\underline{u}_{n-2})}{(\int_\Omega f(\bar{u}_{n-2})\,\mathrm{d}x)^p} \geqslant 0,$$

since $\underline{u}_{n-1} \geqslant \underline{u}_{n-2}$ and $\bar{u}_{n-1} \leqslant \bar{u}_{n-2}$. Again, by using the maximum principle for the filtration problem, we get that $\underline{u}_n > \underline{u}_{n-1}$ and $\bar{u}_n < \bar{u}_{n-1}$ for $n = 1, 2, \dots$.

Secondly, we show that $\underline{u}_n < \bar{u}_n$, again by induction:

$$\begin{aligned} \underline{I}_n = \underline{u}_{nt} - \Delta K(\underline{u}_n) - \frac{\lambda f(\underline{u}_{n-1})}{(\int_\Omega f(\bar{u}_{n-1})\,\mathrm{d}x)^p} = 0 \\ = \bar{u}_{nt} - \Delta K(\bar{u}_n) - \frac{\lambda f(\bar{u}_{n-1})}{(\int_\Omega f(\underline{u}_{n-1})\,\mathrm{d}x)^p}. \end{aligned}$$

Thus,

$$[\underline{u}_{nt} - \Delta K(\underline{u}_n)] - [\bar{u}_{nt} - \Delta K(\bar{u}_n)] = \frac{\lambda f(\underline{u}_{n-1})}{(\int_\Omega f(\bar{u}_{n-1})\,\mathrm{d}x)^p} - \frac{\lambda f(\bar{u}_{n-1})}{(\int_\Omega f(\underline{u}_{n-1})\,\mathrm{d}x)^p} \leqslant 0,$$

since $\underline{u}_{n-1} \leqslant \bar{u}_{n-1}$, which holds for $n = 1, 2, \dots$ (for $n = 1$ see Lemma 2.2). This completes the proof. $\square$

Corollary 2.6. *For the iterative schemes of problems (2.4)–(2.7) we have $\underline{u}_n \nearrow \underline{u}$, $\bar{u}_n \searrow \bar{u}$ pointwise as $n \to \infty$ and $\underline{u} \leqslant \bar{u}$.*

Proof. This is an immediate consequence of Proposition 2.5 and the boundedness of the pair (Z, V). $\square$

Next we show the following result.

Proposition 2.7. *The functions $\underline{u}, \bar{u}$ are classical solutions of*

$$S(\underline{u}; \bar{u}) = S(\bar{u}; \underline{u}) = 0, \qquad x \in \Omega, \quad t > 0, \tag{2.8}$$

$$\mathcal{B}(\underline{u}) = \mathcal{B}(\bar{u}) = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{2.9}$$

$$\underline{u}_0(x, 0) = \bar{u}_0(x, 0) = u_0(x), \quad x \in \Omega, \tag{2.10}$$

with $\underline{u}, \bar{u} \in C^{2,1}(\Omega_T)$.

Proof. We write (2.4), (2.6), (2.7) and (2.5)–(2.7) under a (very) weak formulation [**14, 20**]. Actually, we define the following:

$$\begin{aligned} N(z) \equiv \int_\Omega [z(x,s)\eta(x,s)]_0^\tau\,\mathrm{d}x - \int_0^\tau\int_\Omega z(x,s)\eta_t(x,s)\,\mathrm{d}x\,\mathrm{d}s - \int_0^\tau\int_\Omega K(z)\Delta\eta\,\mathrm{d}x\,\mathrm{d}s \\ = \lambda\int_0^\tau\int_\Omega F(z; u)\eta\,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{2.11}$$

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where

$$F(z;u) = f(z)\biggl(\int_\Omega f(u)\,\mathrm{d}x\biggr)^{-p} \in L^2(\Omega_T)$$

and

$$u \in L^2(\Omega_T), \quad K(u) \in \dot{V}_2(\Omega_T) = L^\infty((0,T);L^2(\Omega_T)) \cap L^2((0,T);L^2_{\mathrm{loc}}(\Omega)).$$

The function $\eta \in W^{2,1}_{\mathrm{c}}(\Omega_T)$ (η can also be taken in $C^\infty_{\mathrm{c}}(\Omega_T)$), $\eta(x,t) \geqslant 0$ with $\Delta\eta < 0$ (or $\Delta\eta \geqslant 0$). (For the definitions of V-spaces, see [**14**, p. 419] and [**15**].)

Problem (2.4), (2.6), (2.7) under a (very) weak formulation is written as

$$N(\underline{u}_n) = \lambda \int_0^\tau \int_\Omega F(\underline{u}_{n-1}, \bar{u}_{n-1})\eta\,\mathrm{d}x\,\mathrm{d}s.$$

Now, passing to the limit as $n \to \infty$, using the monotonicity of $\underline{u}_n$, $\bar{u}_n$, the monotone convergence theorem (due to the boundedness of (z,v) we may also use the dominated convergence theorem) and the fact that $\tau < \hat{T}$, with $\hat{T}$ as in Lemma 2.3 (we only need that $\bar{u}_n$ is uniformly bounded), we get

$$N(\underline{u}) = \lambda \int_0^\tau \int_\Omega F(\underline{u}, \bar{u})\eta\,\mathrm{d}x\,\mathrm{d}s$$

and, similarly,

$$N(\bar{u}) = \lambda \int_0^\tau \int_\Omega F(\bar{u}, \underline{u})\eta\,\mathrm{d}x\,\mathrm{d}s.$$

Equivalently, we have (in the distributional sense)

$$S(\bar{u}, \underline{u}) = S(\underline{u}, \bar{u}) = 0 \quad \text{in } \mathcal{D}'(\Omega_T). \tag{2.12}$$

Regularity. In fact, the solution found above is classical. By using standard regularity theory [**14**, p. 419], we see that any bounded (very) weak solution belongs to $C^{\alpha,\alpha/2}(\Omega_T)$ for some $0 < \alpha \leqslant 1$ (Sobolev embedding lemma). By bounded (very) weak solutions $\underline{u}, \bar{u}$, we mean functions which satisfy (2.12) and $\|\underline{u}\|_{L^\infty(\Omega_T)} < \infty$, $\|\bar{u}\|_{L^\infty(\Omega_T)} < \infty$. Now, by bootstrapping arguments and Schauder-type estimates, we obtain that $\underline{u}, \bar{u} \in C^{2+\alpha,1+\alpha/2}(\Omega_T)$. Finally, $\underline{u}$ and $\bar{u}$ are classical solutions, i.e. they satisfy (2.8)–(2.10) and $\underline{u}, \bar{u} \in C^{2,1}(\Omega_T)$. This completes the proof. □

So far we have proved that $\underline{u} \leqslant \bar{u}$. We now prove that $\underline{u} = \bar{u}$.

Lemma 2.8. *Let f be Lipschitz continuous (actually we only need one side Lipschitz: $f(a+b) - f(b) \leqslant La$, where L is a positive constant and $0 < a < R$ for some R) and $\underline{u}, \bar{u} \in C^{2,1}(\Omega_T)$. Then we have $\underline{u} \geqslant \bar{u}$.*

Proof. Let $\underline{u}^\varepsilon = \underline{u} + \varepsilon e^{\sigma t} > \underline{u}$ for some $\varepsilon > 0$ (we define $\bar{u}^\varepsilon$ similarly). Actually, we use $0 < \varepsilon \ll 1$, $\varepsilon e^{\sigma t} < \varepsilon e^{\sigma T} = R$ and L fixed. Then we have (see also Lemma 2.2 and [**2**])

$$\begin{aligned}
S(\underline{u}^\varepsilon; \bar{u}) &= \underline{u}^\varepsilon_t - \Delta K(\underline{u}^\varepsilon) - \lambda \frac{f(\underline{u}^\varepsilon)}{(\int_\Omega f(\bar{u})\,\mathrm{d}x)^p} \\
&= \underline{u}_t + \varepsilon\sigma e^{\sigma t} - K''(\underline{u}^\varepsilon)|\nabla \underline{u}|^2 - K'(\underline{u}^\varepsilon)\Delta \underline{u} - \lambda \frac{f(\underline{u}^\varepsilon)}{(\int_\Omega f(\bar{u})\,\mathrm{d}x)^p} \\
&\geqslant \underline{u}_t - \Delta K(\underline{u}) - \lambda \frac{f(\underline{u}) + L\varepsilon e^{\sigma t}}{(\int_\Omega f(\bar{u})\,\mathrm{d}x)^p} + \varepsilon\sigma e^{\sigma t} \\
&\qquad + (K''(\underline{u}) - K''(\underline{u}^\varepsilon))|\nabla(\underline{u})|^2 + (K'(\underline{u}) - K'(\underline{u}^\varepsilon))\Delta \underline{u} \\
&= S(\underline{u}; \bar{u}) + \varepsilon e^{\sigma t}\biggl[\sigma - \lambda \frac{L}{(\int_\Omega f(\bar{u})\,\mathrm{d}x)^p} - K'''(\underline{u})|\nabla \underline{u}|^2 - K''(\underline{u})\Delta \underline{u}\biggr] + O(\varepsilon^2) \\
&> S(\underline{u}; \bar{u}) = S(\bar{u}; \underline{u}).
\end{aligned}$$

The last inequality is due to the fact that σ is taken sufficiently large, $\underline{u}$ is bounded in $C^{2,1}(\bar{\Omega}_T)$, and $\underline{u} \leqslant \bar{u}$ and $\underline{u}^\varepsilon \leqslant \bar{u}$ for $\varepsilon \ll 1$. Now, by using Lemma 2.2, we have $\underline{u}^\varepsilon = \underline{u} + \varepsilon e^{\sigma t} > \bar{u}$ for any $0 < \varepsilon \ll 1$ and, taking $\varepsilon \to 0$, we get $\underline{u} \geqslant \bar{u}$. This completes the proof. $\square$

Remark 2.9. From the above analysis we obtain that the solution u to (1.1) continues to exist as long as it remains less than or equal to $b = V(\hat{T}) \geqslant \sup v(x,t)$. This argument implies that if u ceases to exist, then this will occur only by blow-up, which means that there exists a sequence $(x_n, t_n) \to (x^*, t^*)$ as $n \to \infty$, with $t^* < \infty$ such that $u(x_n, t_n) \to \infty$ as $n \to \infty$. Actually, we shall see at the end of the proof of Theorem 3.1 that the blow-up of u implies $\|u(\cdot\,, t)\| = \sup_\Omega |u(\cdot\,, t)| \to \infty$ as $t \to t^*- < \infty$.

The following theorem guarantees the local existence and uniqueness.

Theorem 2.10. *Problem (1.1) has a unique classical solution u in $C^{2,1}(\Omega_T)$ for some $T > 0$.*

Proof. The proof is a consequence of the previous lemmas, corollary and propositions. For the uniqueness result in particular, see Corollary 2.6 and Lemma 2.8. $\square$

In the next section we prove blow-up.

3. Blow-up of solutions for the non-local porous medium equation

We now come to the main aim of this work. In what follows, we show under which conditions blow-up of the solution u occurs for any $u_0(x) > 0$.

3.1. The Neumann problem

We assume that $\beta \equiv 0$ i.e. the Neumann problem. We now state our result.

Theorem 3.1. *Let Ω be a bounded domain of $\mathbb{R}^N$, $f(s)$ is taken to satisfy (1.2). Then the solution $u(x,t)$ of (1.4) blows up in finite time for all values of the parameter $\lambda > 0$.*

Proof. We assume the initial-value problem:

$$\theta'(t) = cf^{1-p}(\theta(t)), \quad t \in (0,T),\ \theta(0) = \theta_0, \tag{3.1}$$

where $c = \lambda|\Omega|^{1-p}$ is a constant, for some $T = T(\theta_0) > 0$.

Setting

$$B(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,\mathrm{d}x,$$

integrating (1.4 a) over Ω and using Green's identity we get

$$\dot{B}(t) = \frac{1}{|\Omega|}\int_{\partial\Omega}\frac{\partial u^m}{\partial\hat{n}}\,\mathrm{d}S(x) + \lambda\frac{1}{|\Omega|}\biggl(\int_\Omega f(u)\,\mathrm{d}x\biggr)^{1-p}.$$

The first integral is equal to zero, due to the Neumann boundary condition. Now, using Jensen's inequality we derive

$$\dot{B}(t) \geqslant \lambda|\Omega|^{-p}f^{1-p}(B), \quad t > 0,\ B_0 = B(0), \tag{3.2}$$

which implies

$$\lambda|\Omega|^{-p}t \leqslant \int_{B(0)}^{B(t)}\frac{\mathrm{d}s}{f^{1-p}(s)} < \int_{B_0\geqslant b}^{\infty}\frac{\mathrm{d}s}{f^{1-p}(s)} = T(B_0) < T(\theta_0) < \infty, \quad \theta_0 < B_0.$$

Taking $\theta_0 < B_0 < \|u_0\|$ and from a standard comparison between (3.1) and (3.2), we have

$$\theta(t) \leqslant B(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)\,\mathrm{d}x \leqslant \|u(\cdot\,,t)\|,$$

where

$$\|u(\cdot\,,t)\| \quad \text{and} \quad B(t) \to \infty \quad \text{as } t \to T^*- = \frac{|\Omega|^p}{\lambda}T(B_0)-,$$

since $\theta(t) \to \infty$ as $t \to T(\theta_0)-$. Hence $\|u(\cdot\,,t)\| \to \infty$ as $t \to t^* \leqslant T^* < \infty$, which implies that u blows up in finite time (see also Remark 2.9; the non-extendability implies blow-up). □

3.2. The Dirichlet and Robin problems

Here we have that $0 < \beta(x) \leqslant \infty$ on $\partial\Omega$. Moreover, we make the following assumption:

$$\Omega \text{ is convex.} \tag{3.3}$$

Also we consider the following auxiliary eigenvalue problem:

$$\Delta\phi + \mu\phi = 0, \quad x \in \Omega, \tag{3.4 a}$$

$$\frac{\partial\phi}{\partial\hat{n}} + \beta(x)\phi = 0, \quad x \text{ on } \partial\Omega. \tag{3.4 b}$$

Then, for the eigenpair (μ,ϕ), we have that the first eigenvalue $\mu > 0$ and $\phi = \phi(x)$ is positive and bounded, i.e. $0 \leqslant \min_\Omega \phi(x) \leqslant \phi \leqslant \bar{k} = \max_\Omega \phi(x)$. We take

$$\int_\Omega \phi(x)\,\mathrm{d}x = 1$$

for our version of Jensen's inequality to hold.

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Theorem 3.2. *Let (1.2), (3.3) hold and, in addition,*

$$\int_{\Omega}(f^{1-p}(u(x,t)) - u^m(x,t))\phi(x)\,\mathrm{d}x > 0. \tag{3.5}$$

Then, for sufficiently large λ ($\lambda > \lambda_0 = \mu|\Omega|^p[(\gamma+1)/k]^p > 0$), there exists a $t^* < \infty$ such that the solution of (1.4) blows up in finite time.

Proof. In order to prove blow-up we use Kaplan's method. We begin by multiplying ($1.4\,a$) by eigenfunction ϕ, integrating over Ω and setting

$$A(t) = \int_{\Omega}\phi(x)u(x,t)\,\mathrm{d}x.$$

We obtain

$$\dot{A}(t) = \frac{\partial}{\partial t}\int_{\Omega}\phi u\,\mathrm{d}x = \int_{\Omega}\phi\Delta u^m\,\mathrm{d}x + \lambda\frac{\int_{\Omega}\phi f(u)\,\mathrm{d}x}{(\int_{\Omega}f(u)\,\mathrm{d}x)^p}.$$

Assuming that u is global in time, applying Green's identity and problem (3.4), we have

$$\dot{A}(t) = -\mu\int_{\Omega}u^m\phi\,\mathrm{d}x + \lambda\frac{\int_{\Omega}\phi f(u)\,\mathrm{d}x}{(\int_{\Omega}f(u)\,\mathrm{d}x)^p}. \tag{3.6}$$

Here is where the convexity of Ω is used. Due to the convexity of Ω and the fact that f is positive and increasing, by using the method of moving parallel planes [**9**, **11**, **16**], a relative compact set $\Omega_0 \subset \Omega(\bar{\Omega}_0 \subset \Omega)$ can be constructed, so that

$$\int_{\Omega}f(u)\,\mathrm{d}x \leqslant (\gamma+1)\int_{\Omega_0}f(u)\,\mathrm{d}x \tag{3.7}$$

for some $\gamma = \gamma(\Omega) \in \mathbb{N}^*$. Let $k = \inf_{x\in\Omega_0}\phi(x)$. Then, by using the fact that $\bar{\Omega}_0 \subset \Omega$ and the maximum principle for problem (3.4), we have $k > 0$; thus, (3.7) implies that

$$\int_{\Omega}f(u)\,\mathrm{d}x \leqslant \frac{\gamma+1}{k}\int_{\Omega_0}f(u)\phi(x)\,\mathrm{d}x \leqslant \frac{\gamma+1}{k}\int_{\Omega}f(u)\phi(x)\,\mathrm{d}x$$

and so

$$\biggl(\int_{\Omega}f(u)\phi(x)\,\mathrm{d}x\biggr)\biggl(\int_{\Omega}f(u)\,\mathrm{d}x\biggr)^{-p} \geqslant \biggl(\frac{k}{\gamma+1}\biggr)^p\biggl(\int_{\Omega}f(u)\phi(x)\,\mathrm{d}x\biggr)^{1-p}. \tag{3.8}$$

Now, on using (3.5), (3.6), (3.8), we get

$$\dot{A}(t) \geqslant -\mu|\Omega|\oint_{\Omega}f^{1-p}(u)\phi(x)\,\mathrm{d}x + \lambda\biggl(\frac{k}{\gamma+1}\biggr)^p|\Omega|^{1-p}\biggl(\oint_{\Omega}f(u)\phi(x)\,\mathrm{d}x\biggr)^{1-p},$$
$$\text{where } \oint = \biggl(\frac{1}{|\Omega|}\biggr)\int.$$

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Applying Jensen's inequality, then taking sufficiently large λ ($\lambda > \lambda_0$) and again using Jensen, we derive

$$\begin{aligned}\dot{A}(t) &\geqslant \biggl[\lambda\biggl(\frac{k}{\gamma+1}\biggr)^{p}|\Omega|^{1-p} - \mu|\Omega|\biggr]\biggl(\oint_{\Omega} f(u)\phi(x)\,\mathrm{d}x\biggr)^{1-p} \\ &\geqslant \biggl[\lambda\biggl(\frac{k}{\gamma+1}\biggr)^{p}|\Omega|^{1-p} - \mu|\Omega|\biggr] f^{1-p}(A).\end{aligned}$$

The previous relation with the help of (1.2), as in Theorem 3.1, implies blow-up. □

4. Blow-up for sufficiently large initial data

Blow-up also occurs for sufficiently large initial data. The following results are valid for any $\lambda > 0$.

Theorem 4.1. *Let (1.2), (3.3) and*

$$\int_{\Omega}(f^{1-q}(u(x,t)) - u^{m}(x,t))\phi(x)\,\mathrm{d}x > 0 \quad \textit{for } 0 < p < q < 1 \tag{4.1}$$

hold. Then the solution $u(x,t)$ of (1.4), with $0 < \beta(x) \leqslant \infty$, i.e. the Dirichlet or Robin problem, blows up in finite time for sufficiently large initial data u_0:

$$\int_{\Omega} u_0\phi\,\mathrm{d}x = A_0 \geqslant A^* > \max\{0, \max\{\delta : g(\delta) = 0\}\},$$

where g is defined in (4.2).

Proof. Following the steps in the proof of Theorem 3.2, we get (3.6) and using (4.1) and Jensen's inequality again, we get

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} \geqslant \lambda\biggl(\frac{k}{\gamma+1}\biggr)^{p}|\Omega|^{1-p}f^{1-p}(A) - \mu|\Omega|f^{1-q}(A) := g(A), \quad t > 0,\ A_0 = A(0). \tag{4.2}$$

Let $\delta_0 > 0$ be the largest root (otherwise $g(s) > 0$ for any $s > 0$) of the equation

$$g(s) = |\Omega|f^{1-p}(s)\biggl\{\lambda\biggl[\frac{k}{(\gamma+1)|\Omega|}\biggr]^{p} - \mu f^{p-q}(s)\biggr\} = 0.$$

Then $g(s) \geqslant 0$ for all $s \geqslant \delta_0$. Furthermore, on taking $g(s) \geqslant \Lambda f^{1-p}(s)$, we have

$$t \leqslant \int_{A_0}^{A(t)} \frac{\mathrm{d}s}{g(s)} < \int_{A_0}^{\infty} \frac{\mathrm{d}s}{g(s)} \leqslant \frac{1}{\Lambda}\int_{A_0}^{\infty} \frac{\mathrm{d}s}{f^{1-p}(s)} < \infty,$$

provided that

$$0 < \Lambda = |\Omega|\biggl\{\lambda\biggl[\frac{k}{(1+\gamma)|\Omega|}\biggr]^{p} - \mu f^{p-q}(A^*)\biggr\} < \infty.$$

Now, for positive initial data $u_0(x) \in L^1(\Omega)$ such that

$$A_0 = \int_\Omega u_0(x)\phi(x)\,\mathrm{d}x \geqslant A^*,$$

relations (1.2 b) and (4.2) imply that the solution of (1.4) blows up at finite time

$$t^* \leqslant T^* \leqslant \int_{A_0}^{\infty} \frac{\mathrm{d}s}{g(s)} < \infty.$$

□

Remark 4.2. For $\beta(x) \equiv 0$, i.e. the Neumann problem, Theorem 3.1 holds for any initial data and for any $\lambda > 0$.

Remark 4.3. The same results hold for the filtration equation. The proofs are similar.

5. Blow-up for the non-local filtration equation, $\lambda \gg 1$

5.1. The Neumann problem

Here we have the same result as in the porous medium equation, for any positive initial data, so we can state a similar theorem.

Theorem 5.1. *Let Ω be a bounded domain of $\mathbb{R}^N$. $f(s)$ is taken to satisfy (1.2). Then the solution $u(x,t)$ of (1.1) blows up in finite time for all values of the parameter $\lambda > 0$.*

The proof is as for the porous medium problem (Theorem 3.1).

5.2. The Dirichlet and Robin problem

Again with the same method used in the porous medium problem we state the following theorem.

Theorem 5.2. *Let (1.2), (3.3) hold and let*

$$\int_\Omega (f^{1-p}(u(x,t)) - K(u(x,t)))\phi(x)\,\mathrm{d}x > 0 \quad \text{for } 0 < p < 1. \tag{5.1}$$

On taking sufficiently large values of λ,

$$\lambda > \lambda_0 = \mu\left(\frac{\gamma+1}{k}\right)^p |\Omega|^p > 0,$$

there exists a $t^* < \infty$ such that the solution of (1.1) blows up in finite time t^*.

Remark 5.3. Condition (5.1) holds, for instance, for any $f(u) > (K(u))^{1/(1-p)}$, $u > 0$ (this also implies (1.2 b) for some $K(u)$). Similarly, condition (3.5) holds, for instance, for any function such that $f(u) > u^{m/(1-p)}$, $u > 0$; the latter also implies the validity of (1.2 b).

6. Discussion

In this work, we firstly prove local existence and uniqueness of some non-local initial boundary-value problems for a porous medium equation (see (1.4)) and filtration equation (see (1.1)). We consider positive initial data ($u_0(x) > 0$ in Ω), in order to avoid degenerating solutions. For both problems we prove blow-up of solutions either for $\lambda \gg 1$ and any $u_0(x) > 0$ or for large—in some sense ($A_0 \gg 1$)—initial data $u_0(x)$ and any $\lambda > 0$. The method we use is Kaplan's method. For the proofs of Theorems 3.2, 4.1 and 5.2, we restrict Ω to be a convex domain.

In the case where we allow $u_0 \geqslant 0$ in Ω, we have to work with the very weak formulation of solutions, i.e.

$$N(u) = \lambda \int_0^\tau \int_\Omega F(u;u)\eta \, \mathrm{d}x \, \mathrm{d}s$$

(see (2.11)) and, again, on using comparison methods, we obtain similar existence and uniqueness results to those for the case when $u_0 > 0$ in Ω.

We propose some very interesting open questions for future work, such as

(a) the study of the blow-up, but now for $u_0(x) \geqslant 0$ in Ω, which results in the appearance of degeneracy of the solutions,

(b) showing the blow-up of solutions for any $\lambda > \lambda^*$ (while there are no steady-state solutions for $\lambda > \lambda^*$, for $0 < \lambda < \lambda^*$ there are classical steady-state solutions), and

(c) finding the rate of the growth of solution in some special cases, for instance, in the radial symmetric case, when the blow-up takes place at the origin, for $f(s) = \mathrm{e}^s$ and by using formal asymptotics for proper dimensions N (e.g. $N = 1, 2$).

Finally, we point out that the existence and uniqueness results are valid for any $p > 0$.

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