BIDUALS OF BANACH SPACES WITH BASES

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R. C. James [**2**] (or see p. 7ff of [**3**]) gave a useful representation of the bidual of any space with a shrinking basis. This note gives a representation of the bidual of any space with a basis.

Our notation follows that of [**3**], where undefined terms can be found. Let $\{e_n\}$ be a basic sequence with coefficient functionals $\{f_n\}$. We will assume $\{e_n\}$ is *bimonotone*; that is $\left\|\sum_N^M a_n e_n\right\| \leq \|\sum a_n e_n\|$. The space $\{e_n\}^{\text{LIM}}$ is the set of scalar sequences $\{a_n\}$ so that $\|\{a_n\}\| = \sup \left\|\sum_1^N a_n e_n\right\| < \infty$. We will abuse notation and equate such $\{a_n\}$ with the formal sum $\sum a_n e_n$. We have $[e_n]^*$ is $\{f_n\}^{\text{LIM}}$. The basic sequence $\{e_n\}$ is *boundedly complete* if $\{e_n\}^{\text{LIM}} = [e_n]$. The basic sequence $\{e_n\}$ is *shrinking* if $\{f_n\}$ is boundedly complete. James's result mentioned above is that for shrinking $\{e_n\}$, we have $[e_n]^{**} = \{e_n\}^{\text{LIM}}$.

A basic sequence $\{e_n\}$ is *wild* if there is $\{a_n\} \in \{e_n\}^{\text{LIM}}$ and $\{b_n\} \in \{f_n\}^{\text{LIM}}$ so that $\left\{\sum_1^N a_n b_n\right\}_N$ does not converge. Define $Z(e_n)$ to be the space $\{e_n\}^{\text{LIM}}/[e_n]$ and let $Z^*(e_n)$ be $\phi^*(Z(e_n)^*)$, where ϕ is the induced quotient map.

If $\mathcal{U}$ is a free ultrafilter on N (i.e. $\bigcap \mathcal{U} = \emptyset$) and $\{a_n\}$ is a bounded scalar sequence, then $\lim_{\mathcal{U}} a_n = L$ means that $\{n \in N : |a_n - L| < \varepsilon\} \in \mathcal{U}$ for all $\varepsilon > 0$. Note that this limit must exist and there is a subsequence $\{n(i)\}$ such that $a_{n(i)} \to L$. Conversely, if $a_{n(i)} \to L$, then there is a free ultrafilter $\mathcal{U}$ such that $\lim a_n = L$. Note that by using an ultrafilter instead of a subsequence we get convergence for all bounded $\{a_n\}$. Define $\sum_{\mathcal{U}} a_n$ to be $\lim_{\mathcal{U}} \sum_1^n a_i$.

THEOREM. *Let $\mathcal{U}$ be a free ultrafilter on N. Define $T_{\mathcal{U}} : \{e_n\}^{\text{LIM}} \to [e_n]^{**} = (\{f_n\}^{\text{LIM}})^*$ by $\langle T_{\mathcal{U}}\{a_n\}, \{b_n\}\rangle = \sum_{\mathcal{U}} a_n b_n$. Let $P_{\mathcal{U}} : [e_n]^{**} \to [e_n]^{**}$ be defined by $P_{\mathcal{U}} x^{**} = T_{\mathcal{U}}\{x^{**}(f_n)\}$. Then $T_{\mathcal{U}}$ is an isometry, $\|P_{\mathcal{U}}\| = \|I - P_{\mathcal{U}}\| = 1$ and $[e_n]^{**} = T_{\mathcal{U}}(\{e_n\}^{\text{LIM}}) \oplus Z^*(f_n)$.*

Proof. Since $\left|\sum_1^N a_n b_n\right| \leq \left\|\sum_1^N a_n e_n\right\| \cdot \left\|\sum_1^N b_n f_n\right\| \leq \|\{a_n\}\| \cdot \|\{b_n\}\|$, $\sum_{\mathcal{U}} a_n b_n$ is well-defined. Thus $T_{\mathcal{U}}\{a_n\}$ is a linear functional on $\{f_n\}^{\text{LIM}}$ and $\|T_{\mathcal{U}}\{a_n\}\| \leq \|\{a_n\}\|$. Actually the norms are equal since if $b_n = 0$ for $n \geq N$, then $\sum_{\mathcal{U}} a_n b_n = \sum_1^N a_n b_n$ and since $\{e_n\}^{\text{LIM}}$ is normed by $[f_n]$.

If $x^{**} \in [e_n]^{**}$ and $a_n = x^{**}(f_n)$, then $\left\|\sum_1^n a_i e_i\right\|$ is equal to the norm of x^{**} restricted to

$[f_i]_1^n$ which is $\leq \|x^{**}\|$. Hence $P_{\mathcal{U}}$ is a norm one projection onto $T_{\mathcal{U}}(\{e_n\}^{\mathrm{LIM}})$ with kernel $[f_n]^{\perp} = Z^*(f_n)$. Suppose that $z^* \in Z^*(f_n)$ with $\|z^*\| = 1$ and $\{a_n\} \in \{e_n\}^{\mathrm{LIM}}$; let $x^{**} = T_{\mathcal{U}}\{a_n\} + z^*$. Let $z \in Z(f_n)$ so that $\|z\| = 1$ and $z^*(z) > 1 - \varepsilon$. Pick $\{b_n\} \in \{f_n\}^{\mathrm{LIM}}$; so that $\|\sum b_n f_n\| < 1 + \varepsilon$ and $\phi(\sum b_n f_n) = z$. Choose N so that $\left|\sum_1^N a_n b_n - \sum_{\mathcal{U}} a_n b_n\right| < \varepsilon$ and thus $\left|\langle T_{\mathcal{U}}\{a_n\}, \sum_{N+1}^{\infty} b_n f_n\rangle\right| < \varepsilon$. Now $\phi\left(\sum_{N+1}^{\infty} b_n f_n\right) = z$ and the bimonotoneness condition implies $\left\|\sum_{N+1}^{\infty} b_n f_n\right\| < 1 + \varepsilon$. Hence $\|x^{**}\| > (1 - 2\varepsilon)/(1 + \varepsilon)$ and $\|I - P_{\mathcal{U}}\| = 1$.

REMARKS. 1. Note that in general $\{e_n\}^{\mathrm{LIM}}$ is just a quotient of $[e_n]^{**}$, and since there are wild bases (see below) something like this ultrafilter sum is needed.

2. This is similar to embedding $[e_n]^{**}$ in the nonstandard hull of $[e_n]$ and is where the author obtained the original proof.

3. If $\mathcal{U}(\alpha)$, $\alpha \in \Gamma$, are different ultrafilters so that $\sum_{\mathcal{U}(\alpha)} a_n b_n$, $\alpha \in \Gamma$, are all distinct for some fixed $\{a_n\} \in \{e_n\}^{\mathrm{LIM}}$ and $\{b_n\} \in \{f_n\}^{\mathrm{LIM}}$, then for non-negative scalars t_α with $\sum_\alpha t_\alpha = 1$ we have $\left\|\sum_\alpha t_\alpha T_{\mathcal{U}(\alpha)}\{a_n\}\right\| = \|\{a_n\}\|$.

4. The Theorem immediately generalizes to spaces with a Schauder decomposition [**3**, p. 47ff] with all but finitely many of the factors being reflexive, but not infinitely many [**1**].

5. In light of [**3**, p. 26] or [**1**], $Z^*(f_n)$ could be almost anything. No representation theorem can say much about this space.

EXAMPLE. We now exhibit a space with a wild basis. Let X be a Banach space with a bimonotone basis $\{e_n\}$. Let $\{b_n\}$ be a bounded sequence and let F be the "formal" function $\sum b_n f_n$. Let $\|.\|_X$ be the norm on X and define

$$\left\|\sum a_n e_n\right\|_F = \max\left\{\left\|\sum a_n e_n\right\|_X, \sup\left\{\left|\sum_N^M a_n b_n\right| : N \leq M\right\}\right\}.$$

Let $Y = \{x \in X : \|x\|_F < \infty\}$ and note that $\{e_n\}$ is a bimonotone basis for Y and that F is a continuous linear functional on Y. Thus $X = Y$ if and only if $F \in \{f_n\}^{\mathrm{LIM}}$.

If X is c_0 and $b_n = (-1)^n$, then the resulting space Y still has $\sum e_n \in \{e_n\}^{\mathrm{LIM}}$. Hence the basis $\{e_n\}$ is wild in Y. Let the set of even integers belong to $\mathcal{U}$ and the set of odd integers belong to $\mathcal{U}'$. Then $(T_{\mathcal{U}} + T_{\mathcal{U}'})/2$ is an isometry of $\{e_n\}^{\mathrm{LIM}}$ into Y^{**} which is not $T_{\mathcal{V}}$ for any ultrafilter $\mathcal{V}$.

If instead we choose b_n to be the sequence $1, -1, 2^{-1}, 2^{-1}, -2^{-1}, -2^{-1}, 2^{-2}, 2^{-2}, 2^{-2}, 2^{-2}, -2^{-2}, -2^{-2}, -2^{-2}, -2^{-2}, \ldots$ then there are uncountably many ultrafilters $\mathcal{U}(\alpha)$ so that $T_{\mathcal{U}(\alpha)}(\sum e_n) \neq T_{\mathcal{U}(\beta)}(\sum e_n)$, for $\alpha \neq \beta$.

If $f \notin \{f_n\}^{\mathrm{LIM}}$, then the space Y always has a subspace isomorphic to c_0. However, since being a wild basis is self-dual, this isn't true of all wild bases.

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