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Linear Dispersive Decay Estimates for the 3+1 Dimensional Water Wave Equation with Surface Tension

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Abstract. We consider the linearization of the three-dimensional water waves equation with surface tension about a flat interface. Using oscillatory integral methods, we prove that solutions of this equation demonstrate dispersive decay at the somewhat surprising rate of $t^{-5/6}$. This rate is due to competition between surface tension and gravitation at $O(1)$ wave numbers and is connected to the fact that, in the presence of surface tension, there is a so-called "slowest wave". Additionally, we combine our dispersive estimates with L^2 type energy bounds to prove a family of Strichartz estimates.

1 Introduction

The Korteweg–de Vries equation ($u_t = u_{xxx} + uu_x$), nonlinear Schrödinger equation ($iu_t = \Delta u + N(u)$), and nonlinear wave equation ($\Box u = N(u)$) each serve as modulation equations for the free-surface water wave problem in various physical scenarios; see, for example, [3,4,6]. In fact, it was this purpose that led to the original derivation of KdV. It is well known that solutions of the linearized versions of these equations exhibit dispersion, characterized in part by the fact that the amplitude of solutions decays algebraically in time while an L^2 based norm remains constant. This observation, coupled with powerful techniques from harmonic analysis, has resulted in a large number of breakthroughs in the existence theory for the nonlinear problems; see, for example, [12]. Despite these successes, there is a lack of analogous (linear) estimates for problems involving the motion of free-surface fluid interfaces.

In this paper we prove rigorous dispersive estimates for the linearized water wave problem in 3 + 1 dimensions with surface tension. Specifically, we consider the motion of the interface between an ideal fluid below and a vacuum above, which together occupy all of $\mathbf{R}^3$. We take z as the vertical coordinate and x and y as the horizontal. We assume that the fluid velocity field is irrotational (in the bulk), that surface tension is present, and that gravity acts downwards. The fluid velocity field is given by $\mathbf{u}(x, y, z, t)$, and the interface is assumed to be the graph of a function $\eta(x, y, t)$. The equations of motion for this scenario are well known ([5, 11]):

- $\Delta\phi = 0$ in the fluid domain;
- $\phi_z \to 0$ as $z \to -\infty$;
- $\mathbf{u} = \nabla\phi$ in the fluid domain;

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- $\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z$ on the surface;
- $\phi_t + g\eta + 1/2|\nabla\phi|^2 = \tau\,\mathrm{div}\Big(\frac{\nabla\eta}{\sqrt{1+\eta_x^2+\eta_y^2}}\Big)$ on the surface.

Here $g > 0$ is the acceleration due to gravity, and τ is the surface tension constant. Note that one can reformulate this problem purely in terms of the variables $\eta(x, y, t)$ and $\varphi(x, y, t) = \phi(x, y, \eta(x, y, t), t)$ ([2]). If one does so, and then linearizes about the equilibrium $\eta = \phi = 0$, the resulting system is

$$\eta_t = 1/2(\mathcal{R}_x\partial_x + \mathcal{R}_y\partial_y)\varphi, \quad \varphi_t = \tau(\eta_{xx} + \eta_{yy}) - g\eta,$$

where $\mathcal{R}_j$ are the two dimensional Riesz transforms. Solutions of this system of equations can be determined by means of the Fourier transform. In particular,

$$\eta(\mathbf{x}, t) = \int_{\mathbf{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}} \cos(\lambda(|\mathbf{k}|)t)\widehat{\eta}_0(\mathbf{k})d\mathbf{k} + \int_{\mathbf{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}} \sin(\lambda(|\mathbf{k}|)t)\frac{|\mathbf{k}|}{\lambda(|\mathbf{k}|)}\widehat{\varphi}_0(\mathbf{k})d\mathbf{k},$$

$$\varphi(\mathbf{x}, t) = \int_{\mathbf{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}} \cos(\lambda(|\mathbf{k}|)t)\widehat{\varphi}_0(\mathbf{k})d\mathbf{k} - \int_{\mathbf{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}} \sin(\lambda(|\mathbf{k}|)t)\frac{\lambda(|\mathbf{k}|)}{|\mathbf{k}|}\widehat{\eta}_0(\mathbf{k})d\mathbf{k}.$$

Thus, the solution of the linearized problem is given by linear combinations of the operators S_j, $j = 1, 2, 3$:

$$S_j(t)f(\mathbf{x}) := \int_{\mathbf{R}^2} \frac{e^{i(\mathbf{x}\cdot\mathbf{k}+\lambda(|\mathbf{k}|)t)}}{\sigma_j(|\mathbf{k}|)}\widehat{f}(\mathbf{k})d\mathbf{k}. \tag{1.1}$$

Here $\lambda(r) := \sqrt{gr + \tau r^3}$, where g is the acceleration due to gravity, and τ is the surface tension constant. Additionally, $\sigma_1(r) = 1$, $\sigma_2(r) = \lambda(r)/r$ and $\sigma_3(r) = 1/\sigma_2(r)$. Our main results are an amplitude dispersive decay estimate

$$\|S_j(t)f\|_{L^\infty} \leq C|t|^{-5/6}\|f\|_{B^{s_j}_{1,1}}$$

and a set of associated Strichartz estimates; see Theorem 1.2.

The time decay exponent of $-5/6$ here may seem unnatural but is formally the optimal rate of decay, as we now demonstrate. Consider the group speed

$$\lambda'(r) = \frac{g + 3\tau r^2}{2\sqrt{gr + \tau r^3}}.$$

Note that $\lambda'(r) = O(r^{-1/2})$ for $r \sim 0$ and $\lambda(r) = O(r^{1/2})$ for $r \to \infty$. Since this quantity tends to infinity as $r \to 0$ and $r \to \infty$, it clearly has a minimum value at some point r_s. That is to say, we will have a degenerate stationary point when $r = r_s$., And so have for $r \sim r_s$

$$\lambda(r) \sim \lambda(r_s) + \lambda'(r_s)(r - r_s) + 1/6\lambda'''(r_s)(r - r_s)^3.$$

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If we observe waves that move with speed equal to the minimum wave speed (that is $\mathbf{x} = (x_1, 0) = (-\lambda'(r_s)t, 0)$), we have

$$\begin{aligned}
&\left| \int_{|\mathbf{k}|\sim r_s} e^{i(\mathbf{x}\cdot\mathbf{k}+\lambda(|\mathbf{k}|)t)} d\mathbf{k} \right| \\
&\quad \sim \left| r_s \int_{\theta=-\pi}^{\pi} \int_{r\sim r_s} e^{i(rx_1\cos(\theta)+\lambda(r)t)} dr d\theta \right| \\
&\quad \sim C\left| \int_{\theta\sim 0} \int_{r\sim r_s} e^{i(rx_1\cos(\theta)+\lambda(r)t)} dr d\theta \right| \\
&\quad \sim C\left| \int_{\theta\sim 0} \int_{r\sim r_s} e^{i(-r\lambda'(r_s)t(1-\theta^2/2)+\lambda(r_s)t+\lambda'(r_s)(r-r_s)t+1/6\lambda'''(r_s)(r-r_s)^3 t)} \right| \\
&\quad \sim \left| \int_{\theta\sim 0} e^{iCt\theta^2} d\theta \right| \left| \int_{r\sim r_s} e^{iCt(r-r_s)^3} dr \right| \\
&\quad \sim Ct^{-1/2} \left| \int_{\theta\sim 0} e^{iC(t^{1/2}\theta)^2} d(t^{1/2}\theta) \right| t^{-1/3} \left| \int_{r\sim r_s} e^{iC(t^{1/3}(r-r_s))^3} d(t^{1/3}r) \right| \\
&\quad \sim Ct^{-5/6}.
\end{aligned}$$

(See also [7].)

Though this formal calculation is in some sense the heart of our estimate, there are several technical complications that arise when making it rigorous. The first is that as $r \to \infty$, $\lambda''(r) \to 0$. Typical stationary phase estimates that come into play for estimating (1.1) require lower bounds on this quantity for large r. Moreover, the Fourier multiplier operators $\sigma_j(r)$ complicate the stationary phase argument. We circumvent both problems by requiring additional regularity of the initial data utilizing the Besov norms. We have substantially improved the methods used in [9] wherein we proved similar estimates for the 2+1 dimensional water wave problem that required sizable regularity of the initial data.

1.1 Setup

Let $\{\widehat{\phi}_n\}$ be a partition of unity subordinate to the regions

$$U_0 := \left\{ |\mathbf{k}| \in [0, 1/2] \right\} \quad \text{and} \quad U_n := \left\{ |\mathbf{k}| \in [2^{n-2}, 2^{n-1}] \right\}$$

for $n \geq 1$. In particular, we assume the support of $\widehat{\phi}_n$ is in $U_{n-1} \cup U_n \cup U_{n+1}$ if $n \geq 1$ and in $U_0 \cup U_1$ if $n = 0$. We let $\chi_n(\mathbf{k})$ be the characteristic function of $U_{n-1} \cup U_n \cup U_{n+1}$ if $n \geq 1$ and the characteristic function of $U_0 \cup U_1$ otherwise. Let $I_0 := [0, 1/2]$ and $I_n := [2^{n-2}, 2^{n-1}]$, $n \geq 1$.

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Breaking up $\mathbf{R}^2$ into $\cup_{n\geq 0} U_n$ and using the convolution estimate, we find

$$\begin{aligned}\left|S_j(t)f(\mathbf{x})\right| &\leq \sum_{n=0}^{\infty}\left|\int_{\mathbf{R}^2}\frac{e^{i(\mathbf{x}\cdot\mathbf{k}+\lambda(|\mathbf{k}|)t)}}{\sigma_j(|\mathbf{k}|)}\chi_n(\mathbf{k})\widehat{\phi}_n(\mathbf{k})\widehat{f}(\mathbf{k})d\mathbf{k}\right| \\ &\leq \sum_{n=0}^{\infty}\sup_{\mathbf{x}}\left|\int_{\mathbf{R}^2}\frac{e^{i(\mathbf{x}\cdot\mathbf{k}+\lambda(|\mathbf{k}|)t)}}{\sigma_j(|\mathbf{k}|)}\chi_n(\mathbf{k})d\mathbf{k}\right|\|\phi_n\star f\|_{L^1}.\end{aligned}$$

The following proposition is our crucial decay estimate.

Proposition 1.1 *Let $s_1 = 3/4$, $s_2 = 1/4$, and $s_3 = 5/4$. Then*

$$\sup_{\mathbf{x}}\left|\int_{U_n}\frac{e^{i(\mathbf{x}\cdot\mathbf{k}+\lambda(|\mathbf{k}|)t)}}{\sigma_j(|\mathbf{k}|)}d\mathbf{k}\right| \leq Ct^{-5/6}2^{s_j n}.$$

Here C is a constant independent of n.

Recall that the Besov space $B^s_{p,q}$ can be defined by its norm

$$\|f\|_{B^s_{p,q}} := \left(\sum_{n=0}^{\infty}\left(2^{sn}\|\phi_n\star f\|_{L^p}\right)^q\right)^{1/q}.$$

We show that Proposition 1.1 implies the following set of estimates.

Theorem 1.2 *The linear operators S_j satisfy the following dispersive estimate:*

$$\left\|S_j(t)f\right\|_{L^\infty} \leq Ct^{-5/6}\|f\|_{B^{s_j}_{1,1}},$$

where $s_j = \{\frac{3}{4}, \frac{1}{4}, \frac{5}{4}\}$ and the following Strichartz-type estimates:

$$\|S_j g\|_{L_t^{22/5}(B^{\mu}_{22/5,2})} \leq C\|g\|_{H^{\mu+\gamma_j}}.$$

for any $\mu \in \mathbf{R}$, and $\gamma_j = \{\frac{9}{44}, \frac{-6}{11}, \frac{21}{22}\}$, respectively.

2 Stationary Phase Estimates

We prove Proposition 1.1 by means of the method of stationary phase, and so it is important to understand the group speed $\lambda'(r)$ and its derivatives. The following calculus lemma tells us all we need to know about the group speed as well as the phase speed $\sigma_2(r)$.

Lemma 2.1 (i) *The minimum value of $\lambda'(r)$ occurs at*

$$r_s = \sqrt{\sqrt{4/3}-1} < 1/2 \quad \textit{and} \quad \kappa_s := \lambda'(r_s) > 0.$$

(ii) *There exists $C > 1$ such that for all $r \geq 1/2$, $C^{-1}r^{1/2} \leq \lambda'(r) \leq Cr^{1/2}$.*

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(iii) *There exists* $C > 1$ *such that for all* $r \geq 1$, $C^{-1}r^{1/2} \leq \sigma_2(r) \leq Cr^{1/2}$.
(iv) *There exists* $C > 1$ *such that for all* $r \geq 1$, $C^{-1}r^{-1/2} \leq \lambda''(r) \leq Cr^{-1/2}$.
(v) $\lambda'''(1) = 0$ *and is the only zero of* λ'''. *In particular, if* $0 < r \leq 1/2$, *then* $|\lambda'''(r)| \geq C > 0$.

We will make repeated use of the van der Corput estimate, whose proof can be found in [10].

Lemma 2.2 *Let* $h(r)$ *be* C^k *on* $[a, b]$ *with* $-\infty \leq a < b \leq +\infty$ *and* $k \geq 2$. *Suppose that* $h^{(k)}(r)$ *is either always positive or always negative on* $[a, b]$. *Then*

$$\left| \int_a^b e^{it(\kappa r + h(r))} dr \right| \leq Ct^{-1/k} \left\{ \min_{[a,b]} |h^{(k)}| \right\}^{-1/k},$$

where C *is a positive constant that depends only on* k *(and not on* a *or* b*).*

Moreover, if $|\kappa + h'(r)| \geq c_0 > 0$ *and* $h''(r)$ *has a finite number of zeros in* $[a, b]$, *then*

$$\left| \int_a^b e^{it(\kappa r + h(r))} dr \right| \leq Ct^{-1} \left\{ \min_{[a,b]} |\kappa + h'| \right\}^{-1}.$$

The next lemma is also modified from one in [10].

Lemma 2.3 *For any* h *and* ψ *sufficiently smooth,*

$$\left| \int_a^b e^{ih(r)} \psi(r) dr \right| \leq \sup_{r \in [a,b]} \left| \int_a^r e^{ih(r)} dr \right| \left(\left| \psi(b) \right| + \int_a^b \left| \psi'(r) \right| dr \right).$$

Proof Let $F(r) = \int_a^r e^{ih(\rho)} d\rho$. Then the integral is $\int_a^b F'(r)\psi(r)$. Integrating by parts yields the estimate. ∎

Remark 2.4 We will frequently encounter integrals of the form $\int_a^b |\psi'(r)| dr$. It is straightforward to show that if $\psi'(r) = 0$ at a finite number of values of $r \in \mathbf{R}$, then

$$\int_a^b \left| \psi'(r) \right| dr \leq C \sup_{[a,b]} |\psi|,$$

where C does not depend on explicitly on the size of $[a, b]$, but only on the number of zeros. (This fact is of course used to prove the second part of Lemma 2.2.)

3 Proof of Proposition 1.1

Without loss of generality, we set $\mathbf{x} = (0, y)$. We convert to polar coordinates in the integral and set $\kappa = y/t$:

$$\sup_{\mathbf{x}} \left| \int_{U_n} \frac{e^{i(\mathbf{x}\cdot\mathbf{k} + \lambda(|\mathbf{k}|)t)}}{\sigma_j(|\mathbf{k}|)} d\mathbf{k} \right| = \sup_{\kappa} \left| \int_0^{2\pi} \int_{I_n} \frac{e^{it(\kappa r \sin(\theta) + \lambda(r))}}{\sigma_j(r)} r dr d\theta \right|.$$

We will proceed by cases.

Case 1: $|\kappa| \leq 1/2\kappa_{\rm s}$.

No wave packet can move with speed less than $\kappa_{\rm s}$; this is well known experimentally for (small amplitude) surface water waves; see [11]. The estimate we prove here quantifies this. Consider

$$\begin{aligned} I_{\rm s} &:= \sup_{|\kappa|\leq\kappa_{\rm s}/2} \left| \int_0^{2\pi} \int_{I_n} \frac{e^{it(\kappa r\sin(\theta)+\lambda(r))}}{\sigma_j(r)} r dr d\theta \right| \\ &\leq C \sup_{|\kappa|\leq\kappa_{\rm s}/2} \left| \int_{I_n} \frac{e^{it(\kappa r+\lambda(r))}}{\sigma_j(r)} r dr \right|. \end{aligned}$$

Note that $r/\sigma_j(r)$ remains bounded as $r \to 0$ for all j, and so there are no additional difficulties when treating the integral over low frequencies, $r \in I_0$.

Applying Lemma 2.3 and Remark 2.4 to this integral gives

$$\begin{aligned} &\left| \int_{I_n} \frac{e^{it(\kappa r+\lambda(r))}}{\sigma_j(r)} r dr \right| \\ &\quad \leq \sup_{b\in I_n} \left| \int_{\min I_n}^b e^{it(\kappa r+\lambda(r))} dr \right| \left(\left| \frac{\max I_n}{\sigma_j(\max I_n)} \right| + \int_{I_n} \left| \left(\frac{r}{\sigma_j(r)} \right)' \right| dr \right) \\ &\quad \leq C\left(\frac{2^n}{\sigma_j(2^n)} \right) \sup_{b\in I_n} \left| \int_{\min I_n}^b e^{it(\kappa r+\lambda(r))} dr \right|. \end{aligned}$$

Now we estimate the remaining oscillatory integral by means of the second part of Lemma 2.2, since we know $\kappa + \lambda'(r)$ is bounded away from zero

$$\left| \int_{\min I_n}^b e^{it(\kappa r+\lambda(r))} dr \right| \leq \frac{C}{t} \left(\min_{I_n} |\kappa + \lambda'| \right)^{-1} \leq \frac{C}{t\lambda'(2^n)}.$$

Therefore, we have

$$I_{\rm s} \leq \frac{C}{t} \left(\frac{2^n}{\lambda'(2^n)\sigma_j(2^n)} \right) \leq Ct^{-1} 2^{s_j n}.$$

Remark 3.1 In fact, the rate of decay for $\kappa \leq 1/2\kappa_{\rm s}$ is faster than any power of $1/t$. This follows by noting that, for any $N \in \mathbf{N}$,

$$e^{it(\kappa r+\lambda(r))} = \left(\frac{1}{t(\kappa + \lambda'(r))} \frac{\partial}{\partial r} \right)^N \left(e^{it(\kappa r+\lambda(r))} \right)$$

and then integrating by parts N times inside the oscillatory integral.

Case 2: $|\kappa| \geq 1/2\kappa_{\rm s}$.

In this situation, there may be stationary points at which $\kappa + \lambda'(r) = 0$. In particular, we notice that at $\lambda''(r_{\rm s}) = 0$, meaning that we will get quite slow decay when

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$\kappa \sim \kappa_s$. For the 2+1 dimensional water wave equation, this results in an overall time decay of $t^{-1/3}$; see [9]. The additional space dimension will increase the rate here.

The zeroth Bessel function is given by

$$J_0(z) := \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin(\theta)}d\theta.$$

We rewrite the integral to be estimated:

$$\begin{aligned} I_{\text{fast}} &:= \sup_{|\kappa|\geq 1/2\kappa_s}\left|\int_0^{2\pi}\int_{I_n}\frac{e^{it(\kappa r\sin(\theta)+\lambda(r))}}{\sigma_j(r)}rdrd\theta\right| \\ &= \sup_{|\kappa|\geq 1/2\kappa_s} C\left|\int_{I_n} J_0(tr\kappa)e^{it\lambda(r)}\frac{r}{\sigma_j(r)}dr\right|. \end{aligned}$$

The following estimate for $J_0(z)$ is well known (see [1]):

$$\sup_{z\in\mathbf{R}}|z|^{3/2}\left|J_0(z)-\sqrt{\tfrac{2}{\pi z}}\cos(z-\pi/4)\right| < \infty.$$

And so we have

$$\begin{aligned} I_{\text{fast}} \leq &\sup_{|\kappa|\geq 1/2\kappa_s}\left|\int_{I_n}\sqrt{\tfrac{2}{\pi tr\kappa}}\cos(tr\kappa-\pi/4)e^{it\lambda(r)}\frac{r}{\sigma_j(r)}dr\ biggr\right| \\ &+\sup_{|\kappa|\geq 1/2\kappa_s} C\int_{I_n}\left|\left(J_0(tr\kappa)-\sqrt{\tfrac{2}{\pi tr\kappa}}\cos(tr\kappa-\pi/4)\right)\frac{r}{\sigma_j(r)}\right|dr. \end{aligned}$$

We handle the first piece by the method of stationary phase. In fact, this integral is nearly identical to those we studied in [9]. We employ Lemma 2.3 exactly as in our estimate for I_s and find

$$\begin{aligned} &\left|\int_{I_n}\sqrt{\frac{2}{\pi tr\kappa}}\cos(tr\kappa-\pi/4)e^{it\lambda(r)}\frac{r}{\sigma_j(r)}dr\right| \\ &\quad = C\sup_{\pm}\sqrt{\frac{1}{\kappa t}}\left|\int_{I_n}e^{it(r\kappa\pm\lambda(r))}\frac{\sqrt{r}}{\sigma_j(r)}dr\right| \\ &\quad \leq C\sqrt{\frac{1}{\kappa t}}\left(\frac{2^{n/2}}{\sigma_j(2^n)}\right)\sup_{\substack{b\in I_n\\ \pm}}\left|\int_{\min I_n}^{b}e^{it(\kappa r\pm\lambda(r))}dr\right|. \end{aligned}$$

Notice that $r_s \in I_0$ and $\lambda''(r_s) = 0$. However, $\lambda'''(r)$ is bounded away from zero in U_0. And so, in this interval, we apply the van der Corput estimate with $k = 3$ to find

$$\sup_{\substack{b\in[0,1/2]\\ \pm}}\left|\int_0^b e^{it(\kappa r\pm\lambda(r))}dr\right| \leq Ct^{-1/3}\left(\min_{r\in I_n}|\lambda'''(r)|\right)^{-1/3} \leq Ct^{-1/3}.$$

For all $n \geq 1$, $\lambda''(r) \neq 0$ for $r \in I_n$. Thus we use the van der Corput estimate with $k = 2$:

$$\sup_{\substack{b\in I_n\\ \pm}} \left| \int_{\min I_n}^{b} e^{it(\kappa r \pm \lambda(r))} dr \right| \leq Ct^{-1/2} \Big(\min_{r\in I_n} |\lambda''(r)| \Big)^{-1/2} \leq Ct^{-1/2} 2^{n/4}.$$

And so, we can conclude, for any n,

$$\sup_{|\kappa|\geq 1/2\kappa_s} \left| \int_{I_n} \sqrt{\tfrac{2}{\pi t r\kappa}} \cos(tr\kappa - \pi/4) e^{it\lambda(r)} \frac{r}{\sigma_j(r)} dr \right| \leq Ct^{-5/6} \frac{2^{3n/4}}{\sigma_j(2^n)} \leq Ct^{-5/6} 2^{ns_j}.$$

Now we estimate the error made by approximating J_0. First we consider $n \geq 1$. Then

$$\sup_{|\kappa|\geq 1/2\kappa_s} C \int_{I_n} \left| \Big(J_0(tr\kappa) - \sqrt{\tfrac{2}{\pi t r\kappa}} \cos(tr\kappa - \pi/4) \Big) \frac{r}{\sigma_j(r)} \right| dr$$
$$\leq \sup_{|\kappa|\geq 1/2\kappa_s} C \int_{I_n} \left| (tr\kappa)^{-3/2} \frac{r}{\sigma_j(r)} \right| dr \leq \frac{C}{t^{3/2}} \int_{I_n} \frac{1}{\sqrt{r}\sigma_j(r)} dr \leq \frac{C}{t^{3/2}} 2^{ns_j}.$$

If $n = 0$, then

$$\sup_{|\kappa|\geq 1/2\kappa_s} C \int_{I_0} \left| \Big(J_0(tr\kappa) - \sqrt{\tfrac{2}{\pi t r\kappa}} \cos(tr\kappa - \pi/4) \Big) \frac{r}{\sigma_j(r)} \right| dr$$
$$\leq \sup_{|\kappa|\geq 1/2\kappa_s} C \int_0^{1/2} \left| (tr\kappa)^{-5/6} \frac{r}{\sigma_j(r)} \right| dr \leq \frac{C}{t^{5/6}} \int_0^{1/2} \frac{1}{r^{1/6}\sigma_j(r)} dr \leq \frac{C}{t^{5/6}}.$$

4 Strichartz Estimates

We can now use the L^∞ estimate, along with an L^2 estimate below, to establish a family of Strichartz type estimates on the operators S_j. Our first result follows.

Proposition 4.1 *Let $1 \leq r \leq 2$ and $\frac{1}{q} = 1 - \frac{1}{r}$, then*

$$\big\| S_j(t)\mathcal{F}^{-1}[\chi_n \widehat{\phi}_n f] \big\|_{L^q} \leq Ct^{-\frac{5}{6}(\frac{2}{r}-1)} 2^{\alpha_j n} \|\phi_n \star f\|_{L^r}, \tag{4.1}$$

which in turn implies

$$\|S_j(t) f\|_{L^q} \leq Ct^{-\frac{5}{6}(\frac{2}{r}-1)} \|f\|_{B^{\alpha_j}_{r,1}}.$$

Here $\alpha_1 = \frac{3}{2r} - \frac{3}{4}$, $\alpha_2 = \frac{3}{2r} - \frac{5}{4}$, and $\alpha_3 = \frac{3}{2r} - \frac{1}{4}$.

Proof We will use interpolation between our L^∞ estimates above and an L^2 bound to prove (4.1). To establish the L^2 bound we have the following straightforward estimate:

$$\|S_j(t) f\|_{L^2} \leq \sum_{n=0}^{\infty} \Big\| \mathcal{F}^{-1} \Big[\frac{e^{i\lambda t}}{\sigma_j} \chi_n \widehat{\phi}_n \widehat{f} \Big] \Big\|_{L^2} \leq \sum_{n=0}^{\infty} \Big\| \frac{\chi_n}{\sigma_j} \widehat{\phi}_n \widehat{f} \Big\|_{L^2}.$$

For each individual wavelet we have

$$\Big\| \frac{\chi_n}{\sigma_j}\widehat{\phi}_n\widehat{f}\Big\|_{L^2} \leq \Big\| \frac{\chi_n}{\sigma_j}\Big\|_{L^\infty}\big\| \widehat{\phi}_n\widehat{f}\big\|_{L^2} \leq \frac{C}{\sigma_j(2^n)}\|\phi_n \star f\|_{L^2}.$$

Our operator $S_j f$ satisfies the following for $f_n = \mathcal{F}^{-1}[\chi_n\phi_n f]$

$$\|S_j f_n\|_{L^\infty} \leq Ct^{-5/6}2^{s_j n}\|f_n\|_{L^1}$$

$$\|S_j f_n\|_{L^2} \leq \frac{C}{\sigma_j(2^n)}\|f_n\|_{L^2}.$$

Recall the Riesz–Thorin Interpolation Theorem, which implies that for $\|Tg\|_{L^\infty} \leq M_0\|g\|_{L^1}$ and $\|Tg\|_{L^2} \leq M_1\|g\|_{L^2}$, then $\|Tg\|_{L^r} \leq M\|g\|_{L^q}$ with $\frac{1}{r} = 1-\frac{\theta}{2}$ and $\frac{1}{q} = \frac{\theta}{2}$ with $M \leq M_0^{1-\theta}M_1^{\theta}$. Therefore, if we take $\frac{1}{r}+\frac{1}{q} = 1$, then $\|S_j f_n\|_{L^q} \leq M\|f_n\|_{L^r}$, where

$$M \leq M_0^{1-\theta}M_1^{\theta} \leq C\Big(\frac{2^{s_j n}}{t^{\frac{5}{6}}}\Big)^{\frac{2}{r}-1}\Big(\frac{1}{\sigma_j(2^n)}\Big)^{2-\frac{2}{r}}.$$

Since $\sigma_j(2^n) \approx \{1, 2^{n/2}, 2^{-n/2}\}$, respectively, we have

$$\big\| S_j\mathcal{F}^{-1}[\chi_n\phi_n f]\big\|_{L^q} \leq Ct^{-\frac{5}{6}(\frac{2}{r}-1)}2^{\alpha_j n}\|\phi_n \star f\|_{L^r},$$

where $\alpha_1 = \frac{3}{2r}-\frac{3}{4}, \alpha_2 = \frac{3}{2r}-\frac{5}{4}, \alpha_3 = \frac{3}{2r}-\frac{1}{4}$, $1 \leq r \leq 2$, and $\frac{1}{r}+\frac{1}{q} = 1$, as claimed above. ■

We can now prove the Strichartz estimates by using a duality argument; see, for example, [8].

Theorem 4.2 *We have*

$$\|S_j g\|_{L_t^{22/5}(B^{-\mu_j}_{22/5,2})} \leq C\|g\|_{H^{\beta_j}},$$

where $\mu_j = \{\frac{9}{44}, -\frac{1}{22}, \frac{5}{11}\}$ *and* $\beta_j = \{0, -1/2, 1/2\}$, *respectively.*

We compute the norm of $S(t)$ by duality, so we compute for a test function η:

$$\text{(4.2)} \qquad \big|\langle S_j(t)g, \eta\rangle_{L^2(\mathbf{R}^{3+1})}\big| = \Big|\int_t\int_{\mathbf{k}} e^{it\lambda(|\mathbf{k}|)}\sigma_j^{-1}\,\widehat{g}\,\overline{\widehat{\eta}}\,d\mathbf{k}dt\Big| \leq \big\|\sigma_j^{-1/2}\widehat{g}\big\|_{L^2_{\mathbf{k}}}\cdot\Big\|\int_t e^{it\lambda}\sigma_j^{-1/2}\,\overline{\widehat{\eta}}\,dt\Big\|_{L^2}.$$

Therefore, we need an estimate on the second term on the right-hand side. In particular,

$$\text{(4.3)} \qquad \Big\|\int_t e^{it\lambda}\sigma_j^{-1/2}\widehat{\eta}dt\Big\|_{L^2}^2 = \Big\|\sum_{n=0}^{\infty}\widehat{\phi}_n\chi_n\int_t e^{it\lambda}\sigma_j^{-1/2}\,\widehat{\eta}\,dt\Big\|_{L^2}^2 \leq C\sum_{n=0}^{\infty}\Big\|\int_t e^{it\lambda}\sigma_j^{-1/2}\widehat{\eta}\,\widehat{\phi}_n\,\chi_n dt\Big\|_{L^2}^2.$$

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For a particular wavelet we use our dispersive inequality

$$\begin{aligned}
&\Big\| \int_{-\infty}^{\infty} e^{it\lambda}\sigma_j^{-1/2}\chi_n\,\widehat{\phi}_n\,\widehat{\eta}(t)\,dt\Big\|_{L^2_{\mathbf{k}}}^2\\
&\quad= \int_{t=-\infty}^{\infty}\int_{s=-\infty}^{\infty}\big(e^{i\lambda t}\sigma_j^{-1/2}\chi_n\phi_n\widehat{\eta},\, e^{i\lambda s}\sigma_j^{-1/2}\chi_n\widehat{\phi}_n\widehat{\eta}\big)_{L^2_{\mathbf{k}}} dsdt\\
&\quad= \int_{t=-\infty}^{\infty}\Big(\chi_n\widehat{\phi}_n\widehat{\eta},\, \int_{s=-\infty}^{\infty} e^{i\lambda(s-t)}\sigma_j^{-1}\chi_n\widehat{\phi}_n\widehat{\eta}\Big)_{L^2_{\mathbf{k}}} dsdt\\
&\quad= \int\int\Big(\phi_n\star\eta(t),\, \mathcal{F}^{-1}\big[e^{i(s-t)\lambda(|\mathbf{k}|)}\sigma_j^{-1}\chi\widehat{\phi}_n\widehat{\eta}(s)\big]\Big)_{L^2} dsdt\\
&\quad\le \int_t\int_s \|\phi_n\star\eta(t)\|_{L^r}\Big\|\mathcal{F}^{-1}\big[e^{i(s-t)\lambda(|\mathbf{k}|)}\sigma_j^{-1}\chi\widehat{\phi}_n\widehat{\eta}(s)\big]\Big\|_{L^q} dsdt\\
&\quad\le \int_t\int_s \frac{2^{\alpha_j n}}{|t-s|^{\frac{5}{6}(\frac{2}{r}-1)}}\|\phi_n\star\eta(t)\|_{L^r}\|\phi_n\star\eta(s)\|_{L^r}\,dsdt,
\end{aligned}$$

which we combine with (4.3) to get

$$\begin{aligned}
&\Big\|\int_t e^{it\lambda}\sigma_j^{-1/2}\widehat{\eta}dt\Big\|_{L^2_{\mathbf{k}}}^2 dsdt\\
&\quad\le C\sum_{n=0}^{\infty}\int_t\int_s \frac{2^{\alpha_j n}}{|t-s|^{\frac{5}{6}(\frac{2}{r}-1)}}\|\phi_n\star\eta(t)\|_{L^r}\|\phi_n\star\eta(s)\|_{L^r}dsdt\\
&\quad= C\int_t\int_s |t-s|^{-\frac{5}{6}(\frac{2}{r}-1)}\sum_{n=0}^{\infty}2^{\frac{\alpha_j n}{2}}\|\phi_n\star\eta(t)\|_{L^r}2^{\frac{\alpha_j n}{2}}\|\phi_n\star\eta(s)\|_{L^r}dsdt\\
&\quad\le C\int_t\int_s |t-s|^{-\frac{5}{6}(\frac{2}{r}-1)}\Big(\sum_{n=0}^{\infty}\big(2^{\frac{\alpha_j n}{2}}\|\phi_n\star\eta(t)\|_{L^r}\big)^2\Big)^{1/2}\\
&\qquad\times\Big(\sum_{n=0}^{\infty}\big(2^{\frac{\alpha_j n}{2}}\|\phi_n\star\eta(s)\|_{L^r}\big)^2\Big)^{1/2}dsdt\\
&\quad= C\int_t\int_s |t-s|^{-\frac{5}{6}(\frac{2}{r}-1)}\|\eta(t)\|_{B^{\alpha_j/2}_{r,2}}\|\eta(s)\|_{B^{\alpha_j/2}_{r,2}}dsdt.
\end{aligned}$$

We can now use the classical Hardy–Sobolev–Littlewood inequality

$$\Big|\int\int f(s)|s-t|^{-\ell}g(t)dsdt\Big| \le N_{p,\ell,n}\|f\|_{L^p}\|g\|_{L^q}$$

for $1 < p, q < \infty$, $0 < \ell < n$, and $\frac{1}{p}+\frac{1}{q}+\frac{\ell}{n} = 2$. If $p = q$ and $f = g$, then $|\int\int f(x)|x-y|^{-\ell}f(y)dxdy| \le C\|f\|_{L^p}^2$, where $\frac{2}{p} = 2-\frac{\ell}{n} = 2-\ell$, or $p = \frac{2}{2-\ell}$.

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Applying this to our wavelet bound we see

$$\Big\| \int_t e^{it\lambda} \sigma_j^{-1/2} \widehat{\eta} dt \Big\|_{L^2_{\mathbf{k}}}^2 \le C \|\eta\|^2_{L^p_t\left(B^{\alpha_j/2}_{r,2}\right)}$$

so long as $p = \frac{2}{2-(\frac{5}{6}(\frac{2}{r}-1))} = \frac{12r}{17r-10}$. Choosing $p = r$ then $r = \frac{22}{17}$ and so

$$\Big\| \int_{-\infty}^{\infty} e^{it\lambda} \sigma_j^{-1/2} \chi \widehat{\phi}_n \widehat{\eta}(t) dt \Big\|_{L^2_{\mathbf{k}}} \le C \|\eta\|_{L_t^{22/17}\left(B^{\mu_j}_{22/17,2}\right)} \tag{4.4}$$

where $\mu_j = \{\frac{9}{44}, -\frac{1}{22}, \frac{5}{11}\}$, respectively.

We return now to the adjoint estimate (4.2) with our new tool (4.4):

$$\begin{aligned} \big| \langle S_j(t) g, \eta \rangle_{L^2(\mathbf{R}^{3+1})} \big| &\le C \|g\|_{\dot{H}^{\beta_j}} \Big\| \int_t e^{it\lambda(\mathbf{k})} \sigma_j^{-1/2} \widehat{\widetilde{\eta}} dt \Big\|_{L^2_{\mathbf{k}}} \\ &\le C \|g\|_{\dot{H}^{\beta_j}} \|\eta\|_{L_t^{22/17}\left(B^{\mu_j}_{22/17,2}\right)} \end{aligned}$$

and by duality

$$\|S_j g\|_{L^{22/5}\left(B^{-\mu_j}_{22/5,2}\right)} \le C \|g\|_{H^{\beta_j}}$$

where $\beta_j = \{0, -1/2, 1/2\}$, respectively.

If we convolve our initial data g with $(1-\Delta)^{\nu/2}$ for some real number ν, then we find

$$\begin{aligned} \|S_j g\|_{L^{22/5}\left(B^{\nu-\mu_j}_{22/5,2}\right)} &\le C \big\| S_j (1-\Delta)^{\nu/2} g \big\|_{L^{22/5}\left(B^{-\mu_j}_{22/5,2}\right)} \\ &\le C \big\| (1-\Delta)^{\nu/2} g \big\|_{H^{\beta_j}} \le C \|g\|_{H^{\nu+\beta_j}} \end{aligned}$$

This implies that the linear operators S_j satisfy the following Strichartz estimates

$$\|S_j g\|_{L^{22/5}\left(B^{\mu}_{22/5,2}\right)} \le C \|g\|_{H^{\mu+\gamma_j}}$$

for $\mu \in \mathbf{R}$ and $\gamma_j = \{\frac{9}{44}, -\frac{6}{11}, \frac{21}{22}\}$, respectively. This completes Theorem 1.2.

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