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A Reformulation of Divine's Interplanetary Model

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1. Introduction

Divine (1993) developed a mathematical model to use measurements of interplanetary dust to determine the orbital distributions of particles in interplanetary space. The power of the model is that it uses the fact that the dust particles are in Keplerian orbits to correct for the observation biases based on spatial density and velocity effects of the orbits. In order to do this, he creates families of dust orbits; within each of which the particles have mathematically separable distributions of mass, periapsis, eccentricity, and inclination. He then uses a trial-and-error method to vary these distributions until an adequate fit is made to the data. Each of his distributions is loosely based on populations of interplanetary dust that are believed to be present in the Solar System.

2. Phase Space Density

Our investigations of Divine's model indicate a discrepancy with his distributions in orbital parameters. This problem hinges on Divine's definitions of his probability distributions. In Divine's paper, he gives a formula (his equation 2; without derivation) for the six-dimensional phase space density (for the purposes of this discussion, the distribution in mass is ignored) of a family of orbits with a distribution in periapsis (N_1), eccentricity (p_e), and inclination (p_i);

$$(1) \qquad g_0 = \frac{1}{2\pi e}\left(\frac{r_1}{GM_0}\right)^{3/2} N_1\, p_e\, p_i$$

When we attempted to adapt this relation to other models, it became clear that N_1, p_e, and p_i are not the "textbook" definitions of the distributions, e.g., p_i di is not the number of objects with inclinations between i and (i + di).

At this stage, it will be instructive to derive equation 1 with a more rigorous definition of the distributions in orbital parameters. To derive the phase space density for a distribution of orbiting objects, it is best to start with the phase space density for a single orbiting object (note that we have tried to use variable definitions similar to those Divine used in his paper);

$$
\begin{aligned}
\Phi_1(\vec{r},\vec{v}) &= \frac{1}{4}\sum_{i=1}^{4} \rho(\vec{r})\ \delta(\vec{v}-\vec{v}_i(\vec{r})) \\
&= \frac{1}{4}\sum_{j=1}^{4} \rho(\vec{r})\ \delta(v_x - v_{jx}(\vec{r}))\ \delta(v_y - v_{jy}(\vec{r}))\ \delta(v_z - v_{jz}(\vec{r}))
\end{aligned} \tag{2}
$$

where δ is the Dirac delta function, the r vector is the position, the v vector the velocity, and r is the spatial density given by

$$
\rho(\vec{r}) = \frac{(1-e)^{3/2}}{2\pi^3\, r\, r_1 \sqrt{(r-r_1)((1+e)r_1 + (1-e)r)\left(\cos^2\lambda - \cos^2 i\right)}} \tag{3}
$$

Equation 3 is the same as equation C4 in Divine's paper, where r_1 is the periapsis, e the eccentricity, and i the inclination of the orbit, r is the radius from the center of the attracting body, and λ is the latitude angle. The summing in equation 2 is necessary because for a Keplerian orbit with randomized ascending node and argument of periapsis there are four possible velocities (with equal likelihood) at any point in space that the orbiting object visits (Kessler 1981).

The phase space density in equation 2 is given in terms of the velocity variables. These can be transformed into Divine's orbital coordinates by using the following relation to transform the velocity delta functions

$$
\delta(\vec{x}-\vec{x}') = \left(\frac{\partial(x_i')}{\partial(y_i')}\right)^{-1} \delta(\vec{y}-\vec{y}') = J^{-1}\,\delta(\vec{y}-\vec{y}') \tag{4}
$$

J is the Jacobian for this transform and is given by Divine in his equation B8 (GM_0 is the gravitational constant) as

$$
J = \frac{(GM_0)^{3/2}\, e \sin i}{2r\sqrt{r_1\,(r-r_1)[(1+e)r_1 - (1-e)r][\cos^2\lambda - \cos^2 i]}} \tag{5}
$$

The phase space density of a single orbiting object in orbital variables is, then

$$\begin{aligned}\Phi_1(\vec{r}, r_1, e, i) &= J^{-1}\,\rho(\vec{r})\,\delta(r_1 - r_1')\,\delta(e - e')\,\delta(i - i') \\ &= \frac{(1-e)^{3/2}}{\pi^3\, e\sqrt{r_1}\,(GM_0)^{3/2}\sin i}\,\delta(r_1 - r_1')\,\delta(e - e')\,\delta(i - i')\end{aligned} \tag{6}$$

Note that after this transformation process the summing is no longer necessary, because each of the four terms in the summation returns the same relation as given in equation 6, so the summation cancels the value of 1/4.

3. Addition of Orbital Parameter Distributions

Equation 6 may now be integrated over a "textbook" distribution of orbital parameters given by D_1, D_e, and D_i. $D_1\,dr_1$ is the number of objects having periapsis between r_1 and $(r_1 + dr_1)$. $D_e\,de$ is the number of objects having eccentricities between e and (e + de), and $D_i\,di$ is the number of objects having inclinations between i and (i + di). They are normalized by

$$\int_0^\infty dr_1\, D_1(r_1) = 1, \qquad \int_0^1 de\, D_e(e) = 1, \qquad \int_0^\pi di\, D_i(i) = 1 \tag{7}$$

By integrating equation 6 over these distributions, we arrive at the correct form of the phase space density Φ_0 for a family of orbiting objects

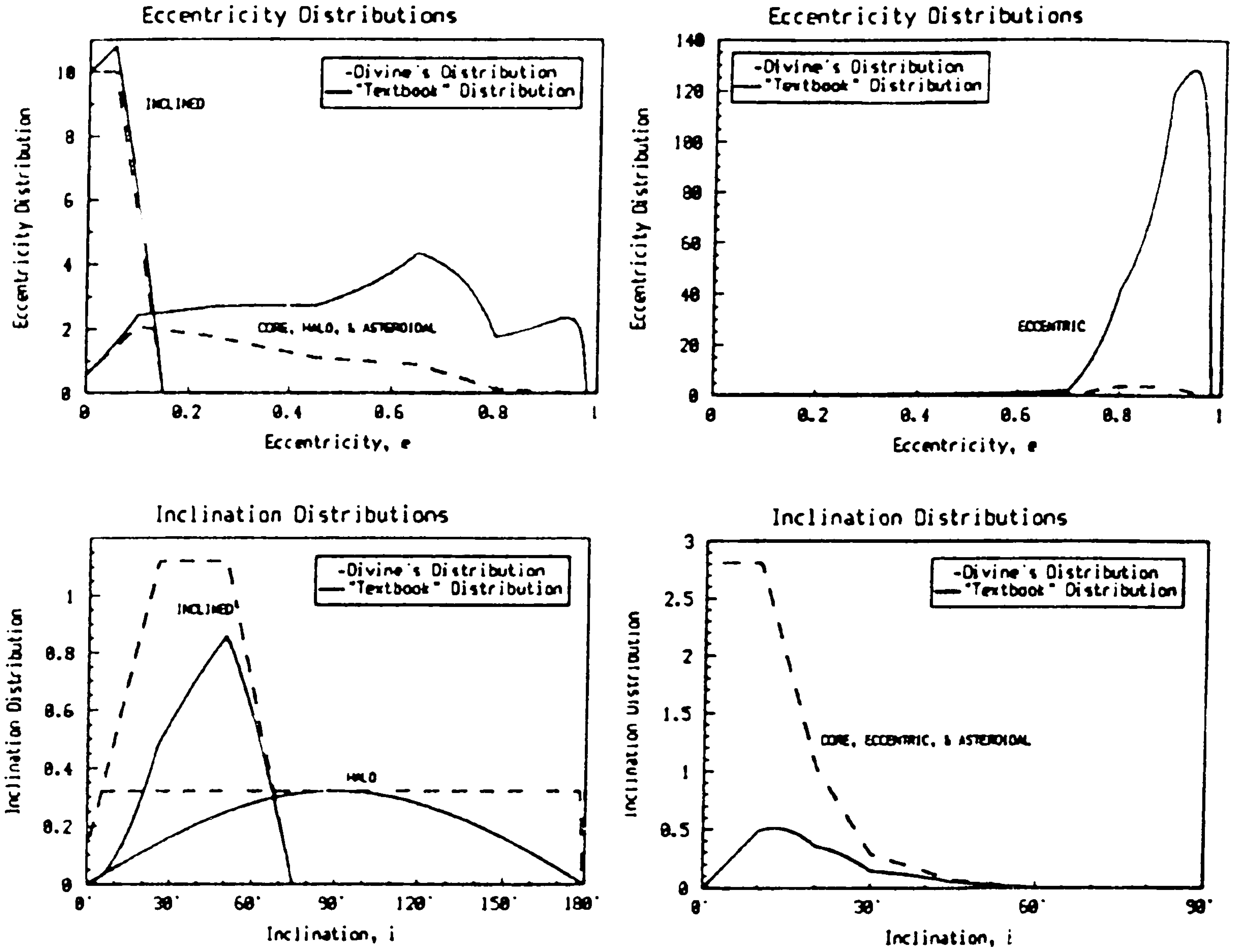
$$\begin{aligned}\Phi_0 &= \int_0^\infty dr_1'\, D_1(r_1') \int_0^1 de'\, D_e(e') \int_0^\pi di'\, D_i(i')\; \Phi_1(\vec{r}, r_1', e', i') \\ &= \frac{(1-e)^{3/2}\, D_1(r_1)\, D_e(e)\, D_i(i)}{\pi^3\, e\sqrt{r_1}\,(GM_0)^{3/2}\sin i}\end{aligned} \tag{8}$$

This equation can be used to replace equation 1 (Divine's equation 2). The three distributions are normalized to unity in equation 7, but to represent the total number of orbiting objects in a family, a mass distribution term can be added to equation 8 (as is done in Divine's paper) and the total number of objects included in it.

As long as Divine's distributions are treated as internal functions to be used to fit measured data and extrapolate to measurements elsewhere, they are internally consistent. The problem comes when any type of physical interpretation of the orbital distributions of the dust families is required; for instance, if an attempt is made to match Divine's distributions to actual sources. In such a case, the "textbook" distributions are needed. The relationship between Divine's distributions and the "textbook" ones can be derived by comparing equation 8 with equation 1,

$$(9) \qquad N_1 = \frac{D_1}{r_1^{\,2}}, \qquad p_i = \frac{D_i}{\sin i}, \qquad p_e = (1-e)^{3/2} D_e$$

The same method outlined above can be used to derive the spatial density of any combination of orbital elements, but it should be noted that different orbital elements (such as apoapsis instead of eccentricity) could give very different results.



Figures: Divine's internal distributions from his paper (Divine 1993) are compared to the true "textbook" distributions they represent as given in equation 9. The axis definitions and units are those used by Divine. While Divine's distributions are internally consistent, any attempt to relate them to the physical distributions and numbers in space requires the use of these "textbook" distributions. Divine's distributions were normalized, so the "textbook" transformations, in general, will not be, and could lead to a misinterpretation of the total number of meteoroids predicted in the environment by his formulae. Note that the "textbook" inclination distribution of the "Halo" family is a sine curve, as would be expected from a spatial density that is independent of latitude.

References

Divine, N. 1993, *JGR*, 98, E9, 17029

Kessler, D. J. 1981, *Icarus*, 48, 39