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Pieri rules for skew dual immaculate functions

Elizabeth Niese, Sheila Sundaram, Stephanie van Willigenburg, and Shiyun Wang

Abstract. In this paper, we give Pieri rules for skew dual immaculate functions and their recently discovered row-strict counterparts. We establish our rules using a right-action analogue of the skew Littlewood–Richardson rule for Hopf algebras of Lam–Lauve–Sottile. We also obtain Pieri rules for row-strict (dual) immaculate functions.

1 Introduction

Schur-like functions are a new and flourishing area since the discovery of quasisymmetric Schur functions in 2011 [11], which led to numerous other similar functions being discovered, for example, [1, 4, 6, 10, 14–17]. In essence, Schur-like functions are functions that refine the ubiquitous Schur functions and reflect many of their properties, such as their combinatorics [2, 9], their representation theory [5, 7, 21, 22], and in the case of quasisymmetric Schur functions have already been applied to resolve conjectures [13]. Of the various Schur-like functions to arise after the quasisymmetric Schur functions, two were naturally related to them: the dual immaculate functions [6] and the row-strict quasisymmetric Schur functions [17]. Recently, a fourth basis that interpolates between these latter two bases, the *row-strict dual immaculate functions*, was discovered [19], thus completing the picture. The representation theory of these functions was revealed in [20], in addition to the fundamental combinatorics in [19]. In this paper, we extend the combinatorics to uncover skew Pieri rules in the spirit of [3, 12, 23] for both row-strict and classical dual immaculate functions.

More precisely, our paper is structured as follows. In Section 2, we establish a right-action analogue of [12, Theorem 2.1] in Theorem 2.6. We then recall required background for the Hopf algebras of quasisymmetric functions, QSym, and noncommutative symmetric functions, NSym, in Section 3. Finally, in Section 4, we give (left) Pieri rules for row-strict immaculate functions and row-strict dual immaculate functions in Corollaries 4.3 and 4.5, respectively. Our final theorem is Theorem 4.7,

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in which we establish Pieri rules for skew dual immaculate functions, and row-strict skew dual immaculate functions.

2 The right-action skew Littlewood–Richardson rule for Hopf algebras

We begin by recalling and deducing general Hopf algebra results that will be useful later. Following Tewari and van Willigenburg [23], let H and H^* be a pair of dual Hopf algebras over a field k with duality pairing $\langle\ ,\ \rangle : H \otimes H^* \to k$ for which the structure of H^* is dual to that of H and vice versa. Let $h \in H, a \in H^*$. By Sweedler notation, we have coproduct denoted by $\Delta h = \sum h_1 \otimes h_2$, and similarly $h_1 h_2 = h_1 \cdot h_2$ denotes product. We define the action of one algebra on the other one by the following:

$$h \rightharpoonup a = \sum \langle h, a_2 \rangle a_1, \tag{2.1}$$

$$a \rightharpoonup h = \sum \langle h_2, a \rangle h_1. \tag{2.2}$$

Let $S : H \to H$ denote the antipode map. Then for $\Delta h = \sum h_1 \otimes h_2$,

$$\sum (Sh_1)h_2 = \varepsilon(h)1_H = \sum h_1(Sh_2), \tag{2.3}$$

where ε and 1 denote counit and unit, respectively. Following Montgomery [18], we can define the convolution product $*$ for f and g in H by

$$(f * g)(a) = \sum \langle f, a_1 \rangle \langle g, a_2 \rangle = \langle fg, a \rangle.$$

Then it follows that

$$\langle g, f \rightharpoonup a \rangle = \langle gf, a \rangle.$$

Similarly, $\langle a \rightharpoonup f, b \rangle = \langle f, ba \rangle$. Since H^* is a left H-module algebra under $\rightharpoonup$, we have that

$$h \rightharpoonup (a \cdot b) = \sum (h_1 \rightharpoonup a) \cdot (h_2 \rightharpoonup b).$$

Lemma 2.1 [12] *For $g, h \in H$ and $a \in H^*$,*

$$(a \rightharpoonup g) \cdot h = \sum (S(h_2) \rightharpoonup a) \rightharpoonup (g \cdot h_1),$$

where $S : H \to H$ is the antipode.

As in Montgomery [18], define a right action by the following:

$$h \leftharpoonup a = \sum \langle h, a_1 \rangle a_2, \tag{2.4}$$

$$a \leftharpoonup h = \sum \langle h_1, a \rangle h_2. \tag{2.5}$$

As before, it follows that $\langle g, f \leftharpoonup a \rangle = \langle fg, a \rangle$ and $\langle a \leftharpoonup f, b \rangle = \langle f, ab \rangle$.

Lemma 2.2 *Let $f \in H$ and $a, b \in H^*$. Then*

$$f \leftharpoonup (a \cdot b) = \sum (f_1 \leftharpoonup a) \cdot (f_2 \leftharpoonup b).$$

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Proof Let $f, g \in H$ and $a, b \in H^*$. Then

$$\begin{aligned}\langle g, f \leftharpoondown (a \cdot b)\rangle &= \langle fg, ab\rangle \\ &= \langle a \leftharpoondown (fg), b\rangle \\ &= \sum \langle f_1 g_1, a\rangle \langle f_2 g_2, b\rangle \\ &= \sum \langle g_1, f_1 \leftharpoondown a\rangle \langle g_2, f_2 \leftharpoondown b\rangle \\ &= \sum \langle g, (f_1 \leftharpoondown a) \cdot (f_2 \leftharpoondown b)\rangle.\end{aligned}$$

Thus, $f \leftharpoondown (a \cdot b) = \sum (f_1 \leftharpoondown a) \cdot (f_2 \leftharpoondown b)$. ■

Lemma 2.3 *Let $a \in H^*$. Then*

$$\varepsilon(h) \cdot 1_H \leftharpoondown a = a$$

for any $h \in H$.

Proof Let $a \in H^*$ and $h \in H$. Then

$$\varepsilon(h) \cdot 1_H \leftharpoondown a = \sum \langle \varepsilon(h) \cdot 1_H, a_1\rangle a_2.$$

This is only nonzero when $a_1 = 1_{H^*}$. ■

Lemma 2.4 *Let $h \in H$ and $a, b \in H^*$. Then*

$$a \cdot (h \leftharpoondown b) = \sum h_1 \leftharpoondown ((S(h_2) \leftharpoondown a) \cdot b).$$

Proof Expand the sum using Lemma 2.2 and coassociativity, $(\Delta \otimes 1) \circ \Delta(h) = (1 \otimes \Delta) \circ \Delta(h) = \sum h_1 \otimes h_2 \otimes h_3$, to get

$$\begin{aligned}\sum h_1 \leftharpoondown ((S(h_2) \leftharpoondown a) \cdot b) &= \sum (h_1 \leftharpoondown (S(h_2) \leftharpoondown a)) \cdot (h_3 \leftharpoondown b) \\ &= \sum (h_1 \cdot S(h_2) \leftharpoondown a) \cdot (h_3 \leftharpoondown b) \text{ since } H^* \text{ is an } H\text{-module} \\ &= ((\varepsilon(h) \cdot 1_H) \leftharpoondown a) \cdot (h \leftharpoondown b) \text{ by (2.3)} \\ &= a \cdot (h \leftharpoondown b) \text{ by Lemma 2.3.}\end{aligned}$$ ■

Lemma 2.5 *Let $g, h \in H$ and $a \in H^*$. Then*

$$h \cdot (a \leftharpoondown g) = \sum (S(h_2) \leftharpoondown a) \leftharpoondown (h_1 \cdot g).$$

Proof Let $g, h \in H$ and $a, b \in H^*$. Then

$$\begin{aligned}\langle h \cdot (a \leftharpoondown g), b\rangle &= \langle a \leftharpoondown g, h \leftharpoondown b\rangle \\ &= \langle g, a \cdot (h \leftharpoondown b)\rangle \\ &= \left\langle g, \sum (h_1 \leftharpoondown ((S(h_2) \leftharpoondown a) \cdot b)\right\rangle \text{ by Lemma 2.4} \\ &= \sum \langle g, h_1 \leftharpoondown ((S(h_2) \leftharpoondown a) \cdot b)\rangle \\ &= \sum \langle h_1 \cdot g, (S(h_2) \leftharpoondown a) \cdot b\rangle \\ &= \sum \langle (S(h_2) \leftharpoondown a) \leftharpoondown (h_1 \cdot g), b\rangle.\end{aligned}$$ ■

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We can use the right action to obtain an algebraic Littlewood–Richardson formula analogous to [12, Theorem 2.1] for those bases whose skew elements appear as the right tensor factor in the coproduct.

Let $\{L_\alpha\} \subset H$ and $\{R_\beta\} \subset H^*$ be dual bases with indexing set $\mathcal{P}$. Then

$$L_\alpha \cdot L_\beta = \sum_\gamma b^\gamma_{\alpha,\beta} L_\gamma \qquad \Delta(L_\gamma) = \sum_{\alpha,\beta} c^\gamma_{\alpha,\beta} L_\alpha \otimes L_\beta, \tag{2.6}$$

$$R_\alpha \cdot R_\beta = \sum_\gamma c^\gamma_{\alpha,\beta} R_\gamma \qquad \Delta(R_\gamma) = \sum_{\alpha,\beta} b^\gamma_{\alpha,\beta} R_\alpha \otimes R_\beta, \tag{2.7}$$

where $b^\gamma_{\alpha,\beta}$ and $c^\gamma_{\alpha,\beta}$ are structure constants. We can also write

$$\Delta(L_\gamma) = \sum_\delta L_\delta \otimes L_{\gamma/\delta} \qquad \Delta(R_\gamma) = \sum_\delta R_\delta \otimes R_{\gamma/\delta}. \tag{2.8}$$

Note that $L_\alpha \leftharpoondown R_\beta = R_{\beta/\alpha}$ and $R_\beta \leftharpoondown L_\alpha = L_{\alpha/\beta}$. Further,

$$\Delta(L_{\alpha/\beta}) = \sum_{\pi,\rho} c^\alpha_{\pi,\rho,\beta} L_\pi \otimes L_\rho \qquad \Delta(R_{\alpha/\beta}) = \sum_{\pi,\rho} b^\alpha_{\pi,\rho,\beta} R_\pi \otimes R_\rho. \tag{2.9}$$

The antipode acts on L_ρ by $S(L_\rho) = (-1)^{\theta(\rho)} L_{\rho^*}$ where $\theta : \mathcal{P} \to \mathbb{N}$ and $* : \mathcal{P} \to \mathcal{P}$.

Theorem 2.6 *For $\alpha, \beta, \gamma, \delta \in \mathcal{P}$,*

$$L_{\alpha/\beta} \cdot L_{\gamma/\delta} = \sum_{\pi,\rho,\nu,\mu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} b^\delta_{\mu,\rho^*} L_{\nu/\mu}.$$

Proof We use Lemma 2.5 and the preceding facts about the product, coproduct, and antipode maps on H and H^* to obtain

$$\begin{aligned}
L_{\alpha/\beta} \cdot L_{\gamma/\delta} &= L_{\alpha/\beta} \cdot (R_\delta \leftharpoondown L_\gamma) \\
&= \sum_{\pi,\rho} c^\alpha_{\pi,\rho,\beta} (S(L_\rho) \leftharpoondown R_\delta) \leftharpoondown (L_\pi \cdot L_\gamma) \\
&= \sum_{\pi,\rho} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} (L_{\rho^*} \leftharpoondown R_\delta) \leftharpoondown (L_\pi \cdot L_\gamma) \\
&= \sum_{\pi,\rho} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} \left(R_{\delta/\rho^*} \leftharpoondown \left(\sum_\nu b^\nu_{\pi,\gamma} L_\nu \right) \right) \\
&= \sum_{\pi,\rho,\nu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} (R_{\delta/\rho^*} \leftharpoondown L_\nu) \\
&= \sum_{\pi,\rho,\nu,\mu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} b^\delta_{\mu,\rho^*} (R_\mu \leftharpoondown L_\nu) \\
&= \sum_{\pi,\rho,\nu,\mu} (-1)^{\theta(\rho)} c^\alpha_{\pi,\rho,\beta} b^\nu_{\pi,\gamma} b^\delta_{\mu,\rho^*} L_{\nu/\mu}.
\end{aligned}$$

∎

3 The dual Hopf algebras QSym and NSym

We now focus our attention on the dual Hopf algebra pair of noncommutative symmetric functions and quasisymmetric functions, and introduce our main objects of study the (row-strict) dual immaculate functions.

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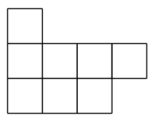
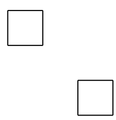
A *composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, denoted by $\alpha \vDash n$ is a list of positive integers such that $\sum_{i=1}^{k} \alpha_i = n$. We call n the *size* of α and sometimes denote it by $|\alpha|$, and call k the *length* of α and sometimes denote it by $\ell(\alpha)$. If $\alpha_{j_1} = \cdots = \alpha_{j_m} = i$, we sometimes abbreviate this to i^m, and denote the *empty composition* of 0 by $\varnothing$. There exists a natural correspondence between compositions $\alpha \vDash n$ and subsets $S \subseteq \{1, \ldots, n-1\} = [n-1]$. More precisely, $\alpha = (\alpha_1, \ldots, \alpha_k)$ corresponds to $\text{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$, and conversely $S = \{s_1, \ldots, s_{k-1}\}$ corresponds to $\text{comp}(S) = (s_1, s_2 - s_1, \ldots, n - s_{k-1})$. We also denote by S^c the set complement of S in $[n-1]$.

Given a composition α, its *diagram*, also denoted by α, is the array of left-justified boxes with α_i boxes in row i from the *bottom*. Given two compositions α, β we say that $\beta \subseteq \alpha$ if $\beta_j \leq \alpha_j$ for all $1 \leq j \leq \ell(\beta) \leq \ell(\alpha)$, and given α, β such that $\beta \subseteq \alpha$, the *skew diagram* α/β is the array of boxes in α but not β when β is placed in the bottom-left corner of α. If, furthermore, $\beta \subseteq \alpha$ and $\alpha_j - \beta_j \in \{0,1\}$ for all $1 \leq j \leq \ell(\beta) \leq \ell(\alpha)$, then we call α/β a *vertical strip*.

Example 3.1 If $\alpha = (3, 4, 1)$, then $|\alpha| = 8, \ell(\alpha) = 3$, and $\text{set}(\alpha) = \{3, 7\}$. Its diagram is

$\alpha =$ 

and if $\beta = (2, 4)$, then

$\alpha/\beta =$ 

is a vertical strip.

Definition 3.2 Given a composition α, a *standard immaculate tableau* T of *shape* α is a bijective filling of its diagram with $1, \ldots, |\alpha|$ such that:

(1) The entries in the leftmost column increase from bottom to top.
(2) The entries in each row increase from left to right.

We obtain a *standard skew immaculate tableau* of shape α/β by extending the definition to skew diagrams α/β in the natural way.

Given a standard (skew) immaculate tableau, T, its *descent set* is

$$\text{Des}(T) = \{i : i + 1 \text{ appears strictly above } i \text{ in } T\}.$$

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Example 3.3 A standard skew immaculate tableau of shape $(3,4,1)/(1)$ is

$T =$

7			
2	3	4	6
	1	5	

with $\mathrm{Des}(T) = \{1,5,6\}$.

We are now ready to define our Hopf algebras and functions of central interest.

Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$ and commuting variables $\{x_1, x_2, \ldots\}$ we define the *monomial quasisymmetric function* M_α to be

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$$

the *fundamental quasisymmetric function* F_α to be

$$F_\alpha = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow j \notin \mathrm{set}(\alpha)}} x_{i_1} \cdots x_{i_n}$$

the *dual immaculate function* $\mathfrak{S}^*_\alpha$ to be

$$\mathfrak{S}^*_\alpha = \sum_T F_{\mathrm{comp}(\mathrm{Des}(T))}$$

and the *row-strict dual immaculate function* $\mathcal{R}\mathfrak{S}^*_\alpha$ to be

$$\mathcal{R}\mathfrak{S}^*_\alpha = \sum_T F_{\mathrm{comp}(\mathrm{Des}(T)^c)},$$

where the latter two sums are over all standard immaculate tableaux T of shape α. These extend naturally to give *skew* dual immaculate and row-strict dual immaculate functions $\mathfrak{S}^*_{\alpha/\beta}$ [6] and $\mathcal{R}\mathfrak{S}^*_{\alpha/\beta}$[19], where α/β is a skew diagram.

Example 3.4 We have that $M_{(2)} = x_1^2 + x_2^2 + x_3^2 + \cdots$ and $F_{(2)} = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots = \mathfrak{S}^*_{(2)} = \mathcal{R}\mathfrak{S}^*_{(1^2)}$ from the following standard immaculate tableau T with $\mathrm{Des}(T) = \varnothing$.

$T =$

1	2

The set of all monomial or fundamental quasisymmetric functions forms a basis for the *Hopf algebra of quasisymmetric functions* QSym, as does the set of all (row-strict) dual immaculate functions. There exists an involutory automorphism ψ defined on fundamental quasisymmetric functions by

$$\psi(F_\alpha) = F_{\mathrm{comp}(\mathrm{set}(\alpha^c))}$$

such that [19]

$$\psi(\mathfrak{S}^*_\alpha) = \mathcal{R}\mathfrak{S}^*_\alpha$$

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for a composition α. This extends naturally to skew diagrams α/β to give

$$\psi(\mathfrak{S}^*_{\alpha/\beta}) = \mathcal{R}\mathfrak{S}^*_{\alpha/\beta}.$$

Dual to the Hopf algebra of quasisymmetric functions is the *Hopf algebra of noncommutative symmetric functions* NSym. Given a composition $\alpha = (\alpha_1, \dots, \alpha_k) \vDash n$ and *noncommuting variables* $\{y_1, y_2, \dots\}$ we define the *nth elementary noncommutative symmetric function* $\boldsymbol{e}_n$ to be

$$\boldsymbol{e}_n = \sum_{i_1 < \cdots < i_n} y_{i_1} \cdots y_{i_n}$$

and the *elementary noncommutative symmetric function* $\boldsymbol{e}_\alpha$ to be

$$\boldsymbol{e}_\alpha = \boldsymbol{e}_{\alpha_1} \cdots \boldsymbol{e}_{\alpha_k}.$$

Meanwhile, we define the *nth complete homogeneous noncommutative symmetric function* $\boldsymbol{h}_n$ to be

$$\boldsymbol{h}_n = \sum_{i_1 \leq \cdots \leq i_n} y_{i_1} \cdots y_{i_n}$$

and the *complete homogeneous noncommutative symmetric function* $\boldsymbol{h}_\alpha$ to be

$$\boldsymbol{h}_\alpha = \boldsymbol{h}_{\alpha_1} \cdots \boldsymbol{h}_{\alpha_k}.$$

The set of all elementary or complete homogeneous noncommutative symmetric functions forms a basis for NSym. The duality between QSym and NSym is given by

$$\langle M_\alpha, \boldsymbol{h}_\alpha \rangle = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise. This induces the bases dual to the (row-strict) dual immaculate functions via

$$\langle \mathfrak{S}^*_\alpha, \mathfrak{S}_\alpha \rangle = \delta_{\alpha\beta} \qquad \langle \mathcal{R}\mathfrak{S}^*_\alpha, \mathcal{R}\mathfrak{S}_\alpha \rangle = \delta_{\alpha\beta}$$

and implicitly defines the bases of *immaculate* and *row-strict immaculate functions*. While concrete combinatorial definitions of these functions have been established [6, 19], we will not need them here. However, what we will need is the involutory automorphism in NSym corresponding to ψ in QSym, defined by $\psi(\boldsymbol{e}_\alpha) = \boldsymbol{h}_\alpha$ that gives $\psi(\mathfrak{S}_\alpha) = \mathcal{R}\mathfrak{S}_\alpha$ [19].

4 The Pieri rules for skew dual immaculate functions

A left Pieri rule for immaculate functions was conjectured in [6, Conjecture 3.7] and proved in [8]. Given a composition $\alpha = (\alpha_1, \dots, \alpha_k)$, we say that $\mathrm{tail}(\alpha) = (\alpha_2, \dots, \alpha_k)$. If $\beta \in \mathbb{Z}^k$, then $\mathrm{neg}(\alpha - \beta) = |\{i : \alpha_i - \beta_i < 0\}|$. Let $\mathrm{sgn}(\beta) = (-1)^{\mathrm{neg}(\beta)}$ with $\mathrm{neg}(\beta) = |\{i : \beta_i < 0\}|$.

Following [8], we define $Z_{s,\alpha}$ to be a set of all $\beta \in \mathbb{Z}^k$ such that:

(1) $\beta_1 + \cdots + \beta_k = s$ and $\beta_1 + \cdots + \beta_i \leq s$ for all $i < k$.
(2) $\alpha_i - \beta_i \geq 0$ for all $1 \leq i \leq k$ and $|i : \alpha_i - \beta_i = 0| \leq 1$.
(3) For all $1 \leq i \leq k$,

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- if $\alpha_i > s - (\beta_1 + \cdots + \beta_{i-1})$, then $0 \le \beta_i \le s - (\beta_1 + \cdots + \beta_{i-1})$,
- if $\alpha_i < s - (\beta_1 + \cdots + \beta_{i-1})$, then $\beta_i < 0$, and
- if $\alpha_i = s - (\beta_1 + \cdots + \beta_{i-1})$, then either $\beta_i < 0$ or $\beta_i = \alpha_i$ and $\beta_{i+1} = \cdots = \beta_k = 0$.

Now we are ready to define the coefficients of the immaculate basis appearing in the left Pieri rule.

Definition 4.1 [8] For a positive integer s and compositions α, γ with $|\alpha| - |\gamma| = s$, let $1 \le j \le k$ be the smallest integer such that $\alpha_i = \gamma_{i-1}$ for all $j < i \le k$ where $j = k$ when $\alpha_k \ne \gamma_{k-1}$. Let $j \le r \le k$ be the largest integer such that $\alpha_j < \alpha_{j+1} < \cdots < \alpha_r$. Let $\alpha^{(i)} = (\alpha_1, \ldots, \alpha_i)$. Then define

$$c^{\gamma}_{s,\alpha} = \begin{cases} \operatorname{sgn}(\alpha - \gamma), & \text{if } \ell(\gamma) = \ell(\alpha) \text{ and } \alpha - \gamma \in Z_{s,\alpha}, \\ \operatorname{sgn}(\alpha^{(j-1)} - \gamma^{(j-1)}), & \text{if } \ell(\gamma) = \ell(\alpha) - 1, \\ & r - j \text{ is even, and} \\ & (\alpha^{(j-1)} - \gamma^{(j-1)}, \alpha_j, 0, \ldots, 0) \in Z_{s,\alpha}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.2 [6, 8] *Let $m > 0$ and α be a composition. Then*

$$\boldsymbol{h}_m \mathfrak{S}_\alpha = \sum_{\substack{\beta \vDash |\alpha| + m \\ \beta_1 \ge m \\ 0 \le \ell(\beta) - \ell(\alpha) \le 1}} c^{\mathrm{tail}(\beta)}_{\beta_1 - m, \alpha} \mathfrak{S}_\beta.$$

Applying ψ to both sides of the left Pieri rule in Theorem 4.2 immediately yields a left Pieri rule for row-strict immaculate functions.

Corollary 4.3 *Let $m > 0$ and α be a composition. Then*

$$\boldsymbol{e}_m \mathcal{R}\mathfrak{S}_\alpha = \sum_{\substack{\beta \vDash |\alpha| + m \\ \beta_1 \ge m \\ 0 \le \ell(\beta) - \ell(\alpha) \le 1}} c^{\mathrm{tail}(\beta)}_{\beta_1 - m, \alpha} \mathcal{R}\mathfrak{S}_\beta.$$

Lemma 3.1 of [8] shows that for $s \ge 0$, $r > 0$ and compositions α, β with $|\alpha| = |\beta| + s$,

$$\langle \mathfrak{S}_\alpha, F_{(s)} \mathfrak{S}^*_\beta \rangle = \langle \boldsymbol{h}_r \mathfrak{S}_\alpha, \mathfrak{S}^*_{(s+r,\beta)} \rangle.$$

This leads to the following Pieri rule for dual immaculate functions.

Theorem 4.4 [8] *Let $s > 0$ and α be a composition. Then*

$$F_{(s)} \mathfrak{S}^*_\alpha = \sum_{\substack{\beta \vDash |\alpha| + s \\ 0 \le \ell(\beta) - \ell(\alpha) \le 1}} c^{\alpha}_{s,\beta} \mathfrak{S}^*_\beta.$$

Again, applying ψ to both sides gives a Pieri rule for row-strict dual immaculate functions.

Corollary 4.5 *Let $s > 0$ and α be a composition. Then*

$$F_{(1^s)}\mathcal{R}\mathfrak{S}^*_\alpha = \sum_{\substack{\beta\models|\alpha|+s\\ 0\le\ell(\beta)-\ell(\alpha)\le 1}} c^\alpha_{s,\beta}\mathcal{R}\mathfrak{S}^*_\beta.$$

We use these results together with Hopf algebra computations to construct a Pieri rule for skew dual immaculate functions. Using the map ψ, this also gives a Pieri rule for row-strict skew dual immaculate functions. But first, we have a small, yet crucial, lemma.

Lemma 4.6 *Let α and γ be compositions. Then $\mathfrak{S}_\gamma \leftharpoonup \mathfrak{S}^*_\alpha = \mathfrak{S}^*_{\alpha/\gamma}$.*

Proof Recall that if $H = \mathrm{QSym}$ and $H^* = \mathrm{NSym}$ are our pair of dual Hopf algebras, then we know $\Delta\mathfrak{S}^*_\alpha = \sum_\beta \mathfrak{S}^*_\beta \otimes \mathfrak{S}^*_{\alpha/\beta}$ and we have that

$$\mathfrak{S}_\gamma \leftharpoonup \mathfrak{S}^*_\alpha = \sum_\beta \langle \mathfrak{S}_\gamma, \mathfrak{S}^*_\beta\rangle \mathfrak{S}^*_{\alpha/\beta} = \mathfrak{S}^*_{\alpha/\gamma}$$

since $\langle \mathfrak{S}_\gamma, \mathfrak{S}^*_\beta\rangle = \delta_{\gamma\beta}$, where $\delta_{\gamma\beta} = 1$ if $\gamma = \beta$ and 0 otherwise. ■

We can now give our Pieri rule for (row-strict) skew dual immaculate functions.

Theorem 4.7 *Let $\gamma \subseteq \alpha$. Then*

$$\mathfrak{S}^*_{(s)}\mathfrak{S}^*_{\alpha/\gamma} = \sum_{\beta/\tau}(-1)^{|\gamma|-|\tau|}\cdot c^\alpha_{|\beta|-|\alpha|,\beta}\,\mathfrak{S}^*_{\beta/\tau},$$

and hence by applying ψ to both sides

$$\mathcal{R}\mathfrak{S}^*_{(s)}\mathcal{R}\mathfrak{S}^*_{\alpha/\gamma} = \sum_{\beta/\tau}(-1)^{|\gamma|-|\tau|}\cdot c^\alpha_{|\beta|-|\alpha|,\beta}\,\mathcal{R}\mathfrak{S}^*_{\beta/\tau},$$

where $|\beta/\tau| = |\alpha/\gamma| + s$, γ/τ is a vertical strip of length at most s, $\ell(\beta) - \ell(\alpha) \in \{0, 1\}$ and $c^\alpha_{|\beta|-|\alpha|,\beta}$ is the coefficient of Definition 4.1. These decompositions are multiplicity-free up to sign.

Proof Note that $\mathfrak{S}^*_{(1^s)} = F_{(1^s)}$ and $\mathfrak{S}^*_{(s)} = F_{(s)}$. Recall that

$$\Delta F_\alpha = \sum_{\substack{(\beta,\gamma)\text{ with}\\ \beta\cdot\gamma=\alpha\text{ or}\\ \beta\odot\gamma=\alpha}} F_\beta \otimes F_\gamma, \tag{4.1}$$

where for $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_l)$, $\beta\cdot\gamma = (\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_l)$ is the *concatenation* of β and γ, and $\beta \odot \gamma = (\beta_1, \ldots, \beta_{k-1}, \beta_k + \gamma_1, \gamma_2, \ldots, \gamma_l)$ is the *near-concatenation* of β and γ.

Then we have that

$$\Delta(F_{(s)}) = \sum_{i=0}^{s} F_{(i)} \otimes F_{(s-i)}.$$

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Thus,

$$\begin{aligned}\mathfrak{S}^*_{(s)}\mathfrak{S}^*_{\alpha/\gamma} &= \mathfrak{S}^*_{(s)}(\mathfrak{S}_\gamma \leftharpoondown \mathfrak{S}^*_\alpha) && \text{by Lemma 4.6}\\ &= F_{(s)}(\mathfrak{S}_\gamma \leftharpoondown \mathfrak{S}^*_\alpha)\\ &= \sum_{i=0}^{s}(S(F_{(s-i)}) \leftharpoondown \mathfrak{S}_\gamma) \leftharpoondown (F_{(i)}\mathfrak{S}^*_\alpha) && \text{by Lemma 2.5.}\end{aligned}$$

We first compute $S(F_{(s-i)}) \leftharpoondown \mathfrak{S}_\gamma$. Since it is well known that $S(F_\alpha) = (-1)^{|\alpha|}F_{\text{comp}(\text{set}(\alpha)^c)}$ we have that $S(F_{(s-i)}) = (-1)^{s-i}F_{(1^{s-i})}$. Furthermore, we can write the coproduct as

$$\Delta(\mathfrak{S}_\gamma) = \sum_{\delta,\tau} b^\gamma_{\delta,\tau}\mathfrak{S}_\delta \otimes \mathfrak{S}_\tau.$$

Thus,

$$\begin{aligned}S(F_{(s-i)}) \leftharpoondown \mathfrak{S}_\gamma &= (-1)^{s-i}F_{(1^{s-i})} \leftharpoondown \mathfrak{S}_\gamma\\ &= \sum_{\delta,\tau}(-1)^{s-i}b^\gamma_{\delta,\tau}\langle F_{(1^{s-i})}, \mathfrak{S}_\delta\rangle\mathfrak{S}_\tau\\ &= \sum_{\delta,\tau}(-1)^{s-i}b^\gamma_{\delta,\tau}\langle \mathfrak{S}^*_{(1^{s-i})}, \mathfrak{S}_\delta\rangle\mathfrak{S}_\tau\\ &= \sum_{\tau}(-1)^{s-i}b^\gamma_{(1^{s-i}),\tau}\mathfrak{S}_\tau.\end{aligned}$$

By the definition of product and coproduct on NSym, we have that

$$b^\gamma_{\delta,\tau} = \langle \Delta\mathfrak{S}_\gamma, \mathfrak{S}^*_\delta \otimes \mathfrak{S}^*_\tau\rangle = \langle \mathfrak{S}_\gamma, \mathfrak{S}^*_\delta \cdot \mathfrak{S}^*_\tau\rangle.$$

To compute this for $\delta = (1^{s-i})$, we use Proposition 3.34 from [6] which states that $F^\perp_{(1^r)}\mathfrak{S}_\alpha = \sum_\beta \mathfrak{S}_\beta$, where $\beta \in \mathbb{Z}^{\ell(\alpha)}$, $\alpha_k - \beta_k \in \{0,1\}$ for all k and $|\beta| = |\alpha| - r$. The operator $F^\perp$ is used throughout [6], and has the property that $\langle F^\perp\mathfrak{S}_\alpha, \mathfrak{S}^*_\beta\rangle = \langle \mathfrak{S}_\alpha, F\mathfrak{S}^*_\beta\rangle$.

Thus,

$$\begin{aligned}b^\gamma_{(1^{s-i}),\tau} &= \langle \mathfrak{S}_\gamma, \mathfrak{S}^*_{(1^{s-i})}\mathfrak{S}^*_\tau\rangle\\ &= \langle \mathfrak{S}_\gamma, F_{(1^{s-i})}\mathfrak{S}^*_\tau\rangle\\ &= \langle F^\perp_{(1^{s-i})}\mathfrak{S}_\gamma, \mathfrak{S}^*_\tau\rangle\\ &= \left\langle \sum_\beta \mathfrak{S}_\beta, \mathfrak{S}^*_\tau\right\rangle\\ &= \delta_{\beta\tau},\end{aligned}$$

where the sum is over all β such that $\beta \in \mathbb{Z}^{\ell(\gamma)}$, $\gamma_k - \beta_k \in \{0,1\}$ for all k, and $|\beta| = |\gamma| - (s-i)$.

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Then using the above calculations, Theorem 4.4 and Lemma 4.6, we have that

$$\begin{aligned}
\mathfrak{S}^*_{(s)}\mathfrak{S}^*_{\alpha/\gamma} &= \mathfrak{S}^*_{(s)}(\mathfrak{S}_\gamma \leftharpoondown \mathfrak{S}^*_\alpha)\\
&= \sum_{i=0}^{s}\left((S(F_{(s-i)}) \leftharpoondown \mathfrak{S}_\gamma) \leftharpoondown (F_{(i)}\mathfrak{S}^*_\alpha)\right)\\
&= \sum_{i=0}^{s}\left((-1)^{(s-i)} \sum_{\substack{\tau\in\mathbb{Z}^{\ell(\gamma)}\\ \gamma_k-\tau_k\in\{0,1\}\\ |\tau|=|\gamma|-(s-i)}} \mathfrak{S}_\tau\right) \leftharpoondown \left(\sum_{\substack{\beta\vDash|\alpha|+i\\ 0\le\ell(\beta)-\ell(\alpha)\le 1}} c^\alpha_{i,\beta}\mathfrak{S}^*_\beta\right)\\
&= \sum_{i=0}^{s} \sum_{\substack{\tau,\beta\\ \tau\in\mathbb{Z}^{\ell(\gamma)}\\ \gamma_k-\tau_k\in\{0,1\}\\ |\tau|=|\gamma|-(s-i)\\ \beta\vDash|\alpha|+i\\ \ell(\beta)-\ell(\alpha)\in\{0,1\}}} (-1)^{(s-i)}\cdot c^\alpha_{i,\beta}\,\mathfrak{S}^*_{\beta/\tau}\\
&= \sum_{\beta/\tau}(-1)^{|\gamma|-|\tau|}\cdot c^\alpha_{|\beta|-|\alpha|,\beta}\,\mathfrak{S}^*_{\beta/\tau},
\end{aligned}$$

where $|\beta/\tau| = |\alpha/\gamma| + s$, γ/τ is a vertical strip of length at most s, and $\ell(\beta) - \ell(\alpha) \in \{0,1\}$. ■

Example 4.8 Let us compute $\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)}$.

First, we need to compute all compositions $\beta \vDash 4 + i$ for $i \in \{0,1,2\}$ and $\ell(\beta) = 3$ or 4. We list all possible choices for β as the set

$$\begin{aligned}
A = \{&(1,1,1,1),(1,1,2),(1,2,1),(2,1,1),(1,1,1,2),(1,1,2,1),(1,2,1,1),\\
&(2,1,1,1),(1,1,3),(1,2,2),(1,3,1),(2,1,2),(2,2,1),(3,1,1),(1,1,1,3),\\
&(1,1,2,2),(1,1,3,1),(1,2,1,2),(1,2,2,1),(1,3,1,1),(2,1,1,2),\\
&(2,1,2,1),(2,2,1,1),(3,1,1,1),(1,1,4),(1,2,3),(1,3,2),(1,4,1),\\
&(2,1,3),(2,2,2),(2,3,1),(3,1,2),(3,2,1),(4,1,1)\}.
\end{aligned}$$

Next, we need to find τ by removing a vertical strip of length at most $s = 2$ from $\gamma = (1,1)$. We list all options for τ as the set $B = \{\varnothing, (1), (1,1)\}$.

By Theorem 4.7, now we expand $\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)}$ by finding all valid pairs (β,τ) such that $|\beta/\tau| = 4$. Thus,

$$\begin{aligned}
\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)} = {}& c^{(1,2,1)}_{0,(1,1,1,1)}\mathfrak{S}^*_{(1,1,1,1)} + c^{(1,2,1)}_{0,(1,1,2)}\mathfrak{S}^*_{(1,1,2)}\\
&+ c^{(1,2,1)}_{0,(1,2,1)}\mathfrak{S}^*_{(1,2,1)} + c^{(1,2,1)}_{0,(2,1,1)}\mathfrak{S}^*_{(2,1,1)}\\
&- c^{(1,2,1)}_{1,(1,1,1,2)}\mathfrak{S}^*_{(1,1,1,2)/(1)} - c^{(1,2,1)}_{1,(1,1,2,1)}\mathfrak{S}^*_{(1,1,2,1)/(1)}\\
&- c^{(1,2,1)}_{1,(1,2,1,1)}\mathfrak{S}^*_{(1,2,1,1)/(1)} - c^{(1,2,1)}_{1,(2,1,1,1)}\mathfrak{S}^*_{(2,1,1,1)/(1)}\\
&- c^{(1,2,1)}_{1,(1,1,3)}\mathfrak{S}^*_{(1,1,3)/(1)} - c^{(1,2,1)}_{1,(1,2,2)}\mathfrak{S}^*_{(1,2,2)/(1)}
\end{aligned}$$

$$
\begin{aligned}
&- c^{(1,2,1)}_{1,(1,3,1)} \mathfrak{S}^*_{(1,3,1)/(1)} - c^{(1,2,1)}_{1,(2,1,2)} \mathfrak{S}^*_{(2,1,2)/(1)}\\
&- c^{(1,2,1)}_{1,(2,2,1)} \mathfrak{S}^*_{(2,2,1)/(1)} - c^{(1,2,1)}_{1,(3,1,1)} \mathfrak{S}^*_{(3,1,1)/(1)}\\
&+ c^{(1,2,1)}_{2,(1,1,1,3)} \mathfrak{S}^*_{(1,1,1,3)/(1,1)} + c^{(1,2,1)}_{2,(1,1,2,2)} \mathfrak{S}^*_{(1,1,2,2)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(1,1,3,1)} \mathfrak{S}^*_{(1,1,3,1)/(1,1)} + c^{(1,2,1)}_{2,(1,2,1,2)} \mathfrak{S}^*_{(1,2,1,2)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(1,2,2,1)} \mathfrak{S}^*_{(1,2,2,1)/(1,1)} + c^{(1,2,1)}_{2,(1,3,1,1)} \mathfrak{S}^*_{(1,3,1,1)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(2,1,1,2)} \mathfrak{S}^*_{(2,1,1,2)/(1,1)} + c^{(1,2,1)}_{2,(2,1,2,1)} \mathfrak{S}^*_{(2,1,2,1)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(2,2,1,1)} \mathfrak{S}^*_{(2,2,1,1)/(1,1)} + c^{(1,2,1)}_{2,(3,1,1,1)} \mathfrak{S}^*_{(3,1,1,1)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(1,1,4)} \mathfrak{S}^*_{(1,1,4)/(1,1)} + c^{(1,2,1)}_{2,(1,2,3)} \mathfrak{S}^*_{(1,2,3)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(1,3,2)} \mathfrak{S}^*_{(1,3,2)/(1,1)} + c^{(1,2,1)}_{2,(1,4,1)} \mathfrak{S}^*_{(1,4,1)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(2,1,3)} \mathfrak{S}^*_{(2,1,3)/(1,1)} + c^{(1,2,1)}_{2,(2,2,2)} \mathfrak{S}^*_{(2,2,2)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(2,3,1)} \mathfrak{S}^*_{(2,3,1)/(1,1)} + c^{(1,2,1)}_{2,(3,1,2)} \mathfrak{S}^*_{(3,1,2)/(1,1)}\\
&+ c^{(1,2,1)}_{2,(3,2,1)} \mathfrak{S}^*_{(3,2,1)/(1,1)} + c^{(1,2,1)}_{2,(4,1,1)} \mathfrak{S}^*_{(4,1,1)/(1,1)}.
\end{aligned}
$$

We can compute all the coefficients $c^{\alpha}_{|\beta|-|\alpha|,\beta}$ using Definition 4.1, and most of them turn out to be zero. Hence, we have the following expansion after simplification:

$$
\begin{aligned}
\mathfrak{S}^*_{(2)} \cdot \mathfrak{S}^*_{(1,2,1)/(1,1)} = \mathfrak{S}^*_{(1,2,1)} - \mathfrak{S}^*_{(1,1,2,1)/(1)} - \mathfrak{S}^*_{(2,2,1)/(1)} + \mathfrak{S}^*_{(2,1,2,1)/(1,1)}\\
+ \mathfrak{S}^*_{(3,2,1)/(1,1)}.
\end{aligned}
$$

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Department of Mathematics and Physics, Marshall University, Huntington, WV 25755, United States
e-mail: elizabeth.niese@gmail.com

School of Mathematics, University of Minnesota–Twin Cities, Minneapolis, MN 55455, United States,
e-mail: shsund@umn.edu wang8406@umn.edu

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
e-mail: steph@math.ubc.ca