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Explicit construction of graph invariant for strongly pseudoconvex compact 3-dimensional rational CR manifolds

HING SUN LUK[1] ⋆ and STEPHEN S.T. YAU[2] ⋆⋆

[1] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong; e-mail: hsluk@math.cuhk.edu.hk*

[2] *Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, M/C 249, 851 S. Morgan St., Chicago, IL 60607-7045, USA; e-mail: yau@uic.edu*

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Abstract. Let X be a strongly pseudoconvex compact 3-dimensional CR manifolds which bounds a complex variety with isolated singularities in some $\mathbb{C}^N$. The weighted dual graph of the exceptional set of the minimal good resolution of V is a CR invariant of X; in case X has a tranversal holomorphic S^1 action, we show that it is a complete topological invariant of except for two special cases. When X is a rational CR manifolds, we give explicit algebraic algorithms to compute the graph invariant in terms of the ring structure of $\bigoplus_{k=0}^{\infty} m^k/m^{k+1}$, where m is the maximal ideal of each singularity. An example is computed explicitly to demonstrate how the algorithms work.

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1. Introduction

In view of an example of Webster [We], it is clear that the problem of studying when two given CR manifolds are analytically equivalent is extremely difficult. In a previous paper [LYY], we introduce the notion of algebraic equivalence relation among CR manifolds. Recall that any compact strongly pseudoconvex CR manifold X in $\mathbb{C}^N$ bounds a complex variety V in $\mathbb{C}^N$ with only isolated singularities at Y [Ha-La]. Let $\tilde{V}$ be the normalization of V.

DEFINITION 1.1. Let X_1,X_2 be two connected compact strongly pseudoconvex embeddable manifolds of dimension $2n-1$. We say that X_1 and X_2 are algebraically equivalent if the corresponding normal varieties $\tilde{V}_1$ and $\tilde{V}_2$, which are bounded by X_1 and X_2 respectively, have isomorphic singularities Y_1 and Y_2, i.e., $(\tilde{V}_1, Y_1) \cong (\tilde{V}_2, Y_2)$ as germs of varieties.

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It was observed that two analytically equivalent CR manifolds are automatically algebraically equivalent. In [LYY], we also introduced some numerical invariants under algebraic equivalence for connected compact strongly pseudoconvex embeddable CR manifolds of real dimension 3. In particular, the geometric genus $p_g(X)$ of the CR manifold X was introduced. A real 3-dimensional connected compact strongly pseudoconvex embeddable CR manifold is called a rational CR manifold if its geometric genus vanishes.

DEFINITION 1.2. In Definition 1.1, we say that X_1 and X_2 are topologically algebraic equivalent or have the same topology up to algebraic equivalence if $(\tilde{V}_1, Y_1) \cong (\tilde{V}_2, Y_2)$ topologically as germs of varieties.

Obviously, in order to understand the analytic classification problem of CR manifolds, a first step is to understand the classification problem of CR manifolds up to topologically algebraic equivalence. Then the second step is to understand the classification problem of CR manifolds up to algebraic equivalence. The purpose of this paper is to understand the first step. We shall only consider connected compact strongly pseudoconvex embeddable rational CR manifolds of real dimension 3 in this paper. Let X be such a CR manifold. In [LYY], we define the graph Γ_X to be the graph of the exceptional set of the minimal good resolution of the complex variety V whose boundary is X. It was shown that Γ_X is an invariant under algebraic equivalence. Let X_1, X_2 be two 3-dimensional CR manifolds. We have shown that $\Gamma_{X_1} = \Gamma_{X_2}$ implies that X_1 is topologically algebraic equivalent to X_2 (cf. Theorem 2.4 of [LYY]). The converse of the above statement is also true except for two explicit cases. Therefore it is important to compute Γ_X explicitly for the topologically algebraic equivalence problem. The main result of this paper is that we have developed explicit algorithms to compute Γ_X for any rational 3-dimensional CR manifolds without computing the resolution of the complex variety V. We would like to remark that for CR manifolds X_1, X_2 with transversal holomorphic S^1-action, X_1 is topologically algebraic equivalent to X_2 if and only if X_1 is topologically equivalent to X_2 in the usual sense. Hence for a 3-dimensional CR manifold X with transversal holomorphic S^1-action, Γ_X is basically a complete topological invariant.

In Section 2, we recall some basic notations and facts about CR manifolds. We show that for CR manifolds with transversal holomorphic S^1-action, topologically algebraic equivalence is the same as topological equivalence. In Section 3, we give explicit algebraic algorithms to compute the weighted dual graph Γ of the minimal good resolution of a rational two-dimensional singularity (V, p) without taking the minimal resolution of (V, p). In fact we show how to use the ring structure of $\bigoplus_{k=0}^{\infty} m^k/m^{k+1}$ to find Γ explicitly, where m is the maximal ideal of the singularity of V. In Section 4, we use the result in Section 3 to compute Γ_X explicitly for any connected compact strongly pseudoconvex embeddable rational CR manifold X of real dimension 3. An example is computed explicitly to demonstrate how the algorithms work in Section 2. In particular, we know how to construct explicitly Γ_X, which is a complete topological invariant except for two special cases, for any 3-dimensional connected compact embeddable CR manifold X with transversal holomorphic S^1-action.

2. Preliminary

In this section, we shall recall some basic notations and facts about CR manifolds that will be needed for later discussion. We also show that for 3-dimensional CR manifolds with transversal holomorphic S^1-action, topologically algebraic equivalence and topological equivalence are the same.

The following proposition is proved in [LYY].

PROPOSITION 2.1 [LYY]. *Let X_1 and X_2 be two strongly pseudoconvex compact connected* CR *manifolds in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. If X_1 is* CR *equivalent to X_2, then X_1 is algebraically equivalent to X_2.*

In 1974 Boutet de Monvel [Bo] (cf. [Ko] also) proved that if X is a compact C^∞ strongly pseudoconvex CR manifold of dimension $2n-1$ and $n \geqslant 3$, then X is CR embeddable in $\mathbb{C}^N$. H. Grauert has constructed compact 3-dimensional strongly pseudoconvex CR manifolds which are not embeddable. Such examples were also studied by H. Rossi [Ro] and D. Burns [Bu]. In this paper we shall only consider connected compact embeddable strongly pseudoconvex CR manifolds.

The following theorem is due to Lawson–Yau [La-Ya].

THEOREM 2.2 [La-Ya]. *Let X be a strongly pseudoconvex* CR *manifold of dimension $2n-1>1$ and suppose that X admits a transversal holomorphic S^1-action. Then there exists a holomorphic equivariant embedding $X \hookrightarrow V$ as a hypersurface in an n-dimensional algebraic variety $V \subset \mathbb{C}^N$ with a linear $\mathbb{C}^*$-action. V has at most one singular point at the origin.*

The following theorem illustrates why topologically algebraic equivalence is important.

THEOREM 2.3. *Let X_1, X_2 be strongly pseudoconvex* CR *manifolds of dimension $2n-1>1$ and suppose that X_1 and X_2 admit transversal holomorphic S^1-action. Then X_1 is topologically algebraic equivalent to X_2 if and only if X_1 is topologically equivalent to X_2.*

Proof. '$\Rightarrow$' In view of Theorem 2.2, there exist holomorphic equivariant embeddings $X_1 \hookrightarrow V_1$, $X_2 \hookrightarrow V_2$ as hypersurfaces in n-dimensional algebraic varieties $V_1 \subset \mathbb{C}^{N_1}$, $V_2 \subset \mathbb{C}^{N_2}$, with linear $\mathbb{C}^*$-actions. V_1 and V_2 each has at most one singular point at the origin. Since X_1 is topologically algebraic equivalent to X_2, there exists a homeomorphism $\varphi\colon (U_1, 0) \to (U_2, 0)$, where U_1 (respectively U_2) is an open neighborhood of 0 in $\tilde{V}_1$ = normalization of V_1 (respectively in $\tilde{V}_2 =$

normalization of V_2) so that ∂U_1 is homeomorphic to ∂U_2. Let $\pi_1: \tilde{V}_1 \to V_1$ and $\pi_2: \tilde{V}_2 \to V_2$ be the normalization maps. Then clearly $\partial(\pi_1(U_1))$ is homeomorphic to $\partial(\pi_2(U_2))$. As V_1 admits a linear $\mathbb{C}^*$-action, it is clear that X_1 is topologically equivalent to $\partial(\pi_1(U_1))$. Similarly, by following the $\mathbb{R}_+ (\subset \mathbb{C}^*)$ action, we see that X_2 is topologically equivalent to $\partial(\pi_2(U_2))$. So X_1 is topologically equivalent to X_2.

'$\Leftarrow$' Let S_{ε_1} (respectively S_{ε_2}) be a sphere of radius ε_1 (respectively ε_2) in $\mathbb{C}^{N_1}$ (respectively $\mathbb{C}^{N_2}$) with center at 0. By a result of Milnor [Mi], we know that $(V_1 \cap B_{\varepsilon_1}, 0)$ is homeomorphic to $(C(V_1 \cap S_{\varepsilon_1}), 0)$, where B_{ε_1} is the ball of radius ε_1 in $\mathbb{C}^{N_1}$ with center at 0, and $C(V_1 \cap S_{\varepsilon_1})$ denotes the cone of $V_1 \cap S_{\varepsilon_1}$ with vertex at 0. Similarly, $(V_2 \cap B_{\varepsilon_2}, 0)$ is homeomorphic to $C(V_2 \cap S_{\varepsilon_2}, 0)$. Since X_1 and X_2 admit transversal holomorphic S^1-action, we see that $V_1 \cap S_{\varepsilon_1}$ is homemorphic to X_1 and $V_2 \cap S_{\varepsilon_2}$ is homeomorphic to X_2. As X_1 is homeomorphic to X_2, it follows that $V_1 \cap S_{\varepsilon_1}$ is homeomorphic to $V_2 \cap S_{\varepsilon_2}$. Therefore $(V_1 \cap B_{\varepsilon_1}, 0)$ is homeomorphic to $(V_2 \cap B_{\varepsilon_2}, 0)$. This means that X_1 is topologically algebraic equivalent to X_2. □

DEFINITION 2.1. Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 which is embeddable in $\mathbb{C}^n$. Let V be the subvariety in $\mathbb{C}^n$ such that the boundary of V is X in the C^∞ sense. Then V has isolated singularities at $Y = \{p_1, \dots, p_m\}$. Let $\pi: M \to V$ be a resolution of singularities of V such that the exceptional set $A = \pi^{-1}(Y)$ has normal crossing, i.e. irreducible components A_i of A are nonsingular, they intersect transversely and no three meet at a point. The topological nature of the embedding of the exceptional set A in M is described by the weighted dual graph Γ_M. The vertices of Γ_M correspond to the A_i's. The edges of Γ_M connecting the vertices corresponding to A_i and A_j, $i \neq j$, correspond to the points of $A_i \cap A_j$. Finally, associated to each A_i is its genus g_i as a Riemann surface, and its weight $A_i \cdot A_i$, the topological self intersection number. Among all the resolutions of V such that the exceptional sets have normal crossings, there is a unique minimal one M_0, which is called the minimal good resolution. Any resolution M of V with normal crossing exceptional set is obtained by applying quadratic transformations successively on M_0. The graph Γ_X of the CR manifold X is defined to be Γ_{M_0}.

The following theorem was shown in [LYY].

THEOREM 2.4 [LYY]. *Let X_1 and X_2 be strongly pseudoconvex compact connected embeddable* CR *manifolds of dimension* 3. *Then*

(a) $\Gamma_{X_1} = \Gamma_{X_2}$ implies that X_1 is topologically algebraic equivalent to X_2.

(b) If X_1 is algebraically equivalent to X_2, then $\Gamma_{X_1} = \Gamma_{X_2}$.

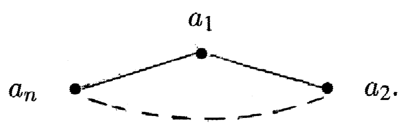
In fact, if X_1 is topologically algebraic equivalent to X_2, then $\Gamma_{X_1} = \Gamma_{X_2}$ except for the following two cases: Let

$$\Gamma_{X_1} = \bigoplus_{j=1}^{m} \Gamma_{X_1}^j \quad \textit{and} \quad \Gamma_{X_2} = \bigoplus_{j=1}^{m} \Gamma_{X_2}^j,$$

where $\Gamma_{X_1}^j$, $\Gamma_{X_2}^j$ *are connected graphs.*

Case (i) *Both* $\Gamma_{X_1}^j$ *and* $\Gamma_{X_2}^j$ *are exactly those of the form below with all* a_i *equal to or smaller than* -2. *The genus of each vertex is zero.*

Case (ii) *Both* $\Gamma_{X_1}^j$ *and* $\Gamma_{X_2}^j$ *are exactly those of the form below with all* a_i *equal to or smaller than* -2 *and one* a_i *equal to or smaller than* -3. *The genus of each vertex is zero.*



DEFINITION 2.2. Let X be a connected compact strongly pseudoconvex embeddable CR manifold of real dimension 3. With the notation in Definition 2.1, the geometric genus of X, $p_g(X)$, is defined to be $\dim H^1(M, \mathcal{O})$.

PROPOSITION 2.5. *Let* X *be a connected compact strongly pseudoconvex* CR *manifold of real dimension* 3 *which is embeddable in* $\mathbb{C}^n$. *Let* V *be the normal variety such that the boundary of* V *is* X *and* V *has isolated singularities at* $Y = \{p_1, \ldots, p_m\}$. *Let* $\pi\colon M \to V$ *be a resolution of singularities of* V. *Let* U_i *be a strongly pseudoconvex neighborhood of* p_i, $1 \leqslant i \leqslant m$, *such that the* U_i*'s pairwise disjoint. Then*

$$\begin{aligned} p_g(X) &= \sum_{i=1}^{m} \dim H^1(\pi^{-1}(U_i), \mathcal{O}) \\ &= \sum_{i=1}^{m} \dim \Gamma(U_i - \{p_i\}, \Omega^2)/L^2(U_i - \{p_i\}, \Omega^2), \end{aligned}$$

where $L^2(U_i - \{p_i\}, \Omega^2)$ *denotes the space of holomorphic* 2*-forms on* $U_i - \{p_i\}$ *which are* L^2*-integrable and* $\Gamma(U_i - \{p_i\}, \Omega^2)$ *is the space of holomorphic* 2*-forms on* $U_i - \{p_i\}$.

Proof. It follows from Lemma 5.3 of [La1] and the main result of [La2]. □

DEFINITION 2.3. Let (V, p) be a two-dimensional irreducible isolated singularity. Let $\pi\colon M \to V$ be a resolution of singularity. The geometric genus of the singularity (V, p), $p_g(V, p)$, is by definition equal to dim $H^1(M, \mathcal{O})$. (V, p) is a rational singularity if its geometric genus vanishes.

DEFINITION 2.4. A connected compact strongly pseudoconvex CR manifold is called a rational CR manifold if $p_g(X)$ vanishes.

In view of Proposition 2.5, it is clear that rational CR manifolds can bound varieties with only rational singularities.

3. Explicit determination of the graphs of rational singularities

In this section, we shall develop explicit algorithms which allow us to determine the weighted dual graphs of minimal resolutions of rational singularities. Let (V, p) be a rational singularity. Let m be the maximal ideal of the local ring $\mathcal{O}_{V,p}$. We shall show that the ring structure of the graded ring $\bigoplus_{k=0}^{\infty} m^k/m^{k+1}$ determines the weighted dual graph explicitly.

DEFINITION 3.1. Let A be the exceptional set in the resolution $\pi: M \to V$ of a normal 2-dimensional singularity p. Suppose that the irreducible components A_i, $1 \leqslant i \leqslant n$, of A are nonsingular. The fundamental cycle Z of A is the minimal cycle $Z = \sum a_i A_i$ such that $Z \neq 0$ and $A_i \cdot Z \leqslant 0$ for all A_i.

It was shown by Artin [Ar] that Z exists and is unique.

PROPOSITION 3.1 [La2]. *Z may be computed as follows. Let $Z_1 = A_{i_0}$ for any A_{i_0}. Having defined $Z_j = \sum a_{ji} A_i$, if there exists an A_{i_j} such that $A_{i_j} \cdot Z_j > 0$, let $Z_{j+1} = Z_j + A_{i_j}$. If $A_i \cdot Z_\ell \leqslant 0$ for all A_i, then $Z = Z_\ell$.*

Proof. We prove by induction that $Z_j \leqslant Z$. This is true if $j = 1$. If $Z_j < Z$, since Z is minimal, there exists A_{i_j} such that $A_{i_j} \cdot Z_j > 0$. However $a_{ji_j} = a_{i_j}$ is impossible for $A_{i_j} \cdot Z \leqslant 0$. Thus $a_{ji_j} = a_{i_j}$ would imply that $A_{i_j} \cdot Z_j \leqslant 0$ since $a_{j_i} \leqslant a_i$ for all i and $A_k \cdot A_\ell \geqslant 0$ if $k \neq \ell$. Hence $a_{ji_j} < a_{i_j}$ if $Z_j < Z$, so that $Z_{j+1} \leqslant Z$. □

DEFINITION 3.2. The sequence $Z_1 = A_{i_0}, Z_2 = Z_1 + A_{i_1}, \ldots, Z_\ell = Z_{\ell-1} + A_{i_{\ell-1}} = Z$ in Proposition 3.1 above is called the computation sequence of the fundamental cycle.

LEMMA 3.1. *Let $Z_1, Z_2, \ldots, Z_\ell = Z$ be the computation sequence of the fundamental cycle. Then $\Gamma(M, \mathcal{O}(-Z_i)/\mathcal{O}(-Z_{i+1})) = 0$, $\Gamma(M, \mathcal{O}_{Z_i}) \cong \mathbb{C}$ and $\Gamma(M, \mathcal{O}) \to \Gamma(M, \mathcal{O}_{Z_i})$ is surjective for all $1 \leqslant i \leqslant \ell - 1$.*

Proof. $\mathcal{O}(-Z_j)/\mathcal{O}(-Z_{j+1})$ represents the sheaf of germs of sections of a line bundle over A_j of Chern class $-A_{i_j} \cdot Z_j < 0$. Hence $\Gamma(M, \mathcal{O}(-Z_j))/\mathcal{O}(-Z_{j+1})) =$ 0 for all $1 \leqslant j \leqslant \ell - 1$. From the exact sheaf sequences

$$
\begin{gathered}
0 \to \mathcal{O}(-Z_1) \to \mathcal{O} \to \mathcal{O}_{Z_1} \to 0 \\
0 \to \mathcal{O}(-Z_1)/\mathcal{O}(-Z_2) \to \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_1} \to 0 \\
\vdots \\
0 \to \mathcal{O}(-Z_j)/\mathcal{O}(-Z_{j+1}) \to \mathcal{O}_{Z_{j+1}} \to \mathcal{O}_{Z_j} \to 0 \\
\vdots
\end{gathered}
$$

One sees inductively $\Gamma(M, \mathcal{O}_{Z_j}) \cong \mathbb{C}$ for $1 \leqslant j \leqslant \ell - 1$. It follows that $\Gamma(M, \mathcal{O}) \to \Gamma(M, \mathcal{O}_{Z_j})$ is surjective for all $1 \leqslant j \leqslant \ell - 1$. □

THEOREM 3.2 (Laufer [La2]). *Let Z be the fundamental cycle of a resolution of p. Then p is a rational singularity if and only if all the A_i have genus 0 and $A_{i_j} \cdot Z_j = 1$ for all Z_j in the computation of Z described in Proposition* 3.1.

Proof. Suppose p is a rational singularity. From the exact sheaf sequence

$$0 \to \mathcal{O}(-A_i) \to \mathcal{O} \to \mathcal{O}_{A_i} \to 0 \tag{3.1}$$

we get the following cohomology exact sequence

$$H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_{A_i}) \to H^2(M, \mathcal{O}(-A_i)), \tag{3.2}$$

where M denotes a neighborhood of A such that $\pi(M)$ is Stein. By the theorem of Siu ([Si]), $H^2(M, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on M. Since $H^1(M, \mathcal{O}) = 0$ also, (3.2) yields $H^1(M, \mathcal{O}_{A_i}) = 0$ which implies that the genus of A_i is equal to zero.

The exact sheaf sequence

$$0 \to \mathcal{O}(-Z_1) \to \mathcal{O} \to \mathcal{O}_{Z_1} \to 0$$

yields

$$
\begin{aligned}
0 &\to \Gamma(M, \mathcal{O}(-Z_1)) \to \Gamma(M, \mathcal{O}) \xrightarrow{\tau} \Gamma(M, \mathcal{O}_{Z_1}) \\
&\to H^1(M, \mathcal{O}(-Z_1)) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_{Z_1}) \to 0.
\end{aligned}
$$

τ is onto by Lemma 3.1. Since $H^1(M, \mathcal{O}) = 0$, $H^1(M, \mathcal{O}(-Z_1)) = H^1(M, \mathcal{O})$. Consider the exact sheaf sequences

$$
\begin{gathered}
0 \to \mathcal{O}(-Z_2) \to \mathcal{O}(-Z_1) \to \mathcal{O}(-Z_1)/\mathcal{O}(-Z_2) \to 0 \\
0 \to \mathcal{O}(-Z_3) \to \mathcal{O}(-Z_2) \to \mathcal{O}(-Z_2)/\mathcal{O}(-Z_3) \to 0 \\
\vdots \\
0 \to \mathcal{O}(-Z_{k+1}) \to \mathcal{O}(-Z_k) \to \mathcal{O}(-Z_k)/\mathcal{O}(-Z_{k+1}) \to 0 \\
\vdots
\end{gathered}
\tag{3.3}
$$

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Let k be the least j such that $A_{i_j} \cdot Z_j > 1$, i.e. $A_{i_j} \cdot Z_j = 1$ for $1 \leqslant j \leqslant k-1$ and $A_{i_k} \cdot Z_k > 1$. Recall that $\mathcal{O}(-Z_j)/\mathcal{O}(-Z_{j+1})$ represents the sheaf of germs of sections of a line bundle over A_{i_j} of Chern class $-A_{i_j} \cdot Z_j$, which is -1 for $j \leqslant k-1$. Hence $\Gamma(M, \mathcal{O}(-Z_j)/\mathcal{O}(-Z_{j+1})) = 0 = H^1(M, \mathcal{O}(-Z_j)/\mathcal{O}(-Z_{j+1}))$ for $j \leqslant k-1$. Thus $H^1(M, \mathcal{O}) \cong H^1(M, \mathcal{O}(-Z_1)) \cong H^1(M, \mathcal{O}(-Z_2)) \cong \cdots \cong H^1(M, \mathcal{O}(-Z_k))$. But at the next exact sequence

$$\to H^1(M, \mathcal{O}(-Z_k)) \to H^1(M, \mathcal{O}(-Z_k)/\mathcal{O}(-Z_{k+1})) \to 0,$$

we have $-A_{i_k} \cdot Z_k \leqslant -2$ so that $H^1(M, \mathcal{O}(-Z_k)/\mathcal{O}(-Z_{k+1})) \neq 0$. Then $H^1(M, \mathcal{O}(-Z_k))$ is mapped onto a nontrivial group and hence $H^1(M, \mathcal{O}) \neq 0$, a contradiction.

Conversely if $A_{i_j} \cdot Z_j = 1$ for all j, the above calculation shows that the map $H^1(M, \mathcal{O}(-Z)) \to H^1(M, \mathcal{O})$ is surjective (in fact an isomorphism). Consider the exact sequences

$$0 \to \mathcal{O}(-Z - Z_{j+1}) \to \mathcal{O}(-Z - Z_j) \to \mathcal{O}(-Z - Z_j)/\mathcal{O}(-Z - Z_{j+1}) \to 0$$

which just continue the sequences listed in (3.3). $\mathcal{O}(-Z - Z_j)/\mathcal{O}(-Z - Z_{j+1})$ represents the sheaf of germs of sections of a line bundle over A_{i_j} of Chern class $-A_{i_j}(Z + Z_j) \geqslant -1$. Hence $H^1(M, \mathcal{O}(-Z - Z_j)/\mathcal{O}(-Z - Z_{j+1})) = 0$ so the map $H^1(M, \mathcal{O}(-Z - Z_j)) \to H^1(M, \mathcal{O})$ is surjective. Continue the argument. We have that the map $H^1(M, \mathcal{O}(-nZ)) \to H^1(M, \mathcal{O})$ is surjective for all n. Hence by [Gr, Sect. 4, Satz 1, p. 355], $H^1(M, \mathcal{O}) = 0$ and p is a rational singularity. $\square$

The following Proposition follows from Lemma 3.1

PROPOSITION 3.3. *Let $\pi: M \to V$ be a resolution of a normal singularity p. Let m_p be the ideal sheaf of p. Then $\pi_0^*(\mathcal{O}(-Z)) = m_p$.*

THEOREM 3.4 (Laufer, [La2]). *Let $\pi: M \to V$ be a resolution of the rational singularity $p \in V$ with V Stein. If F is a line bundle over M with $c_i(F) := c(F|_{A_i}) \geqslant 0$ for all A_i in $A = \pi^{-1}(p)$, then $H^1(M, \mathcal{O}(-Z_i)\mathcal{F}) = 0$ for all Z_i, $i \geqslant 0$, in the computation sequence of Z.*

Proof. As in the proof of Theorem 3.2, we consider the exact sequence

$$\begin{gathered} \vdots \\ 0 \to \mathcal{O}(-Z_{i+1})\mathcal{F} \to \mathcal{O}(-Z_i)\mathcal{F} \to \mathcal{O}(-Z_i)\mathcal{F}/\mathcal{O}(-Z_{i+1})\mathcal{F} \to 0 \\ \vdots \\ 0 \to \mathcal{O}(-Z - Z_{i+1})\mathcal{F} \to \mathcal{O}(-Z - Z_i)\mathcal{F} \to \mathcal{O}(-Z - Z_i)\mathcal{F}/\mathcal{O}(-Z - Z_{i+1})\mathcal{F} \to 0 \\ \vdots \end{gathered}$$

Since $c_i(F) \geqslant 0$ for all i and p is rational, by Theorem 3.2, the quotient sheaves always correspond to line bundles of Chern class at least -1. So, as in the proof of Theorem 3.2, $H^1(M, \mathcal{O}(-Z_i)\mathcal{F}) = 0$. □

THEOREM 3.5. *Let $D = \sum d_i A_i$ and $E = \sum e_i A_i$ be divisors formed from the irreducible components of $A \subset M$, the resolution of a rational singularity. If $A_i \cdot D \leqslant 0$ and $A_i \cdot E \leqslant 0$ for all i, then the canonical map*

$$\Gamma(A, \mathcal{O}(-D)) \otimes_{\mathbb{C}} \Gamma(A, \mathcal{O}(-E)) \to \Gamma(A, \mathcal{O}(-D-E))$$

is surjective.

Proof. (1) Let m_p be the ideal sheaf of the singularity p. We claim that $m_p = \Gamma(A, \mathcal{O}(-Z))$ generates the ideal sheaf $\mathcal{O}(-Z)$ near A. The map $\Gamma(A, \mathcal{O}(-Z)) \to \Gamma(A, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_i))$ is surjective for all i by Theorem 3.4. Sections in $\Gamma(A, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_i))$ correspond to sections of a line bundle on A_i of Chern class $-A_i \cdot Z \geqslant 0$. Thus $\Gamma(A, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_i))$ has no common zeros, as sections of a line bundle. Then at each $q \in A_i$, some element in $\Gamma(A, \mathcal{O}(-Z))$ will vanish to order exactly a_i on A_i (and a_j on A_j if $q = A_i \cap A_j$) but will have no other zero near q. This proves our claim.

(2) We next prove that for any $\ell \geqslant 1$, $\Gamma(A, \mathcal{O}(-Z))^\ell$ contains $\Gamma(A, \mathcal{O}(-(\ell + k)Z))$ for k large enough. Let $z_1, \ldots, z_n$ generate m_p as an $\mathcal{O}$-module. Then in fact $\pi^*(z_1), \ldots, \pi^*(z_n)$ generate the ideal sheaf $\mathcal{O}(-Z)$ near A because $\pi^*(z_1), \ldots, \pi^*(z_n)$ generate $\Gamma(A, \mathcal{O}(-Z))$. Suppose that $f_1, \ldots, f_t$ are the ℓ-fold products of the $z_1, \ldots, z_n$ which generate m_p^ℓ. So $\pi^*(f_1), \ldots, \pi^*(f_t)$ generate the ideal sheaf $\mathcal{O}(-\ell Z)$ near A. The map

$$\lambda: \mathcal{O}^t \to \mathcal{O}(-\ell Z)$$

sending $(b_1, \ldots, b_t) \in \mathcal{O}^t$ to $\sum_{i=1}^t b_i \pi^*(f_i)$ is then surjective. Let $\mathcal{K}$ be the kernel of λ. Since $\mathcal{O}(-kZ)$ is locally free of rank 1,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-kZ)\mathcal{K} \arrow[r] \arrow[d] & \mathcal{O}(-kZ)^t \arrow[r] \arrow[d] & \mathcal{O}(-(k+\ell)Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{O}^t \arrow[r, "\lambda"] & \mathcal{O}(-\ell Z) \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram with exact rows. Taking part of the long exact cohomology sequence, we have

$$\begin{tikzcd}
\Gamma(A, \mathcal{O}(-kZ)^t) \arrow[r] \arrow[d] & \Gamma(A, \mathcal{O}(-(k+\ell)Z)) \arrow[r] \arrow[d, "\beta"] & H^1(A, \mathcal{O}(-kZ)\mathcal{K}) \arrow[d, "\gamma"] \\
\Gamma(A, \mathcal{O}^t) \arrow[r, "\lambda"] & \Gamma(A, \mathcal{O}(-\ell Z)) \arrow[r] & H^1(A, \mathcal{K}).
\end{tikzcd}$$

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By [Gr, Sect. 4 Satz 1, p. 355], γ is the zero map for suitably large k. Hence $im\beta \subset im\lambda = m_p^\ell = \Gamma(A, \mathcal{O}(-Z))^\ell$. Hence $\Gamma(A, \mathcal{O}(-Z))^\ell \supset \Gamma(A, \mathcal{O}(-(k+\ell)Z))$ for suitably large k, as needed.

(3) In this step we shall show that for all u, the map $\Gamma(A, \mathcal{O}(-D)/\mathcal{O}(-uZ - D)) \otimes_{\mathbb{C}} \Gamma(A, \mathcal{O}(-E)/\mathcal{O}(-uZ-E)) \xrightarrow{\tau} \Gamma(A, \mathcal{O}(-D-E)/\mathcal{O}(-uZ-D-E))$ is surjective. In view of Theorem 3.4, $\Gamma(A, \mathcal{O}(-D-E)/\mathcal{O}(-uZ-D-E))$ is successively broken into quotient spaces in the following manner:

$$0 \to \Gamma\left(A, \tfrac{\mathcal{O}(-Z_1-D-E)}{\mathcal{O}(-uZ-D-E)}\right) \to \Gamma\left(A, \tfrac{\mathcal{O}(-D-E)}{\mathcal{O}(-uZ-D-E)}\right) \to \Gamma\left(A, \tfrac{\mathcal{O}(-D-E)}{\mathcal{O}(-Z_1-D-E)}\right) \to 0$$

$$\vdots$$

$$0 \to \Gamma\left(A, \tfrac{\mathcal{O}(-Z_{k+1}-sZ-D-E)}{\mathcal{O}(-uZ-D-E)}\right) \to \Gamma\left(A, \tfrac{\mathcal{O}(-Z_k-sZ-D-E)}{\mathcal{O}(-uZ-D-E)}\right) \to \Gamma\left(A, \tfrac{\mathcal{O}(-Z_k-sZ-D-E)}{\mathcal{O}(-Z_{k+1}-sZ-D-E)}\right) \to 0$$

$$\vdots$$

$$0 \to \Gamma\left(A, \tfrac{\mathcal{O}(-Z_{\ell-1}-(u-1)Z-D-E)}{\mathcal{O}(-uZ-D-E)}\right) \to \Gamma\left(A, \tfrac{\mathcal{O}(-Z_{\ell-2}-(u-1)Z-D-E)}{\mathcal{O}(-uZ-D-E)}\right) \to \Gamma\left(A, \tfrac{\mathcal{O}(-Z_{\ell-2}-(u-1)Z-D-E)}{\mathcal{O}(-Z_{\ell-1}-(u-1)Z-D-E)}\right) \to 0$$

where $s \leqslant u$ and $Z_1, \ldots, Z_\ell = Z$ is the computation sequence of Z. Similarly, $\Gamma(A, \mathcal{O}(-D)/\mathcal{O}(-uZ-D))$ and $\Gamma(A, \mathcal{O}(-E)/\mathcal{O}(-uZ-E))$ may be broken up into quotient spaces in the same manner. Observe that $\Gamma(A, \mathcal{O}(-D)/\mathcal{O}(-uZ-D)) \to \Gamma(A, \mathcal{O}(-D)/\mathcal{O}(-A_{i_k}-D))$ and $\Gamma(A, \mathcal{O}(-E)/\mathcal{O}(-uZ-E)) \to \Gamma(A, \mathcal{O}(-E)/\mathcal{O}(-A_{i_k}-E))$ are surjective by Theorem 3.4. To prove the surjectivity of τ, it suffices to prove for each k the surjectivity of one of the following maps

$$\Gamma\left(A, \frac{\mathcal{O}(-Z_k-sZ-D)}{\mathcal{O}(-Z_{k+1}-sZ-D)}\right) \otimes_{\mathbb{C}} \Gamma\left(A, \frac{\mathcal{O}(-E)}{\mathcal{O}(-A_{i_k}-E)}\right) \to \Gamma\left(A, \frac{\mathcal{O}(-Z_k-sZ-D-E)}{\mathcal{O}(-Z_{k+1}-sZ-D-E)}\right),$$

$$\Gamma\left(A, \frac{\mathcal{O}(-D)}{\mathcal{O}(-A_{i_k}-D)}\right) \otimes_{\mathbb{C}} \Gamma\left(A, \frac{\mathcal{O}(-Z_k-sZ-E)}{\mathcal{O}(-Z_{k+1}-sZ-E)}\right) \to \Gamma\left(A, \frac{\mathcal{O}(-Z_k-sZ-D-E)}{\mathcal{O}(-Z_{k+1}-sZ-D-E)}\right).$$

$Q := \Gamma(A, \mathcal{O}(-Z_k-sZ-D-E)/\mathcal{O}(-Z_{k+1}-sZ-D-E))$ corresponds to sections of a line bundle over A_{i_k} of Chern class $c = -1 + A_{i_k} \cdot (-sZ-D-E)$. If

$c=-1$, $Q=0$. If $c \geqslant 0$, then, say, $-1+A_{i_k}\cdot(-sZ-D) \geqslant 0$. Then Q is the image of $\Gamma(A,\mathcal{O}(-Z_k-sZ-D)/\mathcal{O}(-Z_{k+1}-sZ-D)\otimes_{\mathbb{C}}\Gamma(A,\mathcal{O}(-E)/\mathcal{O}(-A_{i_k}-E))$ as may be seen as follows. $\Gamma(A,\mathcal{O}(-Z_k-sZ-D)/\mathcal{O}(-Z_{k+1}-sZ-D))$ corresponds to sections of a line bundle of Chern class $-1+A_{i_k}(-sZ-D)\geqslant 0$ over A_{i_k} and $\Gamma(A,\mathcal{O}(-E)/\mathcal{O}(-A_{i_k}-E))$ to Chern class $A_{i_k}\cdot(-E)\geqslant 0$. Since A_{i_k} has genus 0, just choose bases for the sections which consist of sections vanishing to different orders at a given point $q\in A_{i_k}$ and observe that Q is indeed given by elements of the tensor product.

(4) We are now ready to finish the proof of Theorem 4.5. Consider the following diagram.

$$\begin{tikzcd}
 & & & 0 \\
\Gamma\left(A,\dfrac{\mathcal{O}(-D)}{\mathcal{O}(-uZ-D)}\right) & \otimes_{\mathbb{C}} & \Gamma\left(A,\dfrac{\mathcal{O}(-E)}{\mathcal{O}(-uZ-E)}\right) \arrow[r, "\alpha"] & \Gamma\left(A,\dfrac{\mathcal{O}(-D-E)}{\mathcal{O}(-uZ-D-E)}\right) \arrow[u] \\
\Gamma(A,\mathcal{O}(-D)) \arrow[u, "\alpha_D"'] & \otimes_{\mathbb{C}} & \Gamma(A,\mathcal{O}(-E)) \arrow[u, "\alpha_E"'] \arrow[r] & \Gamma(A,\mathcal{O}(-D-E)) \arrow[u] \\
 & & & \Gamma(A,\mathcal{O}(-uZ-D-E)) \arrow[u] \\
 & & & 0. \arrow[u]
\end{tikzcd}$$

By Theorem 3.4, α_D and α_E are surjective and the right-hand column sequence is exact. Since α is surjective in view of step 3 above, it remains to show that $\Gamma(A,\mathcal{O}(-D))\otimes_{\mathbb{C}}\Gamma(A,\mathcal{O}(-E))$ contains $\Gamma(A,\mathcal{O}(-uZ-D-E))$ for u sufficiently large. For suitably large v, $vZ>D$ and $vZ>E$ so that $\Gamma(Z,\mathcal{O}(-vZ))\subset\Gamma(A,\mathcal{O}(-D))$ and $\Gamma(A,\mathcal{O}(-vZ))\subset\Gamma(A,\mathcal{O}(-E))$. Therefore $\Gamma(A,\mathcal{O}(-D))\otimes_{\mathbb{C}}\Gamma(A,\mathcal{O}(-E))\supset\Gamma(A,\mathcal{O}(-vZ))\otimes_{\mathbb{C}}\Gamma(A,\mathcal{O}(-vZ))\supset\Gamma(A,\mathcal{O}(-Z))^{2v}\supset\Gamma(A,\mathcal{O}(-uZ))$ for u sufficiently large and bigger than $2v$. It follows that $\Gamma(A,\mathcal{O}(-D))\otimes_{\mathbb{C}}\Gamma(A,\mathcal{O}(-E))$ contains $\Gamma(A,\mathcal{O}(-uZ-D-E))$. $\square$

COROLLARY 3.6 (Artin, [Ar]). *Let Z be the fundamental cycle of a resolution of a rational singularity p of the analytic space V. Then $m_p^n/m_p^{n+1}=$*

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$\Gamma(A, \mathcal{O}(-nZ))/\Gamma(A, \mathcal{O}(-(n+1)Z))$ *and* $\dim_{\mathbb{C}} m_p^n/m_p^{n+1} = -nZ \cdot Z + 1$ *where* m_p *is the ideal sheaf of* V *at* p*. In particular, the Zariski tangent space of* V *at* p *is of dimension* $-Z \cdot Z + 1$.

Proof. dim $m_p^n/m_p^{n+1} = \dim \Gamma(A, \mathcal{O}(-nZ))/\Gamma(A, \mathcal{O}(-(n+1)Z))$ by Theorem 3.5. $\Gamma(A, \mathcal{O}(-nZ))/\Gamma(A, \mathcal{O}(-(n+1)Z)) = \Gamma(A, \mathcal{O}(-nZ)/\mathcal{O}(-(n+1)Z))$ by Theorem 3.4. We have successively

$$0 \to \frac{\mathcal{O}(-Z_1 - nZ)}{\mathcal{O}(-(n+1)Z)} \to \frac{\mathcal{O}(-nZ)}{\mathcal{O}(-(n+1)Z)} \to \frac{\mathcal{O}(-nZ)}{\mathcal{O}(-Z_1 - nZ)} \to 0$$

$$\vdots$$

$$0 \to \frac{\mathcal{O}(-Z_{k+1} - nZ)}{\mathcal{O}(-(n+1)Z)} \to \frac{\mathcal{O}(-Z_k - nZ)}{\mathcal{O}(-(n+1)Z)} \to \frac{\mathcal{O}(-Z_k - nZ)}{\mathcal{O}(-Z_{k+1} - nZ)} \to 0$$

$$\vdots$$

All the first cohomology groups are 0 by Theorem 3.4.

$$\dim \Gamma(A, \mathcal{O}(-nZ)/\mathcal{O}(-Z_1 - nZ)) = -nA_{i_0} \cdot Z + 1.$$

$$\dim \Gamma(A, \mathcal{O}(-Z_k - nZ)/\mathcal{O}(-Z_{k+1} - nZ))$$

$$= -A_{i_k} \cdot (Z_k + nZ) + 1 = -nA_{i_k} \cdot Z.$$

Summing over k, we get dim $\Gamma(A, \mathcal{O}(-nZ)/\mathcal{O}(-(n+1)Z)) = -nZ \cdot Z + 1$. $\square$

Let m be the maximal ideal of $\mathcal{O}_{V,p}$ where p is a rational singularity. Partially order t-tuples of integers by $(b_1, \dots, b_t) \leqslant (b_1', \dots, b_t')$ if $b_i \leqslant b_i'$ for all i. Our b_i will always be nonnegative.

DEFINITION 3.3. Nontrivial subspaces $S_1, \dots, S_t \subset m/m^2$ are distinguished if using graded ring multiplication of $\oplus m^k/m^{k+1}$, $S_1^{b_1} \cdots S_t^{b_t} \equiv 0 \bmod m^{b_1 + \cdots + b_t + 1}$. Moreover, if a minimal $(b_1, \dots, b_t)$ is chosen, $b_i > 0$ for all i.

THEOREM 3.7. *Suppose that* p *is a rational singularity. Distinguished subspaces exist if and only if* p *has more than one curve in* $\pi^{-1}(p)$ *of the minimal resolution.*

Proof. If $\pi^{-1}(p) = A = A_1$, then any $f \in m - m^2$ vanishes to exactly first order on A_1 since $m/m^2 = \Gamma(A, \mathcal{O}(-A)/\mathcal{O}(-2A))$. Hence any t-fold product of such f vanishes to exactly tth order on A and hence is not in $m^{t+1} = \Gamma(A, \mathcal{O}(-(t+1)A))$.

Conversely, suppose that A is minimal, i.e. no $A_i \cdot A_i = -1$, and has at least two irreducible components. We must show that distinguished subspaces exist. Observe that dim $m/m^2 = -Z \cdot Z + 1$, $H^1(A, \mathcal{O}(-A_i - Z)) = 0$ and dim $\Gamma(A, \mathcal{O}(-Z)/\mathcal{O}(-A_i - Z)) = -A_i \cdot Z + 1$. We claim that $\Gamma(A, \mathcal{O}(-A_i - Z)) \neq \Gamma(A, \mathcal{O}(-2Z))$ which, from codimension considerations in $m = \Gamma(A, \mathcal{O}(-Z))$, is equivalent to claiming that $-A_i \cdot Z < -Z \cdot Z$. We may then take $S_i = \Gamma(A, \mathcal{O}(-A_i - Z))/m^2$. Recall $Z = \sum a_i A_i$. $S_1^{a_1} \dots S_n^{a_n} \equiv 0 \bmod m^{a_1+\cdots+a_n+1}$. $(a_1, \dots, a_n)$ may not be minimal. Choose a minimal $(b_1, \dots, b_t)$.

So we need only to show that $-A_i \cdot Z < -Z \cdot Z$. Since $A_j \cdot Z \leqslant 0$, this is certainly true if there exists a $j \neq i$ such that $A_j \cdot Z < 0$ or if $a_i \geqslant 2$. So we assume $Z = A_1 + a_2A_2 + \cdots + a_rA_r$, $A_1 \cdot Z < 0$ but $A_j \cdot Z = 0$, $j \neq 1$ and we must show that $a_2 = \cdots = a_r = 0$. If no $A_j \cdot A_j = -1$, this is exactly the statement of the next lemma.

LEMMA 3.2. *Suppose that the rational singularity p has $r \geqslant 2$ irreducible curves in $\pi^{-1}(p)$ of the minimal resolution. If the fundamental cycle Z is of the form $A_1 + a_2A_2 + \cdots + a_rA_r$ with the property that $A_1 \cdot Z < 0$ but $A_j \cdot Z = 0$, $j \neq 1$, then $a_2 = \cdots = a_r = 0$.*

Proof. The proof is by induction on r and the result is trivially true if $r = 2$. If Y is the union of a subset of the A_i, then any singularity having a connected component of Y as its resolution is rational. The intersection matrix for Y is negative definite and Z may be computed, using Proposition 3.1, by first computing $Z(Y)$, the fundamental cycle for Y. Theorem 3.2 then ensures that the singularity for the component of Y is rational.

Let $C_1, \dots, C_\nu$ be the connected components of $\bigcup_{i>1} A_i$. $C_j \cup A_1$ is rational (i.e. the exceptional set in a resolution of a rational singularity). $Z(C_j \cup A_1) = A_1 + a_2A_2 + \cdots + a_sA_s$, assuming $C_j = \{A_2, \dots, A_s\}$, for in computing Z we may first compute $Z(C_j \cup A_1)$. A_{i_ℓ}, for ℓ past the computation of $Z(C_j \cup A_1)$, is never A_1 since $a_1 = 1$. $A_k \cdot A_{k'} = 0$ for $A_k \in C_j$ and $A_{k'} \notin (C_j \cup A_1)$. $A_k \cdot Z(C_j \cup A_1) \leqslant 0$. So we see by induction that A_{i_ℓ} is never an A_k for $A_k \in C_j$. Hence $Z(C_j \cup A_1)$ satisfies the induction hypothesis. Thus we may assume, by induction, that there is only one connected component C_1.

Since the dual graph for A is a tree, A_1 can meet only one curve, say A_2, in C_1. Thus $Z \cdot Z = A_1 \cdot A_1 + a_2$. We may replace A_1 by a curve B with $B \cdot B = -(a_2+1)$, thereby changing the analytic structure. The new set of curves $B \cup C_1$ has the cycle $B + a_2A_2 + \cdot + a_rA_r$ and so has a negative definite intersection matrix by [Ar, Prop. 2, p. 130] or [Mu, p. 6]. $B \cup C$ occurs as a resolution of some singularity [Gr, p. 367]. By Theorem 3.2, $Z' = B + a_2A_2 + \cdots + a_rA_r$ is the fundamental cycle and the singularity is rational. $Z' \cdot Z = -1$. Hence by Corollary 3.6, Z' is the fundamental cycle of an exceptional set of the first kind. Hence by [Ho, p. 154], $B \cup C$ is the result of a finite iteration of quadratic transformations. Hence $A_k \cdot A_k = -1$ for some k, as we were required to prove. □

COROLLARY 3.8. *If m/m^2 has no distinguished subspaces, then the minimal resolution of p has just one curve A and $-A \cdot A + 1 = \dim m/m^2$.*

COROLLARY 3.9. *Let Z be the fundamental cycle of a resolution of a rational singularity p. The minimal resolution of p has at least two curves if and only if* *either there exist $A_i, A_j,\ i \neq j$ such that $A_i \cdot Z < 0$ and $A_j \cdot Z < 0$ or else $A_i \cdot Z < 0$ and $Z \geqslant 2A_i$. If distinguished subspaces exist, then $\Gamma(A, \mathcal{O}(-A_j - Z))/\Gamma(A, \mathcal{O}(-2Z))$ is, for all j, a nontrivial subspace of m/m^2.*

Proof. '$\Leftarrow$' If there exist $A_i, A_j,\ i \neq j$ such that $A_i \cdot Z < 0$ and $A_j \cdot Z < 0$, then clearly the minimal resolution of p has at least two curves. Suppose next that $A_i \cdot Z < 0$ and $Z \geqslant 2A_i$. Again the minimal resolution of p has at least two curves, otherwise $Z = A_i$.

'$\Rightarrow$' It is an immediate consequence of Lemma 3.2. □

LEMMA 3.3. *Suppose $S_1, \ldots, S_t$ are distinguished subspaces. Moreover a minimal $(b_1, \ldots, b_t)$ is chosen so that $b_i > 0$ for all i. Then $S_i + m^2 \subset \Gamma(A, \mathcal{O}(-A_j - Z))$ for some A_j depending on i.*

Proof. If an $S_i + m^2$ contained for each j a function which vanished to precisely order a_j on A_j, then some linear combination of these functions would vanish to precisely order a_j on A_j for all j. But then we would have $S_1^{b_1} \ldots \hat{S}_i \ldots S_t^{b_t} \equiv 0 \bmod m^{b-b_i}$, where $\hat{S}_i$ indicates omission from the product and $b = b_1 + \cdots + b_t + 1$, contrary to Definition 3.3. □

DEFINITION 3.4. Nontrivial subspaces $S_1, \ldots, S_t$ of m/m^2 are maximal distinguished subspaces if

(i) for some $b_1, \ldots, b_t$, $S_1^{b_1} \ldots S_t^{b_t} \equiv 0 \bmod m^{b_1 + \cdots + b_t + 1}$;

(ii) the $b_1, \ldots, b_t$ are minimal with respect to property (i) and positive

(iii) there do not exist $T_1, \ldots, T_s$ such that $S_{j_i} \subset T_i$ for some S_{j_i} with at least one of the containments non-trivial, T_i a subspace of m/m^2, and positive integers $c_1, \ldots, c_s$ such that $T_1^{c_1} \cdots T_s^{c_s} \equiv 0 \bmod m^{c_1 + \cdots + c_s + 1}$. The $c_1, \ldots, c_s$ are minimal with respect to property (i).

For $A_j \cdot Z < 0$, $\Gamma(A, \mathcal{O}(-A_j - Z))$ may be characterized as a subset of m as follows.

THEOREM 3.10. *Let $Z = \sum a_i A_i$ be the fundamental cycle of the resolution of a rational singularity p. If distinguished subspaces of m/m^2 exist, then maximal distinguished subspaces $S_1, \ldots, S_t$ of m/m^2 exist and are unique. Each S_j corresponds to $W_j = \Gamma(A, \mathcal{O}(-A_j - Z))/\Gamma(A, \mathcal{O}(-2Z))$ for an A_j such that $A_j \cdot Z < 0$. b_i in Definition 3.3 is a_i for $1 \leqslant i \leqslant t$.*

Proof. By Lemma 3.3, any distinguished subspace D_i satisfies $D_i \subset W_k$ for some $W_k = \Gamma(A, \mathcal{O}(-A_k - Z)/\Gamma(A, \mathcal{O}(-2Z))$. The proof of Theorem 3.7 (cf. Corollary 3.9) shows that W_k is a non-trivial subspace of m/m^2. Moreover, for each k, $W_k \supset D_i$ for some i, for otherwise each D_i would have functions vanishing to precisely order a_k on A_k and the D_i could not be distinguished. Hence given any $S_1, \ldots, S_t$ satisfying (i) and (ii) of Definition 3.4, we may choose T_k from among the W's and then choose minimal positive c_ℓ. Thus to prove this theorem, we must show that

(I) if $W_1^{c_1}\dots W_n^{c_n} \equiv 0 \bmod m^{c_1+\cdots+c_n+1}$ and $c_1,\dots,c_n$ are minimal non-negative integers, then

$$c_j = \begin{cases} 0 & \text{if } A_j\cdot Z = 0,\\ a_j & \text{if } A_j\cdot Z < 0.\end{cases}$$

(II) $W_1^{a_1},\dots,W_s^{a_s} \equiv 0 \bmod m^{a_1+\cdots+a_s+1}$, where $j=1,\dots,s$ gives the A_j such that $A_j\cdot Z<0$.

We shall first show that for $A_j\cdot Z<0$, $\Gamma(A,\mathcal{O}(-A_j-Z)) \neq \Gamma(A,\mathcal{O}(-2A_j-Z))$ and $W_i \not\subset W_j$, for $1\leqslant j\leqslant s$ and $i>s$, $W_{j'}\not\subset W_j$ for $1\leqslant j', j\leqslant s$, so that $c_j\geqslant a_j$. The codimension of W_j, $1\leqslant j\leqslant s$, in m/m^2 is $-A_j\cdot Z+1$ which is greater than 1 while the codimension of W_i in m/m^2, $i>s$, equals 1. Thus $W_i\not\subset W_j$ for $1\leqslant j\leqslant s<i$. Now consider, say, the divisor $Z+A_1$. In a manner similar to that used in Proposition 3.1, add successively $B_1=A_{i_1}$, $B_2=A_{i_2},\dots$, such that $B_1\cdot(Z+A_1)>0$, $B_2\cdot(Z+A_1+B_1)>0,\dots$. As the proof of Proposition 3.1 shows, there is a least $E_1\geqslant Z+A_1$ such that $A_k\cdot E_1\leqslant 0$ for all k. Moreover, the process of adding the B's terminates at E_1. $\Gamma(A,\mathcal{O}(-A_1-Z)) = \Gamma(A,\mathcal{O}(-E_1))$ since the successive quotient spaces $\Gamma(A,\mathcal{O}(-Z-A_1-B_1-\cdots-B_{\ell-1})/\mathcal{O}(-Z-A_1-B_1-\cdots-B_\ell))$ correspond to sections of negative bundles and hence are trivial. In adding the B's to $Z+A_1$, we may first add as many as possible of the A_i, $i>s$, such that A_i lies in some connected component Y_ν of $\bigcup_{i>s}A_i$ with $Y_\nu\cap A_1\neq\phi$. Call this cycle E'. $E'-Z$ is a Z_k for some Z_k used in the calculation of Z described in Proposition 3.1. We shall say that $E'-Z$ is a subcalculation of Z. In fact $E'=E_1$, for suppose B existed so that $B\cdot E'>0$. $B\notin Y_\nu$ for any Y_ν such that $Y_\nu\cap A_1\neq\phi$ by our construction of E'. For A_j, $1\leqslant j\leqslant s$, i.e. $A_j\cdot Z<0$, $1\leqslant A_j\cdot E' = A_j\cdot(Z+(E'-Z))$ implies that $A_j\cdot(E'-Z)\geqslant 1-A_j\cdot Z\geqslant 2$ which, by Theorem 3.2, contradicts the rationality of p. Thus $\Gamma(A,\mathcal{O}(-A_1-Z)) = \Gamma(A,\mathcal{O}(-E_1))$ and $\Gamma(A,\mathcal{O}(-E_1))/\Gamma(A,\mathcal{O}(-A_j-E_1)) = \Gamma(A,\mathcal{O}(-E_1)/\mathcal{O}(-E_1-A_j))$ has positive dimension by Theorem 3.4 and the fact that $-A_j\cdot E_1\geqslant 0$. Since E_1-Z has no A_j term for $2\leqslant j\leqslant s$, $W_1\not\subseteq W_j$ for $2\leqslant j\leqslant s$. Also we see that $\Gamma(A,\mathcal{O}(-A_1-Z) = \Gamma(A,\mathcal{O}(-E_1))\not\subseteq \Gamma(A,\mathcal{O}(-2A_1-Z))$ because $\Gamma(A,\mathcal{O}(-E_1)/\mathcal{O}(-A_1-E_1))$ has positive dimension and the coefficients of A_1 in $2A_1+Z$ and A_1+E_1 are $2+a_1$. Thus $c_j\geqslant a_j$, which was the first thing we had to prove.

We next show that $W_1^{a_1}\dots W_s^{a_s}\equiv 0\bmod m^{a_1+\cdots+a_s+1}$. To each A_j, $1\leqslant j\leqslant s$, i.e. $A_j\cdot Z<0$, we associate the cycles E_j above such that $A_k\cdot E_j\leqslant 0$ all k and $\Gamma(A,\mathcal{O}(-A_j-Z)) = \Gamma(A,\mathcal{O}(-E_j))$. Let $D_j=E_j-Z$. We claim that E_j is uniquely determined. Let $\tilde{E}_j$ be another minimal cycle bigger than or equal to $Z+A_j$ such that $A_k\cdot\tilde{E}_j\leqslant 0$. Let $\hat{E}_j$ be the cycle $\min(E_j,\tilde{E}_j)$ by taking minimal of the coefficients of E_j and $\tilde{E}_j$ componentwise. It is clear that

$\hat{E}_j \geqslant A_j + Z$ and $\hat{E}_j \cdot A_k \leqslant 0$ all A_k. So $\hat{E}_j = E_j = \tilde{E}_j$. We must show that $a_1E_1+\cdots+a_sE_s \geqslant (a_1+\cdots+a_s+1)Z$, or more simply, $a_1D_1+\cdots+a_sD_s \geqslant Z$. We have that $D_j \geqslant A_j$. Let us use Proposition 3.1 to compute Z as follows. Let $A_{i_1} = A_1$. Then choose $A_{i_2}, A_{i_3}, \ldots$ to be A_i, $i > s$ for as long as possible. Let F_1 be the resulting cycle. Since $A_i \cdot Z = 0$ for $i > s$, this is just a subcalculation of D_1. It is in fact a complete calculation in this first case. Next, in calculating Z, we must add an A_j, $1 \leqslant j \leqslant s$, since $A_i \cdot F_1 \leqslant 0$ for $i > s$. Now again add A_i with $i > s$ for as long as possible. Since $A_i \cdot F_1 \leqslant 0$, this is just a subcalculation for D_j. Continue in this manner until reaching $Z = a_1A_1 + \cdots + a_sA_s + \cdots$. We perform a_1 subcalculations of D_1, a_2 subcalculations of $D_2, \ldots$, a_s subcalculations of D_s. Hence $a_1D_1 + \cdots + a_sD_s \geqslant Z$ and the theorem is proved. □

LEMMA 3.4. *Let $Z = \sum a_iA_i$ be the fundamental cycle of the resolution of a rational singularity. Suppose $Z \cdot A_i < 0$ for $1 \leqslant i \leqslant s$ and $Z \cdot A_i = 0$ for $i > s$. For $1 \leqslant j \leqslant s$, let $D_j = E_j - Z$ where E_j is the least cycle greater than or equal to $Z + A_j$ such that $A_k \cdot E_j \leqslant 0$ for all k. If A_i, $i > s$, appears in D_j and $A_i \cdot A_\ell = 1$ for some $1 \leqslant \ell \leqslant s$ and $\ell \neq j$, then A_i has coefficient 1 in D_j.*

Proof. Suppose on the contrary that the coefficient of A_i in D_j is bigger than 1. Then there exists a cycle G in the calculation of E_j such that A_i appears in $G - Z$ with coefficient one and $A_i \cdot G = 1$. So $A_i \cdot (G - Z) = 1$. $G - Z$ is a cycle appearing in a subcalculation of Z. $A_\ell \cdot (G - Z) = 1$ since A_i occurs in $G - Z$. Then $A_\ell + G - Z$ appears in a subcalculation of Z and $A_i \cdot (A_\ell + G - Z) = 2$, contradicting the fact that p is a rational singularity. □

COROLLARY 3.11. *Let $Z = \sum a_iA_i$ be the fundamental cycle of the resolution of a rational singularity. Suppose $Z \cdot A_j < 0$ for $1 \leqslant j \leqslant s$ and $Z \cdot A_i = 0$ for $i > s$. For $1 \leqslant j \leqslant s$, let $D_j = E_j - Z$ where E_j is the least cycle greater than or equal to $Z + A_j$ such that $A_k \cdot E_j \leqslant 0$ for all k.*

(1) *For $1 \leqslant j \leqslant s$, let $S_j = \Gamma(A, \mathcal{O}(-A_j - Z))/\Gamma(A, \mathcal{O}(-2Z))$. Then* codim $S_j = -A_j \cdot Z + 1$. *Here* codim S_j *= codimension of S_j in $m/m^2 = \Gamma(A, \mathcal{O}(-Z))/\, \Gamma(A, \mathcal{O}(-2Z))$.*
(2) *Let $|D_j|$ be the union of the curves appearing in D_j with non-zero coefficient. Then $|D_j|$ consists of A_j and those components Y_ν of $\bigcup_{i>s} A_i$ such that $Y_\nu \cap A_j \neq \phi$. Moreover $|D_j| \cap |D_{j_1}| \neq \phi$ if and only if* codim $S_j \cap S_{j_1} <$ codim S_j + codim S_{j_1} *if and only if $A_{j_1} \cap |D_j| \neq \phi$.*
(3) *E_j is obtained in a manner similar to Proposition* 3.1. *Add successively $B_1 = A_{i_1}, B_2 = A_{i_2}, \ldots$ such that $B_1 \cdot (Z + A_j) > 0$, $B_2 \cdot (Z + A_j + B_1) > 0, \ldots$. The process of adding the B's terminates at E_j. In adding the B's to $Z + A_j$, we only need to add those A_i, $i > s$, such that A_i lies in some connected component Y_ν of $\bigcup_{i>s} A_i$ with $Y_\nu \cap A_j \neq \phi$. $D_j = E_j - Z$ is a Z_k for some Z_k used in the calculation of Z described in Proposition* 3.1. *Moreover $a_1D_1 + \cdots + a_sD_s \geqslant Z$.*

Proof. (3) was already contained in the proof of Theorem 3.10. For (1), we first observe that $m/m^2 \cong \Gamma(A,\mathcal{O}(-Z))/\Gamma(A,\mathcal{O}(-2Z))$ by Corollary 3.6. From the short exact sequence

$$0 \to \frac{\Gamma(A,\mathcal{O}(-A_j-Z))}{\Gamma(A,\mathcal{O}(-2Z))} \to \frac{\Gamma(A,\mathcal{O}(-Z))}{\Gamma(A,\mathcal{O}(-2Z))} \to \frac{\Gamma(A,\mathcal{O}(-Z))}{\Gamma(A,\mathcal{O}(-A_j-Z))} \to 0,$$

we deduce that

$$\begin{aligned}\operatorname{codim} S_j &= \frac{\dim \Gamma(A,\mathcal{O}(-Z))}{\Gamma(A,\mathcal{O}(-A_j-Z))} \\ &= \frac{\dim \Gamma(A,\mathcal{O}(-Z)}{\mathcal{O}(-A_j-Z))} \quad \text{by Theorem 3.4} \\ &= -A_j\cdot Z+1.\end{aligned}$$

For (2), we observe that

$$\begin{aligned}S_j\cap S_{j_1} &= \frac{\Gamma(A,\mathcal{O}(-A_j-A_{j_1}-Z))}{\Gamma(A,\mathcal{O}(-2Z))} \\ &= \frac{\Gamma(A,\mathcal{O}(-A_{j_1}-D_j-Z))}{\Gamma(A,\mathcal{O}(-2Z))} \quad \text{by part (3) of the Corollary} \\ &\subseteq \frac{\Gamma(A,\mathcal{O}(-D_j-Z))}{\Gamma(A,\mathcal{O}(-2Z))}.\end{aligned}$$

Recall that

$$S_j = \frac{\Gamma(A,\mathcal{O}(-A_j-Z))}{\Gamma(A,\mathcal{O}(-2Z))} = \frac{\Gamma(A,\mathcal{O}(-D_j-Z))}{\Gamma(A,\mathcal{O}(-2Z))}.$$

From the short exact sequence

$$\begin{aligned}0 &\to \frac{\Gamma(A,\mathcal{O}(-A_{j_1}-D_j-Z))}{\Gamma(A,\mathcal{O}(-2Z))} \to \frac{\Gamma(A,\mathcal{O}(-D_j-Z))}{\Gamma(A,\mathcal{O}(-2Z))} \\ &\to \frac{\Gamma(A,\mathcal{O}(-D_j-Z))}{\Gamma(A,\mathcal{O}(-A_{j_1}-D_j-Z))} \to 0\end{aligned}$$

we deduce that

$$\dim S_j - \dim S_j\cap S_{j_1} = \dim \frac{\Gamma(A,\mathcal{O}(-D_j-Z))}{\Gamma(A,\mathcal{O}(-A_{j_1}-D_j-Z))}.$$

Hence

$$\begin{aligned}\operatorname{codim} S_j\cap S_{j_1} - \operatorname{codim} S_j &= \dim \frac{\Gamma(A,\mathcal{O}(-D_j-Z))}{\Gamma(A,\mathcal{O}(-A_{j_1}-D_j-Z))} \\ &= 1 - A_{j_1}\cdot(D_j+Z) = 1 - A_{j_1}\cdot Z - A_{j_1}\cdot D_j \\ &= \operatorname{codim} S_{j_1} - (A_{j_1}\cdot D_j).\end{aligned}$$

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As shown in the proof of Theorem 3.10, $|D_j|$ consists of A_j and those A_i lying in some connected component Y_ν of $\bigcup_{i>s} A_i$ with $Y_\nu \cap A_j \neq \phi$. Thus $A_{j_1} \cdot D_j > 0$ if and only if $|D_{j_1}| \cap |D_j| \neq \phi$. (2) follows from the equality $A_{j_1} \cdot D_j = \operatorname{codim} S_{j_1} + \operatorname{codim} S_j - \operatorname{codim} S_j \cap S_{j_1}$. □

Thus so far, in our goal of determining the weighted dual graph for the minimal resolution of p, we have found those A_j such that $A_j \cdot Z < 0$ and we know which A_j's can be joined by cycles A_i such that $A_i \cdot Z = 0$. Also, since $a_1 + \cdots + a_s + 1 \leqslant -Z \cdot Z + 1 =$ dimension of Zariski tangent space of the singularity p, we have an apriori estimate on what part of the graded ring structure is needed to determine if distinguished and hence maximal distinguished subspaces exist. We now must determine the graded ring structure for the singularities of the Y_ν, the connected components of $\cup A_i$, $i > s$, so that we can apply Theorem 3.10 and Corollary 3.11 to find more of the curves in the resolution.

LEMMA 3.5. *Let $Z = \sum a_i A_i$ be the fundamental cycle of the minimal resolution of a rational singularity. Suppose $Z \cdot A_i < 0$ for $1 \leqslant i \leqslant s$ and $Z \cdot A_i = 0$ for $i > s$. For $1 \leqslant j \leqslant s$, let $D_j = E_j - Z$ where E_j is the least cycle greater than or equal to $Z + A_j$ such that $A_k \cdot E_j \leqslant 0$ for all k. Let $\bigcup_{i>s} A_i = \bigcup_\nu Y_\nu$ where Y_ν's are connected components of $\bigcup_{i>s} A_i$. Then $a_1 D_1 + \cdots + a_s D_s \geqslant Z + \sum_\nu Z_\nu$ where $Z_\nu = Z(Y_\nu)$ is the fundamental cycle on $|Y_\nu|$.*

Proof. By Corollary 3.11, we know that $a_1 D_1 + \cdots + a_s D_s \geqslant Z$. We shall first prove that for any Y_ν, there exists an irreducible component $A_k^\nu \subset Y_\nu$ such that its coefficient in $a_1 D_1 + \cdots + a_s D_s - Z$ is nonzero.

Suppose on the contrary that for all irreducible components $A_k^\nu \subset Y_\nu$, the coefficient of A_k^ν in $a_1 D_1 + \cdots + a_s D_s - Z$ are zero. Observe that for all irreducible components $A_k^\nu \subseteq Y_\nu$ and all D_j, $A_k^\nu \cdot D_j \leqslant 0$ because of the statement (3) of Corollary 3.1. We claim that actually $A_k^\nu \cdot D_j = 0$ for all $A_k^\nu \subseteq Y_\nu$ and for all D_j.

Suppose $A_k^\nu \cdot D_j < 0$ for some A_k^ν and some D_j. Compute Z by Proposition 3.1, starting with $Z_1 = A_j$. The first stage of adding A_i, $i > s$, gives D_j. We must then add some $A_{j'}$, $1 \leqslant j' \leqslant s$ with $A_{j'} \cap Y_\nu \neq \phi$. Subsequently adding as many A_i, $i > s$, as possible gives a subcalculation D' of some $D_{j'}$. $A_\ell^\nu \cdot D_{j'} \leqslant 0$ for all $A_\ell^\nu \subseteq Y_\nu$. Since A_k^ν does not appear in $a_1 D_1 + \cdots + a_s D_s - Z$, the subcalculation D' of $D_{j'}$ is to include A_k^ν with the same coefficient as does $D_{j'}$. So $A_k^\nu \cdot D' \leqslant 0$. Recall that $A_k^\nu \cdot D_j < 0$ and Z is the sum of D_j and these D' by the end of the proof of Theorem 3.10. We deduce that $A_k^\nu \cdot Z < 0$, contradicting the choice of Y_ν. This proves our claim that $A_k^\nu \cdot D_j = 0$ for all $A_k^\nu \subseteq Y_\nu$ and for all D_j.

Let $F \neq 0$ be a divisor obtained from some D_j by setting equal to zero the coefficients of $A_\ell \not\subseteq Y_\nu$ (i.e. $F = D_j / Y_\nu$) for some j such that $A_j \cap Y_\nu \neq \phi$. $A_j + F$ can only fail to be $Z(A_j \cup Y_\nu)$ (fundamental cycle of $A_j \cup Y_\nu$) if $A_j \cdot (A_j + F) > 0$ since A_j appears once in D_j and $A_j + F$ is a subcalculation of $Z(A_j \cup Y_\nu)$ by the construction of D_j. Since we are only determining some property of the intersection matrix, we are free to disregard the complex structure.

Thus replace A_j by a cycle B with $B \cdot B$ negative enough so that $B \cdot (B+F) < 0$. $A_k^\nu \cdot (B+F) = A_k^\nu(A_j+F) = A_k^\nu \cdot D_j = 0$ for all $A_k^\nu \subseteq Y_\nu$. Then, as before, applying Proposition 3.2, p. 130 of [Ar] to $B+F$, $B \cup Y_\nu$ has a negative definite intersection matrix. $A_j \cup Y_\nu$ is rational so that a computation as in Theorem 3.2 of $A_j \cup Y_\nu$ involves only $+1$'s so that also $B \cup Y_\nu$ is rational and $B+F = Z(B \cup Y_\nu)$. Then by Corollary 3.9, $B \cup Y_\nu$ has only one curve in its minimal resolution. Hence if $Y_\nu \neq \phi$, either B or some A_k^ν has $A_k^\nu \cdot A_k^\nu = -1$. But $B \cdot B$ is very negative. Hence $A_k^\nu \cdot A_k^\nu = -1$, contradicting the minimality of A. This finishes the proof that $Y_\nu \cap |a_1D_1 + \cdots + a_sD_s - Z| \neq \phi$ for any connected component Y_ν of $\bigcup_{i>s} A_i$.

As observed above, $A_k^\nu \cdot D_j \leqslant 0$ for all $A_k^\nu \subseteq Y_\nu$ and all D_j. By definition, $A_k^\nu \cdot Z = 0$. Therefore we have $A_k^\nu \cdot (a_1D_1 + \cdots + a_sD_s - Z) \leqslant 0$ for all $A_k^\nu \subseteq Y_\nu$. Since $a_1D_1 + \cdots + a_sD_s - Z \geqslant 0$ and $Y_\nu \cap |a_1D_1 + \cdots + a_sD_s - Z| \neq \phi$ for any connected component Y_ν of $\bigcup_{i>s} A_i$, we conclude that $a_1D_1 + \cdots + a_sD_s - Z \geqslant \sum Z_\nu$ in view of the definition of fundamental cycle. $\square$

We may now characterize $W_i = \Gamma(A, \mathcal{O}(-A_i - Z))/\Gamma(A, \mathcal{O}(-2Z))$ for $i > s$, i.e. for A_i such that $A_i \cdot Z = 0$.

PROPOSITION 3.12. *Let $Z = \sum a_iA_i$ be the fundamental cycle of the minimal resolution of a rational singularity. Suppose $Z \cdot A_i < 0$ for $1 \leqslant i \leqslant s$ and $Z \cdot A_i = 0$ for $i > s$. For $1 \leqslant j \leqslant s$, let $D_j = E_j - Z$ where E_j is the least cycle greater than or equal to $Z + A_j$ such that $A_k \cdot E_j \leqslant 0$ for all k. Let $\bigcup_{i>s} A_i = \bigcup_{\nu=1}^r Y_\nu$ where Y_ν's are connected components of $\bigcup_{i>s} A_i$. Let $W_i = \Gamma(A, \mathcal{O}(-A_i - Z))/\Gamma(A, \mathcal{O}(-2Z))$. Then*

(1) *For $i \leqslant s$, $W_i = \Gamma(A, \mathcal{O}(-D_i - Z))/\Gamma(A, \mathcal{O}(-2Z))$ and has codimension $-A_i \cdot Z + 1$ in m/m^2.*
(2) *For $i > s$ and $A_i \subseteq Y_\nu$, $W_i = \Gamma(A, \mathcal{O}(-Z_\nu - Z))/\Gamma(A, \mathcal{O}(-2Z))$ where $Z_\nu = Z(Y_\nu)$ is the fundamental cycle with support on Y_ν.*
(3) *$W_i \supset W_j$, $i > s$, $j \leqslant s$, if and only if A_j meets the component Y_ν which contains A_i.*
(4) *W_i, $i > s$, are those subspaces of codimension 1 in m/m^2 such that for $d_{s+1}, \ldots, d_{s+r}$, letting $a = a_1 + \cdots + a_s$ and $d = d_{s+1} + \cdots + d_{s+r}$ where r is the number of Y_ν,*

$$\begin{aligned}&(W_{s+1} + m^2)^{d_{s+1}} \ldots (W_{s+r} + m^2)^{d_{s+r}} m^{a+1}\\ &\quad \subseteq m^d (W_1 + m^2)^{a_1} \ldots (W_s + m^2)^{a_s}\end{aligned} \tag{3.4}$$

and when a minimal set $(d_{s+1}, \ldots, d_{s+r})$ is chosen, all the d_i are positive.

(5) $$m^{a+1} \subseteq (W_1 + m^2)^{a_1} \ldots (W_s + m^2)^{a_s}, \tag{3.5}$$

where $a = a_1 + \cdots + a_s$ implies $A = A_1 \cup \cdots \cup A_s$.

Proof. (1) follows from Corollary 3.11 and the proof of Theorem 3.10.

For $i > s$ and $A_i \subseteq Y_\nu$, then $\Gamma(A, \mathcal{O}(-A_i - Z)) = \Gamma(A, \mathcal{O}(-Z_\nu - Z))$ where $Z_\nu = Z(Y_\nu)$ since $A_k^\nu \cdot Z = 0$ for $A_k^\nu \subseteq Y_\nu$. So (2) follows.

(3) follows immediately from the fact that D_j, $1 \leqslant j \leqslant s$, involves precisely A_j and those A_k^ν appearing in $Y_\nu \cap A_j \neq \phi$.

By Theorem 3.5, $m^d(W_1 + m^2)^{a_1} \ldots (W_s + m^2)^{a_s} = \Gamma\ (A, \mathcal{O}(-(d+a)Z - a_1 D_1 - \cdots - a_s D_s))$ and $(W_{s+1} + m^2)^{d_{s+1}} \ldots (W_{s+r} + m^2)^{d_{s+r}}\ m^{a+1} = \Gamma(A, \mathcal{O}\ (-(d+a+1)Z - d_{s+1}\ Z_{s+1} - \cdots - d_{r+s}\ Z_{r+s}))$. Therefore $(W_{s+1} + m^2)^{d_{s+1}} \ldots (W_{s+r} + m^2)^{d_{s+r}} m^{a+1} \subseteq m^d(W_1 + m^2)^{a_1} \ldots (W_s + m^2)^{a_s}$ if and only if $d_{s+1}Z_{s+1} + \cdots + d_{r+s}Z_{r+s} \geqslant a_1 D_1 + \cdots + a_s D_s - Z$. Since the support of $a_1 D_1 + \cdots + a_s D_s - Z$ is precisely $\bigcup_{i>s} A_i$ by Lemma 3.5, $d_{s+1}, \ldots, d_{r+s}$ can be found such that the above inequality holds and when a minimal set $(d_{s+1}, \ldots, d_{s+r})$ is chosen, all the d_i are positive. If a subspace $T + m^2$ appeared on the left side of (3.4) and had a function $f \in T + m^2$, with $f \notin \Gamma(A, \mathcal{O}(-A_k^\nu - Z))$ for all $A_k^\nu \subseteq Y_\nu$, then f would vanish to exactly order a_ℓ on all $A_\ell \subseteq A$ such that $A_\ell \cap Y_\nu \neq \phi$. Then the exponent for $(T + m^2)$ could be set equal to 0. If $T + m^2 \not\subseteq \Gamma(A, \mathcal{O}(-A_k^\nu - Z))$ for all A_k^ν, then there would exist an $f \in T + m^2$ with $f \notin \Gamma(A, \mathcal{O}(-A_k^\nu - Z))$ for all A_k^ν since $T + m^2$ is closed under linear combination. Since such an f cannot exist, $T + m^2 \subseteq \Gamma(A, \mathcal{O}(-A_k^\nu - Z))$ for some A_k^ν where $A_k^\nu \subseteq Y_\nu$. As $(T + m^2)/m^2$ and $\Gamma(A, \mathcal{O}(-A_k^\nu - Z))/\Gamma(A, \mathcal{O}(-2Z))$ are both codimension 1 subspaces of m/m^2, we conclude that $(T + m^2)/m^2 = W_k$.

Statement (5) is obvious. □

Thus we may determine the graded ring structure for the singularity which has Y_ν as its resolution as follows.

PROPOSITION 3.13. *Let $Z = \sum a_i A_i$ be the fundamental cycle of the minimal resolution of a rational singularity. Suppose $Z \cdot A_i < 0$ for $1 \leqslant i \leqslant s$ and $Z \cdot A_i = 0$ for $i > s$. Let $\bigcup_{i>s} A_i = \bigcup_{\nu=1}^r Y_\nu$ where Y_ν's are connected components of $\bigcup_{i>s} A_i$ and $Z_\nu = Z(Y_\nu)$ be the fundamental cycle with support on Y_ν. Each Y_ν can be blown down to an isolated singularity q_ν. Let m_ν be the maximal ideal of $\mathcal{O}_{q_\nu}$. Then $m_\nu/m_\nu^2 \approx \Gamma(A, \mathcal{O}(-Z_\nu - 2Z))/\Gamma(A, \mathcal{O}(-2Z - 2Z_\nu))$. In general $m_\nu^\lambda/m_\nu^{\lambda+1} \approx \Gamma(A, \mathcal{O}(-\lambda Z_\nu - 2\lambda Z))/\Gamma(A, \mathcal{O}(-(\lambda+1)Z_\nu - 2\lambda Z))$ and this isomorphism preserves multiplication in the graded rings.*

Proof. In view of Theorem 3.4, for any ν and any $A_i^\nu \subset Y_\nu$, the following sequence is exact

$$0 \to \Gamma(A, \mathcal{O}(-A_i^\nu - Z)) \to \Gamma(A, \mathcal{O}(-Z)) \to \Gamma\left(\frac{A, \mathcal{O}(-Z)}{\mathcal{O}(-A_i^\nu - Z)}\right) \to 0.$$

There exists function $f \in \Gamma(A, \mathcal{O}(-Z)) - \Gamma(A, \mathcal{O}(-A_i^\nu - Z))$ which represents an element $\tilde{f}$ in $\Gamma(A, \mathcal{O}(-Z)/\mathcal{O}(-A_i^\nu - Z))$, as a section of the corresponding line bundle. Since $A_i \cdot Z = 0$, so this bundle has Chern class 0 so that $\tilde{f} \in$

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$\Gamma(A, \mathcal{O}(-Z)/\mathcal{O}(-A_i^\nu - Z))$ has no zeros. Hence the zero set of f near A_i^ν is just A_i^ν and those A_k such that $A_k \cap A_i^\nu \neq \phi$. However, $\Gamma(A, \mathcal{O}(-A_i^\nu - Z)) = \Gamma(A, \mathcal{O}(-Z_\nu - Z)) = \Gamma(A, \mathcal{O}(-A_{i'}^\nu - Z))$ for any two A_i^ν, $A_{i'}^\nu \subseteq Y_\nu$. Thus f vanishes to order a_k for $A_k \cap Y_\nu \neq \phi$ and f has no other zeros near Y_ν. Thus multiplication by $f^{2\lambda}$ induces an isomorphism

$$\begin{aligned} m_\nu^\lambda / m_\nu^{\lambda+1} &\approx \Gamma(Y_\nu, \mathcal{O}(-\lambda Z_\nu))/\Gamma(Y_\nu, \mathcal{O}(-(\lambda+1)Z_\nu)) \\ &= \Gamma\left(Y_\nu, \frac{\mathcal{O}(-\lambda Z_\nu)}{\mathcal{O}(-(\lambda+1)Z_\nu)}\right) \\ &\approx \Gamma\left(A, \frac{\mathcal{O}(-2\lambda Z - \lambda Z_\nu)}{\mathcal{O}(-2\lambda Z - (\lambda+1)Z_\nu)}\right) \\ &\approx \Gamma(A, \mathcal{O}(-2\lambda Z - \lambda Z_\nu))/\Gamma(A, \mathcal{O}(-2\lambda Z - (\lambda+1)Z_\nu)). \end{aligned}$$

The first isomorphism follows from Corollary 3.6 and Theorem 3.4 while the last isomorphism follows from Theorem 3.4. Also all these isomorphisms preserve multiplication in the graded ring as needed. □

COROLLARY 3.14. *Let $Z = \sum a_i A_i$ be the fundamental cycle of the minimal resolution of a rational singularity p. Suppose $Z \cdot A_i < 0$ for $1 \leqslant i \leqslant s$ and $Z \cdot A_i = 0$ for $i > s$. Let $\bigcup_{i>s} A_i = \bigcup_{\nu=1}^r Y_\nu$ where Y_ν's are connected components of $\bigcup_{i>s} A_i$ and $Z_\nu = Z(Y_\nu)$ be the fundamental cycle with support on Y_ν. Each Y_ν can be blown down to an isolated singularity q_ν. Then the graded ring structure for the singularity q_ν of Y_ν is determined by the graded ring structure for the ring $\mathcal{O}_p$. Moreover, any finite part of the grading of the ring $\mathcal{O}_{q_\nu}$ is determined by a suitably large finite part of the grading for the ring $\mathcal{O}_p$.*

Proof. Let m_ν be the maximal ideal of $\mathcal{O}_{q_\nu}$ and m be the maximum ideal of $\mathcal{O}_p$. In view of Proposition 3.13

$$m_\nu^\lambda / m_\nu^{\lambda+1} \approx \frac{\Gamma(A, \mathcal{O}(-\lambda Z_\nu - 2\lambda Z))}{\Gamma(A, \mathcal{O}(-(\lambda+1)Z_\nu - 2\lambda Z))}.$$

Let $W_\nu = \Gamma(A, \mathcal{O}(-Z_\nu - Z))/\Gamma(A, \mathcal{O}(-2Z))$. By Theorem 2.5, $\Gamma(A, \mathcal{O}(-\lambda Z_\nu - 2\lambda Z))$ is spanned by $m^\lambda (W_\nu + m^2)^\lambda$ and $\Gamma(A, \mathcal{O}(-(\lambda+1)Z_\nu - 2\lambda Z))$ is spanned by $m^{\lambda-1}(W_\nu + m^2)^{\lambda+1}$. Thus the graded ring structure for the singularity q_ν of Y_ν is determined by the graded ring structure for the ring $\mathcal{O}_p$. □

THEOREM 3.15. *Let p be a rational singularity and m the ideal of p. There exists an explicit algorithm to determine the weighted graph of the minimal resolution of p.*

Proof. Let $Z = \sum a_i A_i$ be the fundamental cycle of the minimal resolution of a rational singularity p. Suppose $Z \cdot A_i < 0$ for $1 \leqslant i \leqslant s$ and $Z \cdot A_i = 0$ for $i > s$. Let $\bigcup_{i>s} A_i = \bigcup_{\nu=1}^r Y_\nu$ where Y_ν's are connected components of

$\bigcup_{i>s} A_i$ and $Z_\nu = Z(Y_\nu)$ be the fundamental cycle with support on Y_ν. We may apply our previous results Proposition 3.13, Corollary 3.14, and Theorem 3.10 to algebraically determine those $A_i \subset Y_\nu$ such that $A_i \cdot Z_\nu < 0$ and also determine the existence of components $Y_{\nu,\tau}$ of $\bigcup_\ell A_\ell$ with $A_\ell \cdot Z_\nu = 0$. Continuing in this manner, we will eventually find all the A_k in a minimal resolution A. We must still determine which A_k intersect and what $A_k \cdot A_k$ equals. Let us suppose that we know which A_k intersect, then we may determine the weights $A_k \cdot A_k$ as follows.

The above calculations of the form $A_\ell \cdot Z < 0$ group the A_k as follows. $X_1 = \{A_1, \ldots, A_s\}$, where $A_i \cdot Z < 0$ if and only if $1 \leqslant i \leqslant s$. The next part of the grouping is $X_{2,(s+1)} = \{A_{(s+1),j}\colon A_{(s+1),j} \cdot Z(Y_{s+1}) < 0\}, \ldots, X_{2,(t_1)} = \{A_{(t_1),j}\colon A_{(t_1),j} \cdot Z(Y_{(t_1)}) < 0\}, \ldots, X_{2,(s+r)}$, where $Y_{s+1}, \ldots, Y_{s+r}$ are connected components of $\bigcup_{i>s} A_i$ and $A_{(t_1),j}$ are those curves in Y_{t_1} such that $A_{(t_1),j} \cdot Z(Y_{t_1}) < 0$. We next consider connected components Y_{t_1,t_2} of $\cup A_\ell$, $A_\ell \notin X_1 \cup X_{2,(s+1)} \cup \cdots \cup X_{2,(s+r)}$. $X_{3,(t_1,t_2)} = \{A_{(t_1,t_2),j} \subseteq Y_{t_1,t_2}\colon A_{(t_1,t_2),j} \cdot Z(Y_{(t_1,t_2)}) < 0\}$. After a finite number r of steps all of the A_k are listed. Let $X_1, X_2, \ldots, X_r$ be the curves listed at each step. Thus $X_2 = \bigcup_t X_{2,(t)}$. Theorem 3.10 algebraically gives the fundamental cycle of each connected component of $\cup A_\ell$, $A_\ell \in X_r$. Part (1) of Corollary 3.11 then determines $A_k \cdot A_k$ for $A_k \in X_r$. Next add the curves of X_{r-1}. Knowing, by assumption which curves intersect, knowing the weights in X_r and knowing from Theorem 3.10 the coefficient of $A_k \in X_{r-1}$ which appears in the fundamental cycle of each connected component R of $\cup A_\ell$, $A_\ell \in X_{r-1} \cup X_r$, we may compute the fundamental cycle of each connected component R (using a computation as in Proposition 3.1). Part (1) of Corollary 3.11 then determines $A_k \cdot A_k$ for $A_k \in X_{r-1}$. We next add the cycles in X_{r-2} and repeat the computation. In this way we work back to X_1 and determine $A_k \cdot A_k$ for all curves A_k.

It thus remains to algebraically determine which A_k intersect. Suppose A_i, $A_{j'} \in X_1$. Corollary 3.11 tells when $A_i \cap |D_{j'}| \neq \phi$. $|D_{j'}|$ consists of $A_{j'}$ and those components Y_ν of $\bigcup_{i>s} A_i$ such that $Y_\nu \cap A_{j'} \neq \phi$. However, Proposition 3.12 tells when A_i meets a component Y_ν. Thus we know which A_i in X_1 intersect and what components Y_ν a given A_i in X_1 meets. $A_k \in X_2$ corresponds to $\Gamma(Y_\nu, \mathcal{O}(-Z_\nu - A_k))/\Gamma(Y_\nu, \mathcal{O}(-2Z_\nu))$ by Theorem 3.10, which in turn corresponds to $\Gamma(A, \mathcal{O}(-2Z - Z_\nu - A_k))/\Gamma(A, \mathcal{O}(-2Z - 2Z_\nu))$ by Proposition 3.13, for some Y_ν such that $A_k \subseteq Y_\nu$. Let E_k be the least cycle E such that $A_j \cdot E \leqslant 0$ for all j and $E \geqslant 2Z + Z_\nu + A_k$. Then $E_k \leqslant 2Z + 2Z_\nu$ since $A_k \leqslant Z_\nu$ by the choice of ν. Thus $E_k - 2Z - Z_\nu$ does not involve any $A_i \in X_1$. In fact $E_k = 2Z + Z_\nu + D_k^{Y_\nu}$ where $|D_k^{Y_\nu}|$ consists of A_k and those components $Y_{\nu,\tau}$ of $\cup A_\ell$, $A_\ell \notin X_1 \cup X_2$ such that $Y_{\nu,\tau} \cap A_k \neq \phi$. E_k is obtained in a manner similar to Proposition 3.1: Add successively $B_1 = A_{i_1}$, $B_2 = A_{i_2}, \ldots$, such that $B_1 \cdot (2Z + Z_\nu + A_k) > 0$, $B_2 \cdot (2Z + Z_\nu + A_k + B_1) > 0, \ldots$. The process of adding the B's terminates at E_k. $\Gamma(A, \mathcal{O}(-2Z - Z_\nu - A_k - A_i)) = \Gamma(A, \mathcal{O}(-2Z - Z_\nu - D_\nu^{Y_k} - A_i))$ since the successive quotient spaces $\Gamma(A, \mathcal{O}(-2Z - Z_\nu - A_i - A_k - B_1 - \cdots - B_{\ell-1})/\mathcal{O}(-2Z - Z_\nu - A_i - A_k - B_1 - \cdots - B_\ell))$ correspond to sections of negative bundles and hence are trivial. In adding B's to $2Z + Z_\nu + A_k$, we may first add as many as possible of the A_ℓ, $\ell > s$, such that A_ℓ lies in some connected component $Y_{\nu,\tau}$ of $\cup A_\ell$, $A_\ell \notin X_1 \cup X_2$, where $Y_{\nu,\tau} \cap A_k \neq \phi$. Call this cycle E'_k. $E'_k - 2Z - Z_\nu$ is a Z_k for some Z_k used in the calculation of Z described in Proposition 3.1. In fact $E'_k = E_k$, for suppose B existed so that $B \cdot E'_k > 0$. $B \notin Y_{\nu,\tau}$ for any $Y_{\nu,\tau}$ such that $Y_{\nu,\tau} \cap A_k \neq \phi$ by our construction of E'_k. For $A_j \in X_1 \cup X_2$, $A_j \cdot (2Z + Z_\nu) < 0$ by Theorem 3.2. $1 \leqslant A_j \cdot E'_k = A_j \cdot [2Z + Z_\nu + (E'_k - (2Z + Z_\nu))]$ implies $A \cdot (E'_k - 2Z - Z_\nu) \geqslant 1 - A_j \cdot (2Z + Z_\nu) \geqslant 2$ which, by Theorem 3.2, contradicts the rationality of p. Thus $E_k = E'_k = 2Z + Z_\nu + D_k^{Y_\nu}$ as claimed. Consider the sheaf exact sequence

$$0 \to \frac{\mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu})}{\mathcal{O}(-2Z - Z_\nu - A_i - D_k^{Y_\nu})} \to \frac{\mathcal{O}(-2Z - Z_\nu)}{\mathcal{O}(-2Z - Z_\nu - A_i - D_k^{Y_\nu})} \to \frac{\mathcal{O}(-2Z - Z_\nu)}{\mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu})} \to 0.$$

Since $H^1(A, \mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu})) = 0$ by Theorem 3.4, we have the following short exact sequence

$$0 \to \Gamma\left(A, \frac{\mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu})}{\mathcal{O}(-2Z - Z_\nu - A_i - D_k^{Y_\nu})}\right) \to \Gamma\left(A, \frac{\mathcal{O}(-2Z - Z_\nu)}{\mathcal{O}(-2Z - Z_\nu - A_i - D_k^{Y_\nu})}\right) \to \Gamma\left(A, \frac{\mathcal{O}(-2Z - Z_\nu)}{\mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu})}\right) \to 0. \tag{3.5}$$

As $H^1(A, \mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu} - A_i)) = 0$, $H^1(A, \mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu})) = 0$ by Theorem 3.4, we have

$$\Gamma\left(A, \frac{\mathcal{O}(-2Z - Z_\nu)}{\mathcal{O}(-2Z - Z_\nu - A_i - D_k^{Y_\nu})}\right) = \frac{\Gamma(A, \mathcal{O}(-2Z - Z_\nu))}{\Gamma(A, \mathcal{O}(-2Z - Z_\nu - A_i - D_k^{Y_\nu})} \tag{3.6}$$

$$\Gamma\left(A, \frac{\mathcal{O}(-2Z - Z_\nu)}{\mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu})}\right) = \frac{\Gamma(A, \mathcal{O}(-2Z - Z_\nu))}{\Gamma(A, \mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu}))} = \frac{\Gamma(A, \mathcal{O}(-2Z - Z_\nu))}{\Gamma(A, \mathcal{O}(-2Z - Z_\nu - A_k))}. \tag{3.7}$$

In view of (3.5), (3.6) and (3.7), we have

$$\dim \frac{\Gamma(A, \mathcal{O}(-2Z - Z_\nu))}{\Gamma(A, \mathcal{O}(-2Z - Z_\nu - A_i - D_k^{Y_\nu})} = \dim \Gamma\left(A, \frac{\mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu})}{\mathcal{O}(-2Z - Z_\nu - D_k^{Y_\nu} - A_i)}\right)$$

$$+\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_k))}$$

$$=-A_i\cdot(2Z+Z_\nu)+\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_k))}-A_i\cdot D_k^{Y_\nu}$$

$$=\dim\Gamma\left(A,\frac{\mathcal{O}(-2Z-Z_\nu)}{\mathcal{O}(-2Z-Z_\nu-A_i)}\right)$$

$$+\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_k))}-A_i\cdot D_k^{Y_\nu}$$

$$=\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_i))}$$

$$+\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_k))}-A_i\cdot D_k^{Y_\nu}.$$

Therefore

$$\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_i))\cap\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_k))}$$

$$=\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_i-A_k))}$$

$$=\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_i-D_k^{Y_\nu}))}$$

$$=\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_i))}$$

$$+\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_k))}-A_i\cdot D_k^{Y_\nu}. \tag{3.8}$$

It is clear from (3.8) that

$$\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_i))\cap\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_k))}$$

$$<\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_i))}+\dim\frac{\Gamma(A,\mathcal{O}(-2Z-Z_\nu))}{\Gamma(A,\mathcal{O}(-2Z-Z_\nu-A_k))} \tag{3.9}$$

if and only if $A_i\cdot D_k^{Y_\nu}>0$; if and only if $A_i\in X_1$ will meet either $A_k\in X_2$ or some connected component $Y_{\nu,\tau}$ in $\cup A_\ell$, $A_\ell\notin X_1\cup X_2$, such that $Y_{\nu,\tau}\cap A_k\neq\phi$.

In X_3, we have similar considerations in $\Gamma(A,\mathcal{O}(-4Z-2Z_\nu-Z_{\nu,\tau}))$ for appropriate $Z_{\nu,\tau}=Z(Y_{\nu,\tau})$ where $Y_{\nu,\tau}$ is a component of $\cup A_\ell$, $A_\ell\subseteq Y_\nu$ but

$A_\ell \notin X_2$. Thus our final step is to algebraically distinguish, for example $U_i = \Gamma(A, \mathcal{O}(-4Z - 2Z_\nu - Z_{\nu,\tau} - A_i))$ for $A_i \in X_1$. No Z_ν or $Z_{\nu,\tau}$ involves an $A_j \in X_1$. Recall that in view of Proposition 3.12, for $k \geqslant s$, $W_k + m^2 = \Gamma(A, \mathcal{O}(-A_k - Z)) = \Gamma(A, \mathcal{O}(-Z_\nu - Z))$ where $Z_\nu = Z(Y_\nu)$ and $A_k \subseteq Y_\nu$. For any $A_i \in X_1$, $-A_i \cdot (Z_\nu + Z) \geqslant -1 - A_i \cdot Z \geqslant 0$, so $\Gamma(A, \mathcal{O}(-Z_\nu - Z)/\mathcal{O}(-A_i - Z_\nu - Z))$ is nontrivial. By Theorem 3.5, $H^1(A, \mathcal{O}(-A_i - Z_\nu - Z)) = 0$. Hence the map $\Gamma(A, \mathcal{O}(-Z_\nu - Z)) \to \Gamma(A, \mathcal{O}(-Z_\nu - Z)/\mathcal{O}(-A_i - Z_\nu - Z))$ is surjective. Since $W_k + m^2$ is closed under linear combination, $W_k + m^2$ contains functions that vanish to exactly order a_i on A_i, $A_i \in X_1$. Thus the U_i, $1 \leqslant i \leqslant s$, are characterized by being maximal subspaces of $\Gamma(A, \mathcal{O}(-Z_{\nu,\tau} - 2Z_\nu - 4Z))$ such that

$$U_1^{a_1} \dots U_s^{a_s} (W_{s+1} + m^2)^{e_{s+1}} \dots (W_{r+s} + m^2)^{e_{r+s}} \subseteq m^e, \tag{3.10}$$

where

$$e = 4a_1 + \cdots + 4a_s + e_{s+1} + \cdots + e_{r+s} + 1$$

the e_k may be arbitrarily large and $(a_1, \dots, a_s)$ are the minimal possible exponents for $U_1, \dots, U_s$.

To see this, we observe that by Theorem 3.5, $U_1^{a_1} \dots U_s^{a_s}\ (W_{s+1} + m^2)^{e_{s+1}} \dots (W_{r+s} + m^2)^{e_{r+s}} = \Gamma(A, \mathcal{O}(-\sum_{i=1}^{s} a_i(4Z + 2Z_\nu + Z_{\nu,\tau} + A_i) - \sum_{\mu=1}^{r} e_{\mu+s}(Z_{\mu+s} + Z))$ and $m^e = \Gamma(A, \mathcal{O}(-eZ))$. Therefore (3.8) holds if and only if $\sum_{i=1}^{s} a_i(4Z + 2Z_\nu + Z_{\nu,\tau} + A_i) + \sum_{\mu=1}^{r} e_{\mu+s}(Z_{\mu+s} + Z) \geqslant (4a_1 + \cdots + 4a_s + e_{s+1} + \cdots + e_{r+s} + 1)Z$ which, in turn, is equivalent to $\sum_{i=1}^{s} a_i(A_i + 2Z_\nu + Z_{\nu,\tau}) + \sum_{\mu=1}^{r} e_{\mu+s} Z_{\mu+s} \geqslant Z$. Since the support of $Z - \sum_{i=1}^{s} a_i(A_i + 2Z_\nu + Z_{\nu,\tau})$ is contained in $\bigcup_{i>s} A_i$, $e_{s+1}, \dots, e_{r+s}$ can be found and may be arbitrarily large such that the above inequality holds. It is also clear that $(a_1, \dots, a_s)$ are the minimal possible exponents for $U_1, \dots, U_s$.

If a subspace T of $\Gamma(A, \mathcal{O}(-4Z - 2Z_\nu - Z_{\nu,\tau}))$ appeared on the left side of (3.8) and had a function $f \in T$, with $f \notin \Gamma(A, \mathcal{O}(-4Z - 2Z_\nu - Z_{\nu,\tau} - A_i))$ for all $A_i \in X_1$, then f would vanish to exactly order $-4a_i$ on all $A_i \in X_1$. Then the exponent for T could be set equal to 0. If $T \not\subseteq \Gamma(A, \mathcal{O}(-4Z - 2Z_\nu - Z_{\nu,\tau} - A_i))$ for all $A_i \in X_1$, then there would exist an $f \in T$ with $f \notin \Gamma(A, \mathcal{O}(-4Z - 2Z_\nu - Z_{\nu,\tau} - A_i))$ for all $A_i \in X_1$ since T is closed under linear combinations. Since such an f cannot exist, $T \subseteq \Gamma(A, \mathcal{O}(-4Z - 2Z_\nu - Z_{\nu,\tau} - A_i))$ for some $A_i \in X_1$.

We may get a crude estimate for $e_{s+1}, \dots, e_n$ by considering all possible ways that the A_k can intersect, then determining, as described previously, the possible weighted graph. We can then determine all the cycles used in the computation of (3.8) and take the maximum of the needed $e_{s+1}, \dots, e_n$.

Summarizing all of the above results gives the following.

THEOREM 3.16. *Let p be a rational singularity and m the ideal of p. There exists an explicit algorithm to compute the weighted dual graph of the minimal resolution of p in terms of the ring structure of $\bigoplus_{n=0}^{\infty} m^n/m^{n+1}$.*

4. Determination of the graphs of rational CR manifolds

Let X be a connected compact strongly pseudoconvex CR manifold of real dimension 3 which is embeddable in $\mathbb{C}^n$. Let V be the subvariety in $\mathbb{C}^n$ such that the boundary of V is X and V has isolated singularities at $Y = \{p_1, \ldots, p_m\}$. Let $\tilde{V}$ be the normalization of V. Note that $\tilde{V}$ may not be in $\mathbb{C}^n$. Choose N large enough so that $\tilde{V}$ is embeddable in $\mathbb{C}^N$. Let $\tilde{Y} = \{q_1, \ldots, q_r\}$ be the normal singularities of $\tilde{V}$.

LEMMA 4.1. *The algebra of* CR *functions on X is isomorphic to the algebra of holomorphic functions on $\tilde{V}$.*

Proof. By the strong pseudoconvexity of $X = \partial\tilde{V}$ and the normality of $\tilde{V}$, one easily sees that CR functions on X extend to holomorphic functions on $\tilde{V}$. The natural map from the algbera of CR functions on X to the algebra of holomorphic functions on $\tilde{V}$ is an isomorphism because of the uniqueness of the extension. □

In view of Lemma 4.1, the analytic spectrum of the algebra of CR functions on X is $\tilde{V}$ because $\tilde{V}$ is a strongly pseudoconvex analytic space. Therefore, to compute the graph Γ_X, we only need to apply our theory developed in Section 3 to the singularities $(\tilde{V}, q_1), \ldots, (\tilde{V}, q_r)$. The following example illustrates how our theory works.

EXAMPLE. Let us consider the 3-dimensional compact connected CR manifold $X = \{(x, y, z) \in \mathbb{C}^3 : |x|^2 + |y|^2 + |z|^2 = 1, xy - z^6 = 0\}$. X bounds the complex variety $V = \{(x, y, z) \in \mathbb{C}^3 : xy - z^6 = 0\}$ with isolated singularity at the origin. It is not difficult to show that holomorphic two forms on $V - \{0\}$ are of the form $h \cdot \omega$, where h is a holomorphic function on V and ω is of the following form

$$\frac{dx \wedge dy}{\frac{\partial f}{\partial z}} = \frac{dy \wedge dz}{\frac{\partial f}{\partial x}} = \frac{dz \wedge dx}{\frac{\partial f}{\partial y}}, \quad f = xy - z^6.$$

One can check that ω is a L^2-integrable holomorphic 2-forms on $V - \{0\}$. By Proposition 2.5, we conclude that $p_g(X)$ is zero. So X is a rational CR manifold.

Let $m = (x, y, z)\mathbb{C}\{x, y, z\}/(xy - z^6)\mathbb{C}\{x, y, z\}$. Then

$$\begin{aligned} m^k &= \frac{(x, y, z)^k \mathbb{C}\{x, y, z\}}{(xy - z^6)(x, y, z)^{k-2}\mathbb{C}\{x, y, z\}}, \quad k \geqslant 2, \\ m/m^2 &= \frac{(x, y, z)\mathbb{C}\{x, y, z\}}{[(xy - z^6)\mathbb{C}\{x, y, z\} + (x, y, z)^2\mathbb{C}\{x, y, z\}]} \\ &= \frac{(x, y, z)\mathbb{C}\{x, y, z\}}{(x, y, z)^2\mathbb{C}\{x, y, z\}} = \langle x, y, z\rangle, \end{aligned}$$

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$$\begin{aligned} m^2/m^3 &= \frac{(x,y,z)^2\mathbb{C}\{x,y,z\}}{[(xy-z^6)\mathbb{C}\{x,y,z\}+(x,y,z)^3\mathbb{C}\{x,y,z\}]} \\ &= \langle x^2,y^2,xy,yz,z^2\rangle. \end{aligned}$$

Recall that by Theorem 3.10, if distinguished subspaces of m/m^2 exist, then maximal distinguished subspaces $S_1,\dots,S_t$ of m/m^2 exist and are unique. Each S_j corresponds to $W_j=\Gamma(A,\mathcal{O}(-Z-A_j))/\Gamma(A,\mathcal{O}(-2Z))$ for an A_j such that $A_j\cdot Z<0$. b_i in Definition 3.4 is a_i for $1\leqslant i\leqslant t$. $a_1+\cdots+a_t+1\leqslant -Z\cdot Z+1=\dim m/m^2=3$ implies that $a_1+\cdots+a_t\leqslant 2$. So there exist at most two maximal distinguished subspaces. By Corollary 3.11, codim $S_j=-A_j\cdot Z+1\geqslant 2$. Since dim $m/m^2=3$ and S_j is a nontrivial subspace of m/m^2, we conclude that dim $S_j=1$. Let

$$S_1=(x)+m^2 \quad \text{and} \quad S_2=(y)+m^2.$$

Then

$$\begin{aligned} S_1S_2 &= (xy)+(x,y)m^2+m^4=(z^6)+(x,y)m^2+m^4 \\ &\equiv 0 \ \text{ in } \ m^2/m^3. \end{aligned}$$

We have found two curves A_1, A_2 in the exceptional set such that $A_1\cdot Z<0$ and $A_2\cdot Z<0$. Moreover, $a_1=a_2=1$. All the other curves A_j, $j\geqslant 3$, if they exist, must have the property that $A_j\cdot Z=0$. Since $3=\operatorname{codim} S_1\cap S_2<\operatorname{codim} S_1+\operatorname{codim} S_2=4$, in view of Corollary 3.11, we know that A_1 and A_2 can be joined by cycles A_i such that $A_i\cdot Z=0$. We now must determine the graded ring structure for the singularities of the Y_ν, the connected components of $\bigcup_{i>2}A_i$, so that we can apply Theorem 3.10 and Corollary 3.11 to find more of the curves in the resolution.

Let $W_3=(x,y)+m^2$. Then we claim that

$$((x,y)+m^2)m^3\subset m((x)+m^2)((y)+m^2).$$

The L.H.S. is $(x,y)m^3+m^5$ while the R.H.S. is $(xy)m+(x,y)m^3+m^5$. So the above inclusion is clear. Therefore by (4) of Proposition 3.12, there is only one connected component Y_3 of $\bigcup_{j\geqslant 3}A_j$. In view of (3) of Proposition 3.12, we know that $A_1\cap Y_3\neq\phi$ and $A_2\cap Y_3\neq\phi$. By the proof of Corollary 3.14, we know that the graded ring structure for the singularity q_3 of Y_3 is $\bigoplus_{k=0}^{\infty}m_3^k/m_3^{k+1}$, where

$$\begin{aligned} \frac{m_3^k}{m_3^{k+1}} &\cong \frac{m^k(W_3+m^2)^k}{m^{k-1}(W_3+m^2)^{k+1}}, \quad k\geqslant 1, \\ \frac{m_3}{m_3^2} &\cong \frac{m((x,y)+m^2)}{((x,y)+m^2)^2}=\frac{(xy)m+m^3}{(x,y)^2+(x,y)m^2+m^4}=\langle xz,yz,z^3,\rangle, \end{aligned}$$

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$$\frac{m_3^2}{m_3^3} \cong \frac{m^2((x,y)+m^2)^2}{m((x,y)+m^2)^3} = \frac{(x,y)^2m^2+(x,y)m^3+m^6}{(x,y)^3m+(x,y)^2m^3+(x,y)m^5+m^7},$$
$$\cong \langle x^2z^2, y^2z^2, xz^3, yz^3, z^6\rangle \quad ((xz)(yz)\equiv 0 \text{ in } m_3^2/m_3^3).$$

We see that there are two distinguished subspaces

$$W_{(3),1} = (xz)+m_3^2, \qquad W_{(3),2} = (yz)+m_3^2$$

in m_3/m_3^2 and $((xz)+m_3^2)((yz)+m_3^2) \equiv 0$ in m_3^2/m_3^3. We have found two curves $A_{(3),1}, A_{(3),2}$ in the exceptional set of q_3 such that $A_{(3),1}\cdot Z_3 < 0$ and $A_{(3),2}\cdot Z_3 < 0$. Moreover, the coefficients of $A_{(3),1}$ and $A_{(3),2}$ in Z_3 are one. All the other curves $A_{(3),j}$, $j \geqslant 3$, if they exist, must have the property that $A_{(3),j}\cdot Z_3 = 0$. Since $3 = \operatorname{codim} W_{(3),1}\cap W_{(3),2} < \operatorname{codim} W_{(3),1} + \operatorname{codim} W_{(3),2} = 4$, in view of Corollary 3.11, we know that $A_{(3),1}$ and $A_{(3),2}$ can be joined by cycles $A_{(3),i}$ such that $A_{(3),i}\cdot Z_3 = 0$.

We now must determine the graded ring structure for the singularities of $Y_{(3),\nu}$, the connected components of $\bigcup_{i>2} A_{(3),i}$, so that we can apply Theorem 3.10 and Corollary 3.11 to find more of the curves in the resolution.

Let $W_{(3),3} = (zx, zy) + m_3^2$. Then we claim that

$$(W_{(3),3}+m_3^2)m_3^3 \subset m_3((zx)+m_3^2)((zy)+m_3^2)\ .$$

The L.H.S. is $(zx,zy)m_3^3+m_3^5$ while the R.H.S. is $(zx)(zy)m_3+(zx,zy)m_3^2+m_3^5$. So the above inclusion is clear. Therefore by (4) of Proposition 3.12, there is only one connected component $Y_{(3),3}$ of $\bigcup_{j\geqslant 3} A_{(3),j}$. In view of (3) of Proposition 3.12, we know that $A_{(3),1}\cap Y_{(3),3} \neq \phi$ and $A_{(3),2}\cap Y_{(3),3}\neq\phi$. By the proof of Corollary 3.14, we know that the graded ring structure for the singularity $q_{(3),3}$ of $Y_{(3),3}$ is $\bigoplus_{k=0}^\infty m_{(3),3}^k/m_{(3),3}^{k+1}$ where

$$\frac{m_{(3),3}^k}{m_{(3),3}^{k+1}} \cong \frac{m_3^k(W_{(3),3}+m_3^2)^k}{m_3^{k-1}(W_{(3),3}+m_m^2)^{k+1}}, \quad k\geqslant 1,$$

$$\frac{m_{(3),3}}{m_{(3),3}^2} \cong \frac{m_3(W_{(3),3}+m_3^2)}{(W_{(3),3}+m_3^2)^2} \approx \frac{m_3(zx,zy)+m_3^3}{(zx,zy)^2+(zx,zy)m_3^2+m_3^4}$$
$$\cong \frac{(zx,zy)m_3/m_3^4+m_3^3/m_3^4}{(zx,zy)^2\mathcal{O}_3/m_3^4+(zx,zy)m_3^2/m_3^4},$$

where $\mathcal{O}_3$ is the local ring of the singularity q_3. By the proof of Corollary 3.14, we have

$$\frac{m_3^3}{m_3^4} \cong \frac{m^3(W_3+m^2)^3}{m^2(W_3+m^2)^4}, \qquad \frac{m_3}{m_3^4} \cong \frac{m^5(W_3+m^2)}{m^2(W_3+m^2)^4},$$

$$\frac{\mathcal{O}_3}{m_3^4} \cong \frac{m^6}{m^2(W_3+m^2)^4}, \qquad \frac{m_3^2}{m_3^4} \cong \frac{m^4(W_3+m^2)^2}{m^2(W_3+m^2)^4}.$$

Therefore

$$\begin{aligned}
\frac{m_{(3),3}}{m_{(3),3}^2} &\cong \frac{(zx,zy)m^3(W_3+m^2)+m^2(W_3+m^2)^3}{(zx,zy)^2m^2+(zx,zy)m^2(W_3+m^2)^2+m^2(W_3+m^2)^4}\\
&\cong \{(zx,zy)(x,y)m^3+(zx,zy)m^5+(x,y)^3m^3+(x,y)^2m^5\\
&\quad +(x,y)m^7+m^9\}/\{(zx,zy)^2m^2\\
&\quad +(zx,zy)(x,y)^2m^2+(zx,zy)(x,y)m^4+(zx,zy)m^6\\
&\quad +(x,y)^4m^2+(x,y)^3m^4+(x,y)^2m^6+(x,y)m^8+m^{10}\}\\
&= \langle z^6x, z^6y, z^9\rangle,\\
\frac{m_{(3),3}^2}{m_{(3),3}^3} &\cong \frac{m_3^2(W_{(3),3}+m_3^2)^2}{m_3(W_{(3),3}+m_3^2)^3}\\
&= \frac{(zx,zy)^2m_3^2+(zx,zy)m_3^4+m_3^6}{(zx,zy)^3m^3+(zx,zy)^2m_3^3+(zx,zy)m_3^5+m_3^7}\\
&\cong \frac{(zx,zy)^2m_3^2/m_3^7+(zx,zy)m_3^4/m_3^7+m_3^6/m_3^7}{(zx,zy)^3m_3/m_3^7+(zx,zy)^2m_3^3/m_3^7+(zx,zy)m_3^5/m_3^7}.
\end{aligned}$$

By the proof of Corollary 3.14, we have

$$\frac{m_3^6}{m_3^7} \cong \frac{m^6(W_3+m^2)^6}{m^5(W_3+m^2)^7}, \qquad \frac{m_3^4}{m_3^7} \cong \frac{m^8(W_3+m^2)^4}{m^5(W_3+m^2)^7},$$

$$\frac{m_3^2}{m_3^7} \cong \frac{m^{10}(W_3+m^2)^2}{m^5(W_3+m^2)^7}, \qquad \frac{m_3^5}{m_3^7} \cong \frac{m^7(W_3+m^2)^5}{m^5(W_3+m^2)^7},$$

$$\frac{m_3^3}{m_3^7} \cong \frac{m^9(W_3+m^2)^3}{m^5(W_3+m^2)^7}, \qquad \frac{m_3}{m_3^7} \cong \frac{m^{11}(W_3+m^2)}{m^5(W_3+m^2)^7}.$$

Therefore

$$\begin{aligned}
&\frac{m_{(3),3}^2}{m_{(3),3}^3}\\
&\quad\cong \frac{\begin{array}{c}(zx,zy)^2m^6(W_3+m^2)^2\\ +(zx,zy)m^6(W_3+m^2)^4+m^6(W_3+m^2)^6\end{array}}{\begin{array}{c}(zx,zy)^3m^5(W_3+m^2)+(zx,zy)^2m^5(W_3+m^2)^3\\ +(zx,zy)m^5(W_3+m^2)^5+m^5(W_3+m^2)^7\end{array}}
\end{aligned}$$

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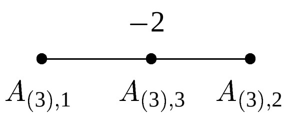
$$
\begin{aligned}
&\cong \{(zx, zy)^2(x, y)^2 m^6 + (zx, zy)^2(x, y)m^8 + (zx, zy)^2 m^{10} \\
&\quad +(zx, zy)(x, y)^4 m^6 + (zx, zy)(x, y)^3 m^8 \\
&\quad +(zx, zy)(x, y)^2 m^{10} + (zx, zy)(x, y)m^{12} + (zx, zy)m^{14} \\
&\quad +(x, y)^6 m^6 + (x, y)^5 m^8 + (x, y)^4 m^{10} + (x, y)^3 m^{12} \\
&\quad +(x, y)^2 m^{14} + (x, y)m^{16} + m^{18}\}/\{(zx, zy)^3(x, y)m^5 + (zx, zy)^3 m^7 \\
&\quad +(zx, zy)^2(x, y)^3 m^5 + (zx, zy)^2(x, y)^2 m^7 \\
&\quad +(zx, zy)^2(x, y)m^9 + (zx, zy)^2 m^{11} \\
&\quad +(zx, zy)(x, y)^5 m^5 + (zx, zy)(x, y)^4 m^7 \\
&\quad +(zx, zy)(x, y)^3 m^9 + (zx, zy)(x, y)^2 m^{11} \\
&\quad +(zx, zy)(x, y)m^{13} + (zx, zy)m^{15} + (x, y)^7 m^5 \\
&\quad +(x, y)^6 m^7 + (x, y)^5 m^9 + (x, y)^4 m^{11} + (x, y)^3 m^{13} \\
&\quad +(x, y)^2 m^{15} + (x, y)m^{17} + m^{19}\} \\
&\cong \langle z^{12}x^2, z^{12}y^2, z^{15}x, z^{15}y, z^{18}\rangle \quad ((z^6x)(x^6y) = z^{18}).
\end{aligned}
$$

It follows that $m_{(3),3}/m^2_{(3),3}$ has no distinguished subspaces. By Corollary 3.8, the minimal resolution of $q_{(3),3}$ has just one curve $A_{(3),3}$ and $-A^2_{(3),3} + 1 = \dim m_{(3),3}/m^2_{(3),3} = 3$. So $A^2_{(3),3} = -2$.

By our previous discussion, we know that the graph of Y_3 looks like the following



$A_{(3),1}$ does not intersect $A_{(3),2}$ because there is no cycle in a rational singularity graph. We also know that

$$Z_3 = A_{(3),1} + A_{(3),3} + A_{(3),2}.$$

In view of (1) of Corollary 3.11, we have

$$-A_{(3),1} \cdot Z_3 + 1 = \operatorname{codim} W_{(3),1} = 2$$

which implies

$$-A^2_{(3),1} = 2 \quad \text{i.e.} \quad A^2_{(3),1} = -2.$$

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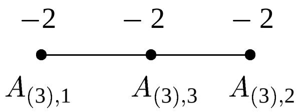
Similarly, we conclude that $A^2_{(3),2} = -2$. So the graph of Y_3 is given by



We now want to determine the intersection properties of A_1 and A_2 with Y_3. According to the proof of Theorem 3.15, we need to characterize $U_1 := \Gamma(A, \mathcal{O}(-2Z - Z_3 - A_1))$ and $U_2 := \Gamma(A, \mathcal{O}(-2Z - Z_3 - A_2))$. They are characterized by being maximal subspaces of $\Gamma(A, \mathcal{O}(-2Z - Z_3)) = mW_3 = (x, y)m + m^3$ such that

$$U_1 U_2 (W_3 + m^2)^{e_3} \subset m^{2+2+e_3+1}, \tag{3.11}$$

where e_3 may be arbitrarily large. Let $U_1 = (x^2, xz) + m^3$ and $U_2 = (y^2, yz) + m^3$. Then

$$\begin{aligned} &U_1 U_2 (W_3 + m^2)^{e_3} \\ &\quad = [(xy)(x, z)(y, z) + (x^2, xz, y^2, yz)m^3 + m^6](W_3 + m^2)^{e_3} \\ &\quad = [(z^6)(x, z)(y, z) + (x^2, xz, y^2, yz)m^3 + m^6](W_3 + m^2)^{e_3} \subset m^{5+e_3}. \end{aligned}$$

We claim that U_1 and U_2 are maximal subspaces of $\Gamma(A, \mathcal{O}(-2Z - Z_3))$ such that (3.11) holds. We need to estimate $\Gamma(A, \mathcal{O}(-2Z - Z_3))/\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_i))$ $i = 1, 2$. In view of Theorem 3.4, we have, for $i = 1, 2$,

$$\frac{\Gamma(A, \mathcal{O}(-2Z - Z_3))}{\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_i))} = \Gamma\left(\frac{A, \mathcal{O}(-2Z - Z_3)}{\mathcal{O}(-2Z - Z_3 - A_i)}\right).$$

The Chern class of the line bundle corresponding to $\mathcal{O}(-2Z - Z_3)/\mathcal{O}(-2Z - Z_3 - A_i)$ is given by $-A_i \cdot (2Z + Z_3) = -2A_i \cdot Z - 1 \geqslant 1$. Therefore dim $\Gamma(A, \mathcal{O}(-2Z - Z_3))/\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_i)) \geqslant 1 + 1 = 2$. So $\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_i))$ is a subspace of codimension at least two in $\Gamma(A, \mathcal{O}(-2Z - Z_3))$. On the other hand, $U_1 = x(x, z) + m^3$, $U_2 = y(y, z) + m^3$ are exactly codimension 2 subspaces in $\Gamma(A, \mathcal{O}(-2Z - Z_3)) = (x, y)m + m^3$. So our claim is proved.

Recall that $W_{(3),1} = \Gamma(Y_3, \mathcal{O}\ (-Z_3 - A_{(3),1})/\mathcal{O}\ (-2Z_3))$ and $W_{(3),2} = \Gamma(Y_3, \mathcal{O}\ (-Z_3 - A_{(3),2})/\mathcal{O}\ (-2Z_3))$ are maximal subspaces in $m_3/m_3^2 = \Gamma(Y_3, \mathcal{O}\ (-Z_3)/\mathcal{O}\ (-2Z_3))$, such that $W_{(3),1} \cdot W_{(3),2} \equiv 0$ in $m_3/m_3^2 = \Gamma\ (Y_3, \mathcal{O}\ (-2Z_3)/\mathcal{O}\ (-3Z_3))$. By the proof of Corollary 3.14, we have

$$\begin{aligned} &\Gamma\left(\frac{Y_3, \mathcal{O}(-Z_3 - A_{(3),1})}{\mathcal{O}(-2Z_3)}\right) \\ &\quad \cong \Gamma\left(A, \frac{\mathcal{O}(-2Z - Z_3 - A_{(3),1})}{\mathcal{O}(-2Z - 2Z_3)}\right) \end{aligned}$$

$$\cong \frac{\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),1}))}{\Gamma(A, \mathcal{O}(-2Z - 2Z_3))},$$

$$\Gamma\left(Y_3, \frac{\mathcal{O}(-Z_3 - A_{(3),2})}{\mathcal{O}(-2Z_3)}\right)$$

$$\cong \Gamma\left(A, \frac{\mathcal{O}(-2Z - Z_3 - A_{(3),2})}{\mathcal{O}(-2Z - 2Z_3)}\right)$$

$$\cong \frac{\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),2}))}{\Gamma(A, \mathcal{O}(-2Z - 2Z_3))}$$

and

$$\Gamma\left(Y_3, \frac{\mathcal{O}(-2Z_3)}{\mathcal{O}(-3Z_3)}\right) \cong \Gamma\left(A, \frac{\mathcal{O}(-4Z - 2Z_3)}{\mathcal{O}(-4Z - 3Z_3)}\right)$$

$$\cong \frac{\Gamma(A, \mathcal{O}(-4Z - 2Z_3))}{\Gamma(A, \mathcal{O}(-4Z - 3Z_3))}$$

$$\cong \frac{m^2(W_3 + m^2)^2}{m(W_3 + m^2)^3}.$$

So $\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),1}))$ and $\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),2}))$ are the greatest subspaces of $\Gamma(A, \mathcal{O}(-2Z - Z_3)) = m(W_3 + m^2)$ such that

$$\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),1})) \cdot \Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),2}))$$
$$\subset m(W_3 + m^2)^3. \tag{3.12}$$

We claim that

$$\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),1}))$$
$$= (x^2, y^2, yz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4,$$
$$\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),2}))$$
$$= (x^2, y^2, xz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4.$$

It is easy to check that

$$[(x^2, y^2, yz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4]$$
$$\times[(x^2, y^2, xz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4]$$
$$\subset (x, y)^3m + (x, y)^2m^3 + (x, y)m^5 + m^7.$$

So (3.12) holds. We now estimate $\Gamma(A, \mathcal{O}(-2Z-Z_3)/\mathcal{O}(-2Z-Z_3-A_{(3),i}))$, $i = 1, 2$. In view of Theorem 3.4, we have for $i = 1, 2$, $\Gamma(A, \mathcal{O}(-2Z-Z_3)/\mathcal{O}(-2Z-Z_3-A_{(3),i})) = \Gamma(A, \mathcal{O}(-2Z-Z_3)/\Gamma(A, \mathcal{O}(-2Z-Z_3-A_{(3),i}))$. The Chern class of the line bundle corresponding to $\mathcal{O}(-2Z-Z_3)/\mathcal{O}(-2Z-Z_3-A_{(3),i})$ is given by $-A_{(3),i}\cdot(2Z+Z_3) = -A_{(3),i}\cdot Z_3 = 1$. Therefore dim $\Gamma(A, \mathcal{O}(-2Z-Z_3))/\Gamma(A, \mathcal{O}(-2Z-Z_3- A_{(3),i})) = 1+1 = 2$. So $\Gamma(A, \mathcal{O}\ (-2Z-Z_3-A_{(3),i}))$ is a subspace of codimension two in $\Gamma(A, \mathcal{O}(-2Z-Z_3))$. On the other hand, $(x^2, y^2, yz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4$ and $(x^2, y^2, xz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4$ are exactly codimension two subspaces in $\Gamma(A, \mathcal{O}(-2Z-Z_3)) = m(W_3+m^2) = (x, y)m + m^3$. So our claim is proved.

$$\dim \frac{\Gamma(A, \mathcal{O}(-2Z-Z_3))}{\Gamma(A, \mathcal{O}(-2Z-Z_3-A_1)) \cap \Gamma(A, \mathcal{O}(-2Z-Z_3-A_{(3),1}))}$$

$$= \dim \frac{m(W_3+m^2)}{[(x^2, xz)+m^3] \cap [(x^2, y^2, yz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4]}$$

$$= \dim \frac{(x, y)m + m^3}{(x^2) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4} = 4,$$

$$\dim \frac{\Gamma(A, \mathcal{O}(-2Z-Z_3))}{\Gamma(A, \mathcal{O}(-2Z-Z_3-A_1))} = \frac{(x, y)m + m^3}{(x^2, xz) + m^3} = 2,$$

$$\dim \frac{\Gamma(A, \mathcal{O}(-2Z-Z_3))}{\Gamma(A, \mathcal{O}(-2Z-Z_3-A_{(3),1}))}$$

$$= \frac{(x, y)m + m^3}{(x^2, y^2, yz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4} = 2.$$

Hence

$$\dim \frac{\Gamma(A, \mathcal{O}(-2Z-Z_3))}{\Gamma(A, \mathcal{O}(-2Z-Z_3-A_1)) \cap \Gamma(A, \mathcal{O}(-2Z-Z_3-A_{(3),1}))}$$

$$= \dim \frac{\Gamma(A, \mathcal{O}(-2Z-Z_3))}{\Gamma(A, \mathcal{O}(-2Z-Z_3-A_1))} + \dim \frac{\Gamma(A, \mathcal{O}(-2Z-Z_3))}{\Gamma(A, \mathcal{O}(-2Z-Z_3-A_{(3),1}))}.$$

By the proof of Theorem 3.15, we conclude that $A_1 \cap A_{(3),1} = \phi$ and $A_1 \cap A_{(3),3} = \phi$.

$$\dim \frac{\Gamma(A, \mathcal{O}(-2Z-Z_3))}{\Gamma(A, \mathcal{O}(-2Z-Z_3-A_1)) \cap \Gamma(A, \mathcal{O}(-2Z-Z_3-A_{(3),2}))}$$

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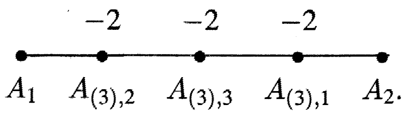
$$= \dim \frac{m(W_3 + m^2)}{[(x^2, xz) + m^3] \cap [(x^2, y^2, xz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4]}$$

$$= \dim \frac{(x, y)m + m^3}{(x^2, xz) + (x^3, x^2z, xz^2, y^3, y^2z, yz^2) + m^4} = 3 < 2 + 2$$

$$= \dim \frac{\Gamma(A, \mathcal{O}(-2Z - Z_3))}{\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_1))} + \dim \frac{\Gamma(A, \mathcal{O}(-2Z - Z_3))}{\Gamma(A, \mathcal{O}(-2Z - Z_3 - A_{(3),2}))}.$$

By the proof of Theorem 3.15, we conclude that $A_1 \cap A_{(3),2} \neq \phi$ or $A_1 \cap A_{(3),3} \neq \phi$. Since we already know $A_1 \cap A_{(3),3} = \phi$, we conclude that $A_1 \cap A_{(3),2} \neq \phi$. Similarly, we can conclude that $A_2 \cap A_{(3),1} \neq \phi$. So the graph of the singularity looks like



We also know that

$$Z = A_1 + A_{(3),2} + A_{(3),3} + A_{(3),1} + A_2.$$

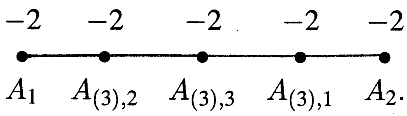
In view of (1) of Corollary 3.11, we have

$$-A_1 \cdot Z + 1 = \operatorname{codim} W_1 = 2$$

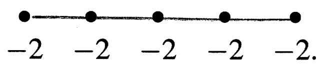
which implies

$$-A_1^2 = 2 \quad \text{i.e.} \quad A_1^2 = -2.$$

Similarly, we can deduce that $A_2^2 = -2$. So the complete weighted dual graph is



Thus, the graph Γ_X is



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