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CONCENTRATION-CANCELLATION AND HARDY SPACES

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Abstract

Let v^ϵ be a sequence of DiPerna-Majda approximate solutions to the 2-d incompressible Euler equations. We prove that if the vorticity sequence is weakly compact in the Hardy space $H^1(R^2)$ then a subsequence of v^ϵ converges strongly in the energy norm to a solution of the Euler equations.

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The convergence of approximate solutions v^ϵ of the 2-d inviscid Euler equations as the regularization parameter ϵ goes to zero has been studied by DiPerna and Majda in a series of papers ([4, 5, 6]). In [4, 5] they give several examples of sequences of compactly supported approximate solutions v^ϵ (as defined in [4, Definition 1.1]) whose vorticity ω^ϵ is bounded in L^1 which fail to be compact in L^2 so that in the limit concentration phenomena occur. Moreover in [4, Theorem 1.3] a criterion which rules concentrations out is proposed: it is shown that a uniform bound on a logarithmic Morrey norm of ω^ϵ yields strong L^2-convergence of the velocity field.

In this note a different criterion for compactness is introduced: we show that strong L^2-compactness of v^ϵ follows from weak compactness of ω^ϵ in the Hardy space $H^1(R^2)$. Since $H^1(R^2)$ is not rearrangement invariant the fine structure of the vorticity plays a crucial role in getting strong L^2-convergence. We recall that by the Dunford-Pettis theorem (see [7, VIII, Theorem 1.3]) a necessary and sufficient condition for a subset Λ of $L^1(R^2)$ to be weakly pre-compact in $L^1(\Omega)$ is that

$$\lim_{s\to+\infty}\left(\sup_{f\in\Lambda}\int_{|x|>s}|f|\,dx\right)=0 \tag{1}$$

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and that there exists a positive function $G(s) : R^+ \to R^+$ such that

$$\lim_{s\to+\infty} G(s)/s = +\infty$$

and

(2) $$\sup_{f\in\Lambda} \int_{R^2} G(|f|)\,dx < +\infty.$$

Let R_i, $i = 1, 2$, denote the Riesz transforms:

$$R_j f(x) = \int_{R^2} \frac{x_j - y_j}{|x-y|^3} f(y)\,dx.$$

We adopt both notation and terminology of [4]. We formulate our result as follows.

THEOREM 1. *Let v^ϵ be a sequence of approximate solutions such that $\omega^\epsilon \in H^1(R^2) \cap C_o^\infty(R^2)$ and that for $t \geq 0$*

(3) $$\|\omega^\epsilon(.,t)\|_{H^1} < C, \qquad 0 < \epsilon \leq \epsilon_0.$$

Moreover let there be a function $G(s) : R^+ \to R^+$ such that (1) *and* (2) *hold for $\Lambda = \{\omega^\epsilon\}, \{R_i\omega^\epsilon\}, i = 1, 2$.*

Then there is a subsequence of v^ϵ which converges strongly in L^2_{loc} to a weak classical solution v of the Euler equations. Moreover $v \in W^{1,1}(R^2)$.

We recall (see [8, Ch. XIV]) that a function f belongs to the Hardy space $H^1(R^2)$ if and only if there is a sequence of numbers λ_j satisfying $\sum_1^\infty |\lambda_j| < \infty$ and a series of functions (*atoms*) a_j such that

(4) $$f = \sum_1^\infty \lambda_j a_j$$

where the $a_j's$ have the following properties:

(a) a_j is supported on a ball B_j and $\|a_j\|_\infty < 1/|B_j|$;
(b) $\int_{R^2} a_j(x)\,dx = 0$.

The H^1-norm of f can be defined as the infimum of the expressions $\sum_1^\infty |\lambda_j|$ on all possible representations of f as in (4).

If Condition (b) were dropped the resulting space would be $L^1(R^2)$. The subtle cancellation effect due to (b) (cf. 'phantom vortices' in [4, 1.A]) is crucial in obtaining strong L^2-compactness.

PROOF OF THEOREM 1. To prove the theorem we introduce the stream function ψ^ϵ such that

(5) $$\Delta\psi^\epsilon = \omega^\epsilon$$

and we proceed as in the proof of [4, Theorem 3.1]. It is known (see [8, Chapter XV]) that for every f in $BMO(R^2)$ there are g_i in $L^\infty(R^2)$, $i = 0, 1, 2$, such that

$$f = g_0 + \sum_{i=1,2} R_i g_i.$$

Hence

$$\int_{R^2} f\omega^\epsilon\, dx = \int_{R^2} \omega^\epsilon \Big(g_0 + \sum_{i=1,2} R_i g_i \Big)\, dx = \int_{R^2} \omega^\epsilon g_0 - \sum_{i=1,2} g_i R_i \omega^\epsilon\, dx.$$

By our Assumption (2) the sequence $\{\omega^\epsilon\}$ and its Riesz transforms admit a weakly convergent subsequence in $L^1(R^2)$. Therefore there is a subsequence such that

$$\omega^\epsilon \rightharpoonup \omega \qquad \text{weakly in } H^1(R^2). \tag{6}$$

The statement of Theorem 1 is guaranteed (see [4, (3.6)]) by showing that for all $\rho \in C_o^\infty(R^2)$

$$\lim_{\epsilon\to 0} \int_{R^2} \rho |v^\epsilon|^2\, dx = \int_{R^2} \rho |v|^2 dx. \tag{7}$$

Indeed after integrating by parts (7) is seen to hold if and only if (see [4, (3.7)–(3.10)])

$$\lim_{\epsilon\to 0} \int_{R^2} \rho \psi^\epsilon \omega^\epsilon\, dx = \int_{R^2} \rho \psi\, d\omega \tag{8}$$

where ψ is the stream function corresponding to ω in (5).

We recall that

$$\frac{\partial^2}{\partial x_j \partial x_k} f = -R_j R_k \Delta f.$$

Hence

$$\frac{\partial^2}{\partial x_j \partial x_k} \psi^\epsilon = -R_j R_k \omega^\epsilon.$$

Since the Riesz transform maps $H^1(R^2)$ continuously into itself we obtain

$$\left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\epsilon \right\|_{L^1} \le \left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\epsilon \right\|_{H^1} \le C \|\omega^\epsilon\|_{H^1} \tag{9}$$

and ψ^ϵ remains bounded in $W^{2,1}(R^2)$.

We recall that for any bounded domain Ω in R^2, by the Gagliardo-Sobolev imbedding theorem, $W^{2,1}(R^2)$ is continuously imbedded in $C(\overline{\Omega})$.

Therefore by (9),

$$\|\psi^\epsilon\|_{C(\overline{\Omega})} \le C \|\omega^\epsilon\|_{H^1}. \tag{10}$$

Moreover (see [1, Lemma 5.8]) if $u \in W^{2,1}(R^2)$ for any $P_o \in R^2$ we have that for $\delta > 0$, and $|\Delta P| < \delta/2$

$$(11)\ |u(P_o+\Delta P)-u(P_o)| \le C\Bigg(\frac{1}{\delta^2}\|u(P+\Delta P)-u(P)\|_{L^1(B_\delta(P_o))} + \frac{1}{\delta}\sum_i \left\| \frac{\partial}{\partial x_i}u(P+\Delta P) - \frac{\partial}{\partial x_i}u(P)\right\|_{L^1(B_\delta(P_o))} + \sum_{i,j}\left\|\frac{\partial^2}{\partial x_j \partial x_i}u(P+\Delta P) - \frac{\partial^2}{\partial x_j \partial x_i}u(P)\right\|_{L^1(B_\delta(P_o))}\Bigg).$$

By (weak) continuity of the Riesz transforms from $H^1(R^2)$ into itself there is a subsequence of $\partial^2\psi^\epsilon/\partial x_j \partial x_k$ that converges weakly in $H^1(R^2)$ to a $\phi_{i,j} \in H^1(R^2)$. On the other hand weak convergence in $H^1(R^2)$ implies weak convergence in $L^1(\Omega)$ (indeed $L^\infty \subset BMO$) so that we have

$$\frac{\partial^2}{\partial x_j \partial x_k}\psi^\epsilon \rightharpoonup \phi_{i,j} \qquad \text{weakly in } L^1(R^2).$$

By the full version of the Dunford-Pettis Theorem, for every $\kappa > 0$ there is a $\delta > 0$ such that for any $P \in \Omega$

$$\left\|\frac{\partial^2}{\partial x_j \partial x_k}\psi^\epsilon\right\|_{L^1(B_\delta(P))} < \kappa$$

uniformly in ϵ.

We observe that if $|\Delta P| < \delta$,

$$\left\|\frac{\partial^2}{\partial x_j \partial x_k}[\psi^\epsilon(P+\Delta P) - \psi^\epsilon(P)]\right\|_{L^1(B_\delta(P_o))} < C\left\|\frac{\partial^2}{\partial x_j \partial x_k}\psi^\epsilon\right\|_{L^1(B_{2\delta}(P_o))}.$$

Therefore for every P_o, given $\kappa_o > 0$ we can find a $\delta_o > 0$ such that if $|\Delta P| < \delta_o/2$

$$\sum_{j,k}\left\|\frac{\partial^2}{\partial x_j \partial x_k}[\psi^\epsilon(P+\Delta P) - \psi^\epsilon(P)]\right\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}$$

uniformly in ϵ. Moreover since $W^{1,1}(\Omega)$ is compactly imbedded in L^p for any $p < 2$, by (9) both $\{\psi^\epsilon\}$ and $\{\partial\psi^\epsilon/\partial x_i\}$ are compact in L^1. Hence by the Kondratchev compactness criterion (see [1]) there is a $\delta_1 > 0$ such that if $|\Delta P| < \delta_1$,

$$\frac{1}{\delta_o{}^2}\|u(P+\Delta P) - u(P)\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3},$$

$$\frac{1}{\delta_o}\sum_i\left\|\frac{\partial}{\partial x_i}u(P+\Delta P) - \frac{\partial}{\partial x_i}u(P)\right\|_{L^1(B_\delta(P_o))} < \frac{\kappa_o}{3},$$

and by (11)

$$|\psi^\epsilon(P_o + \Delta P) - u(P_o)| < \kappa_o \tag{12}$$

uniformly in ϵ. The sequence ψ^ϵ is equicontinuous by (12) and by Ascoli's theorem we can extract a subsequence such that

$$\psi^\epsilon \to \psi \qquad \text{strongly in } C(\Omega) \tag{13}$$

By (6) we have that $\omega^\epsilon \rightharpoonup \omega$ weakly in the space of Radon measures $M(\Omega)$ so that (8) holds and the same argument as in [4, Theorem 1.3] yields the statement of the theorem.

REMARK. The first example in [4, (1, §A)] (phantom vortices) shows a sequence of vorticities which stays bounded in $H^1(R^2)$ whose velocity field fails to converge strongly in L^2; in the second example one has strong $L^1(\Omega)$ convergence of the vorticity but the sequence does not lie in $H^1(R^2)$ and again concentrations occur. By looking at the proof of Delort's recent deep result [3] a condition weaker than weak convergence of ω^ϵ in $L^1(\Omega)$ is sufficient to pass to the limit in the quadratic terms of the Euler equations, due to their special structure, although concentrations may occur. It is interesting that every bounded sequence in $H^1(R^2)$ admits a weakly $(*)$ convergent subsequence whose limit stays in $H^1(R^2)$ (see [2, Lemma (4.2)]). However, since $(VMO)^* = H^1(R^2)$ and $L^\infty \not\subset VMO$, this does not yield weak L^1-convergence.

It is worth observing that Condition (2) for ω^ϵ is rearrangement invariant and so in the time dependent case it is conserved by the particle trajectory map. On the other hand, as for the bounds of [4, Theorem 3.1, (3.4)], it is not clear what happens to the H^1-norm as time goes by, since $H^1(R^2)$ is not rearrangement invariant. It would be interesting to have a less cumbersome characterization of weak compactness in H^1 than the one given here.

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