FACTOR IDEALS OF SOME REPRESENTATION ALGEBRAS

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Throughout this paper $\mathscr{F}$ is an algebraically closed field of characteristic p ($\neq 0$) and $\mathscr{G}$ is a finite group whose order is divisible by p. We define in the usual way an $\mathscr{F}$-representation of $\mathscr{G}$ (or $\mathscr{F}\mathscr{G}$-representation) and its corresponding module. The isomorphism class of the $\mathscr{F}\mathscr{G}$-representation module $\mathscr{M}$ is written $\{\mathscr{M}\}$ or, where no confusion arises, M. $A(\mathscr{G})$ denotes the $\mathscr{F}$-representation algebra of $\mathscr{G}$ over the complex field $\mathscr{C}$ (as defined on pages 73 and 82 of [6]).

J. A. Green [6] and S. B. Conlon [5] have shown that $A(\mathscr{G})$ is semisimple if and only if the algebras $W_{\mathscr{D}}(\mathscr{G})$ (see Section 1) are all semisimple, where $\mathscr{D}$ runs over the set of distinct (to within $\mathscr{G}$-conjugacy) p-subgroups of $\mathscr{G}$.

The semisimplicity of $W_{\mathscr{D}}(\mathscr{G})$ is known to hold when $\mathscr{D}$ is cyclic and in the special case where $p = 2$ and $\mathscr{D}$ is a sylow subgroup of $\mathscr{G}$ and is isomorphic to the elementary abelian group on two generators $\mathscr{V}_4$. I have considered the case where $\mathscr{D}$ is this group but the sylow condition does not hold; it is shown that $W_{\mathscr{D}}(\mathscr{G})$ is semisimple when $\mathscr{G}$ is the alternating group on six or seven symbols ($\mathscr{A}_6$ or $\mathscr{A}_7$). It is also shown that $A(\mathscr{A}_6)$ is semisimple if and only if $A(\mathscr{A}_7)$ is. Use is made of Conlon's results [3] on $A(\mathscr{V}_4)$ and $A(\mathscr{A}_4)$.

Section 1 of the paper is introductory. In the second section certain properties of alternating groups are indicated; the semisimplicity of $A(\mathscr{A}_5)$ is proven and those groups $\mathscr{P}$ for which $W_{\mathscr{P}}(\mathscr{A}_6)$ is isomorphic to $W_{\mathscr{P}}(\mathscr{A}_7)$ are listed. The remainder of the paper is concerned with the semisimplicity of $W_{\mathscr{V}_4}(\mathscr{A}_k)$ for $k = 6$ or 7. Certain results derived in Section 3 concerning the decomposition of induced modules are applied in the next section to the $\mathscr{V}_4$-projective $\mathscr{N}_6$ and $\mathscr{N}_7$-modules ($\mathscr{N}_k$ is the $\mathscr{A}_k$-normalizer of $\mathscr{V}_4$); the generators of $A_{\mathscr{V}_4}(\mathscr{N}_k)$ modulo the projective ideal are calculated. In the final sections the structures of $W_{\mathscr{V}_4}(\mathscr{N}_6)$ and $W_{\mathscr{V}_4}(\mathscr{N}_7)$ are found; semisimplicity follows.

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1. Properties of representation algebras

We shall say an algebra $\mathscr{A}$ over the complex field $\mathscr{C}$ is semisimple[1] if, for every non-zero element A of $\mathscr{A}$, there is an algebra homomorphism $\phi_A : \mathscr{A} \to \mathscr{C}$, with $A\phi_A \neq 0$.[2] If the radical of $\mathscr{A}$ is defined as the intersection $\cap \mathscr{M}$ of the maximal ideals $\mathscr{M}$ of $\mathscr{A}$, such that $\mathscr{A}/\mathscr{M} \simeq \mathscr{C}$, then a necessary and sufficient condition for the semisimplicity of $\mathscr{A}$ is that $\mathscr{A}$ should have zero radical.[3]

If $A(\mathscr{G})$ is as defined above, and $\mathscr{D} \leqq \mathscr{G}$, we define $A_{\mathscr{D}}(\mathscr{G})$ and $W_{\mathscr{D}}(\mathscr{G})$ as in [6]. We write $W_{\mathscr{I}}(\mathscr{G}) = A_{\mathscr{I}}(\mathscr{G})$ where $\mathscr{I}$ is the group with one element. It should be noted that $W_{\mathscr{D}}(\mathscr{G})$ is trivial unless $\mathscr{D}$ is a p-group.

If $\mathscr{H}$ is a subgroup of $\mathscr{G}$, then the following results hold:

(1) Theorem [5]. *$A_{\mathscr{H}}(\mathscr{G}) \cong \oplus W_{\mathscr{D}}(\mathscr{G})$, where $\oplus$ is the algebra direct sum over all the non-$\mathscr{G}$-conjugate p-subgroups $\mathscr{D}$ of $\mathscr{G}$ with $\mathscr{D} \leqq_{\mathscr{G}} \mathscr{H}$.*

(2) Corollary. *$A_{\mathscr{H}}(\mathscr{G})$ is semisimple if and only if each $W_{\mathscr{D}}(\mathscr{G})$ is.*

It follows from (1) and (2) that to discuss semisimplicity we need only discuss the semisimplicity of the algebras W. We make use of the following result.

(3) Theorem [6, p. 81]. *If $\mathscr{D} \leqq \mathscr{N} \leqq \mathscr{H} \leqq \mathscr{G}$, where $\mathscr{N}$ is the $\mathscr{G}$-normalizer of $\mathscr{D}$, then*

$$W_{\mathscr{D}}(\mathscr{H}) \cong W_{\mathscr{D}}(\mathscr{G}).$$

In other words we need only consider the normalizer of $\mathscr{D}$ in $\mathscr{G}$, rather than $\mathscr{G}$.

$W_{\mathscr{I}}(\mathscr{G})$ is always semisimple; in fact as $\mathscr{F}$ is algebraically closed it is the direct sum of r copies of $\mathscr{C}$ where r is the number of p-regular conjugacy classes in $\mathscr{G}$ [3, p. 85].

Finally we note that there is no loss of generality in considering $\mathscr{F}$ algebraically closed. If it were not, then let $\mathscr{F}^*$ be the closure of $\mathscr{F}$, and write $A^*(\mathscr{G})$, $W^*_{\mathscr{D}}(\mathscr{G})$ for the algebras derived in this case. Then from p. 80 of [4] we see that the semisimplicity of A^* (or W^*) ensures that of A (or W).

2. Alternating groups

If we take $\mathscr{A}_r$ as a permutation group on the first r positive integers in the usual way we can write $\mathscr{A}_{r-1} < \mathscr{A}_r$, where $\mathscr{A}_{r-1}$ is obtained by omitting all elements of $\mathscr{A}_r$ which properly permute one specific integer.

[1] Also called G-semisimple [3, p. 84].

[2] All mappings except permutations and representations are written on the right.

[3] See p. 84 of [3].

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Suppose $\mathscr{D} \leqq \mathscr{A}_{r-1}$ and suppose the normalizer $\mathscr{N}(\mathscr{D}; \mathscr{A}_r)$ of $\mathscr{D}$ in $\mathscr{A}_r$ satisfies $\mathscr{N}(\mathscr{D}; \mathscr{A}_r) \leqq \mathscr{A}_{r-1}$. Then, by (3),

$$W_{\mathscr{D}}(\mathscr{A}_r) \cong W_{\mathscr{D}}(\mathscr{A}_{r-1}).$$

This sort of result interests us because it provides a link between $A(\mathscr{A}_r)$ and $A(\mathscr{A}_{r-1})$, so we will investigate when these conditions hold. To do this we introduce the idea of a permutation acting on an integer.

We say the permutation P *acts on* the integer i if, when P is considered as a mapping of the integers, $P(i) \neq i$, and a subgroup $\mathscr{P} < \mathscr{A}_r$ acts on i if there is a member of $\mathscr{P}$ which acts on i. Thus the positive integers $\leqq r$ are divided into two disjoint classes: $\alpha_r(\mathscr{P})$, the set of integers acted on by $\mathscr{P}$, and $\beta_r(\mathscr{P})$, the set of integers invariant under $\mathscr{P}$. Then it is easy to see that $\mathscr{N}(\mathscr{P}; \mathscr{A}_r)$ consists those even permutations N of the form $N = N_1 N_2$, where $N_1 \in \mathscr{N}(\mathscr{P}; \mathscr{S}_a)$ and $N_2 \in \mathscr{S}_{r-a}$, a being the order of $\alpha_r(\mathscr{P})$, $\mathscr{S}_a$ the symmetric group on the elements of $\alpha_r(\mathscr{P})$, and $\mathscr{S}_{r-a}$ the symmetric group on the elements of $\beta_r(\mathscr{P})$.

If $a = r-1$ then $\mathscr{S}_{r-a}$ consists the identity element only, and in this case $\mathscr{N}(\mathscr{P}; \mathscr{A}_r) = \mathscr{N}(\mathscr{P}; \mathscr{A}_{r-1}) \leqq \mathscr{A}_{r-1}$. This is the condition we wanted.

(4) THEOREM. *If $\mathscr{P}$ is a subgroup of $\mathscr{A}_{pn+1}$ which acts on pn symbols then*

$$\mathscr{N}(\mathscr{P}; \mathscr{A}_{pn+1}) = \mathscr{N}(\mathscr{P}; \mathscr{A}_{pn})$$

and so

$$W_{\mathscr{P}}(\mathscr{A}_{pn+1}) \cong W_{\mathscr{P}}(\mathscr{A}_{pn}).$$

($\mathscr{A}_{pn}$ is the alternating group on the symbols acted on by $\mathscr{P}$).

We have considered $\mathscr{P}$ acting on np symbols because of the following result:

(5) LEMMA. *If $\mathscr{P}$ is a permutation group of order p^m, where p is prime, then $\mathscr{P}$ acts on exactly pn symbols for some integer n.*

PROOF. Consider the orbit $\mathscr{P}(i)$ of the symbol i under $\mathscr{P}$.

$$\mathscr{P}(i) = \{j : P(i) = j \text{ for some } P \in \mathscr{P}\}.$$

Then, by theorem 3.2 on p. 5 of [9], the order $|\mathscr{P}(i)|$ of $\mathscr{P}(i)$ divides that of $\mathscr{P}$. From the fact that $\mathscr{P}$ is a p-group, $|\mathscr{P}(i)|$ must be a power of p.

The set of symbols acted upon by $\mathscr{P}$ is, by definition,

$$\{i : |\mathscr{P}(i)| \neq 1\},$$

that is

$$\cup \mathscr{P}(j),$$

the union being taken over all j such that $|\mathscr{P}(j)| \neq 1$. Since any two distinct orbits are disjoint, this set has order

$$\sum |\mathscr{P}(j)|,$$

and every term in the sum is a power of p, not p^0. Thus p divides the sum, giving the result, except for the trivial case where $\mathscr{P}$ acts on no symbols, in which case $m = 0$ and we can put $n = 0$.

In particular a sylow p- subgroup of $\mathscr{A}_r$ must act on np symbols where n is the largest integer such that $np \leqq r$, so a sylow p-subgroup of $\mathscr{A}_{pn+1}$ acts on exactly pn symbols and satisfies the conditions of (4).

We shall consider the case $p = 2$. With $n = 2$ we can apply (4) to $\mathscr{A}_4$ and $\mathscr{A}_5$. Every 2-subgroup of $\mathscr{A}_5$ except $\mathscr{I}$ acts on $4(= pn)$ symbols, so if $\mathscr{P}$ is such a group,

$$W_{\mathscr{P}}(\mathscr{A}_5) \cong W_{\mathscr{P}}(\mathscr{A}_4).$$

$A(\mathscr{A}_4)$ is semisimple [3, p. 97], so each $W_{\mathscr{P}}(\mathscr{A}_4)$ is semisimple by (2), and so each $W_{\mathscr{P}}(\mathscr{A}_5)$ is semisimple. Moreover $W_{\mathscr{I}}(\mathscr{A}_5)$ is semisimple. As any 2-subgroup of $\mathscr{A}_5$ is of the same type as $\mathscr{P}$ or is $\mathscr{I}$, we see by (2) that $A(\mathscr{A}_5)$ is semisimple.

If $n = 3$ we have $\mathscr{A}_6$ and $\mathscr{A}_7$. If we write

$$\begin{aligned} U &= (13)(24) \qquad & X &= (123) \qquad & Z &= (1234)(56) \\ V &= (14)(23) \qquad & Y &= (567) \qquad & T &= (12)(56) \end{aligned}$$

then the distinct 2-subgroups of $\mathscr{A}_7$ are $\mathscr{I}$, $\mathscr{Z}_2$, $\mathscr{V}$, $\mathscr{V}'$, $\mathscr{Z}_4$ and $\mathscr{P}$, and their $\mathscr{A}_7$-conjugates, where

$$\begin{aligned} \mathscr{Z}_2 &= \langle U \rangle \\ \mathscr{V} &= \langle U, V \rangle \\ \mathscr{V}' &= \langle UV, T \rangle \\ \mathscr{Z}_4 &= \langle Z \rangle \\ \mathscr{P} &= \langle Z, V \rangle. \end{aligned}$$

These are also the 2-subgroups of $\mathscr{A}_6$.

We shall prove that $W_{\mathscr{V}}(\mathscr{A}_6)$ and $W_{\mathscr{V}}(\mathscr{A}_7)$ are semisimple. That $W_{\mathscr{D}}(\mathscr{G})$ is semisimple when $\mathscr{D} = \mathscr{I}$ has already been noted; that it is so when $\mathscr{D}$ is cyclic has been shown in [8]. From (4) we see that

$$W_{\mathscr{V}'}(\mathscr{A}_6) \cong W_{\mathscr{V}'}(\mathscr{A}_7) \text{ and } W_{\mathscr{P}}(\mathscr{A}_6) \cong W_{\mathscr{P}}(\mathscr{A}_7),$$

since both these subgroups act on $6(= np)$ symbols. Collecting these results and applying (2) we see that $A(\mathscr{A}_7)$ is semisimple if and only if $A(\mathscr{A}_6)$ is.

Even more can be said. There is an isomorphism between $\mathscr{N}(\mathscr{V}; \mathscr{A}_6)$ and $\mathscr{N}(\mathscr{V}'; \mathscr{A}_6)$ which carries $\mathscr{V}$ onto $\mathscr{V}'$, and so $W_{\mathscr{V}'}(\mathscr{A}_6) \cong W_{\mathscr{V}}(\mathscr{A}_6)$ and is semisimple. Therefore $A(\mathscr{A}_6)$ and $A(\mathscr{A}_7)$ are semisimple if and only if $W_{\mathscr{P}}(\mathscr{A}_6)$ is. As $\mathscr{N}(\mathscr{P}; \mathscr{A}_6)$ is $\mathscr{P}$, we can say $A(\mathscr{A}_6)$ and $A(\mathscr{A}_7)$ are semisimple if and only if $W_{\mathscr{P}}(\mathscr{P})$ is semisimple.

It remains to show that $W_{\mathscr{V}}(\mathscr{A}_6)$ and $W_{\mathscr{V}}(\mathscr{A}_7)$ are semisimple. By (3) we need only consider $W_{\mathscr{V}}(\mathscr{N}_6)$ and $W_{\mathscr{V}}(\mathscr{N}_7)$ where $\mathscr{N}_r$ is $\mathscr{N}(\mathscr{V}; \mathscr{A}_r)$ and

$$\mathscr{N}_6 = \langle U, V, X, T \rangle,$$
$$\mathscr{N}_7 = \langle U, V, X, T, Y \rangle.$$

3. Decomposition of induced modules

Suppose $\mathscr{H}$ is a normal subgroup of $\mathscr{G}$, and let $\mathscr{L}$ be an indecomposable $\mathscr{F}\mathscr{H}$-representation module with $\mathscr{G}$-stabilizer $\mathscr{S}$. Then we know [2, p. 162] that $\mathscr{L}^{\mathscr{G}}$ decomposes as does a certain twisted group algebra $\mathscr{I}(\mathscr{S}/\mathscr{H})$ on $\mathscr{S}/\mathscr{H}$ over $\mathscr{F}$. To be specific, suppose $\mathscr{L}^{\mathscr{G}} \cong \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \cdots \oplus \mathscr{L}_m$ and $\mathscr{I}(\mathscr{S}/\mathscr{H}) \cong \mathscr{I}_1 \oplus \mathscr{I}_2 \oplus \cdots \oplus \mathscr{I}_n$, where the $\mathscr{L}_i$ are indecomposable submodules and the $\mathscr{I}_i$ are indecomposable left ideals. Then we can reorder the $\mathscr{I}_i$ so that

$$\begin{gathered} m = n \\ \mathscr{L}_i \cong \mathscr{L}_j \text{ if and only if } \mathscr{I}_i \cong \mathscr{I}_j \\ \dim_{\mathscr{F}} \mathscr{L}_i = \dim_{\mathscr{F}} \mathscr{I}_i \cdot \dim_{\mathscr{F}} \mathscr{L}. \end{gathered} \tag{6}$$

Furthermore $\mathscr{L}_i^{\mathscr{G}}$ is indecomposable.

(7) THEOREM. *If* $\mathscr{S} = \mathscr{H} \times \mathscr{W}$, *then*

$$\mathscr{L}^{\mathscr{G}} \cong \bigoplus_1^h \mathscr{W}_i \otimes_{\mathscr{F}\mathscr{H}} \mathscr{L}$$

is a decomposition into indecomposables, where

$$\mathscr{F}\mathscr{W} = \bigoplus_1^h \mathscr{W}_i$$

is a decomposition into indecomposable left ideals. ($\mathscr{W}_i \otimes_{\mathscr{F}\mathscr{H}} \mathscr{L}$ is defined as the subset of $\mathscr{F}\mathscr{S} \otimes_{\mathscr{F}\mathscr{H}} \mathscr{L}$ consisting of the elements $W \otimes L$, where $W \in \mathscr{W}_i$ and $L \in \mathscr{L}$.)

PROOF. It is clear that the sum is direct and equals $\mathscr{L}^{\mathscr{G}}$ and that the summands are $\mathscr{F}\mathscr{S}$-representation modules. $\mathscr{I}(\mathscr{S}/\mathscr{H})$ is $\mathscr{F}(\mathscr{S}/\mathscr{H})$ in this case, so by (6) $\mathscr{L}^{\mathscr{G}}$ must split into exactly h indecomposable parts. By the Krull-Schmidt theorem these must be the $\mathscr{W}_i \otimes \mathscr{L}$.

Write $\mathscr{X}$ for the alternating group on $\{5, 6, \cdots, k\}$. Then (7) applies when $\mathscr{G} = \mathscr{V} \times \mathscr{X}$ or $\mathscr{A}_4 \times \mathscr{X}$, which are the cases we will need.

4. $\mathscr{V}$-projective representations of $\mathscr{N}_k$

The indecomposable $\mathscr{V}$-projective $\mathscr{F}\mathscr{N}_k$-representation modules are just the indecomposable parts of the modules $\mathscr{L}^{\mathscr{N}_k}$ where $\mathscr{L}$ is an $\mathscr{F}\mathscr{V}$-representation module.

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The indecomposable $\mathscr{F}\mathscr{V}$-representation module isomorphism classes are known ([1], [7]). In the notation of Conlon [3] the classes are A_0, A_n, B_n, $C_n(f)$ and D, where n ranges through the positive integers and f through $\mathscr{F} \cup \{\infty\}$. Typical representations are given on pp. 86—7 of [3]; we shall denote this representative of A_n by $\mathscr{A}_n$, and similarly for the others. The stabilizers are given in (8):

(8)

Module	Generators of Stabilizer	
	in $\mathscr{N}_6$	in $\mathscr{N}_7$
$\mathscr{A}_n$, $\mathscr{B}_n$, $\mathscr{D}$	U, V, X, T	U, V, X, T, Y
$\mathscr{C}_n(0)$	$U\ V, TX$	U, V, TX, Y
$\mathscr{C}_n(1)$	U, V, T	U, V, T, Y
$\mathscr{C}_n(\infty)$	U, V, TX^2	U, V, TX^2, Y
$\mathscr{C}_n(\omega)$, $\mathscr{C}_n(\omega^2)$	U, V, X	U, V, X, Y
$\mathscr{C}_n(f)$	U, V	U, V, Y

where ω is a primitive cube root of unity in $\mathscr{F}$ and f ranges over all values not already listed. The stabilizer in $\mathscr{N}_k$ may be derived from that in $\mathscr{N}_6$ as noted in Section 3.

The representations of $\mathscr{A}_4$ are also known [3]; their classes are $\bar{A}_0^a$, $\bar{A}_n^a$, $\bar{B}_n^a$, $\bar{C}_n^a(g)$, $\bar{C}_n^*(f)$, $\bar{D}^a$, where n ranges through the positive integers, a through the integers modulo 3, g through $\{\omega, \omega^2\}$ and f through a set of representatives of the equivalence classes of $\mathscr{F}$ under the relation

$$f \sim 1 + \frac{1}{f} \sim \frac{1}{1+f},^{4} \quad \text{except } \omega \text{ and } \omega^2.$$

If we write $\bar{\mathscr{A}}_n^a$ for a representative module of the class $\bar{A}_n^a$, and so on, then $\bar{\mathscr{L}}^a$ may be taken as an extension of $\mathscr{L}$; $\bar{\mathscr{C}}_n^*(f)$ may be taken as $(\mathscr{C}_n(f))^{\mathscr{A}_4}$. We follow Conlon's convention that $\bar{\mathscr{A}}_0^a$ yields the representation

$$U \to 1, V \to 1, X \to \omega^a$$

and

$$\bar{\mathscr{A}}_0^a \otimes \bar{\mathscr{L}}^b \cong \bar{\mathscr{L}}^{a+b}.$$

Then if $\bar{\mathscr{L}}^0$ gives the representation $X \to \lambda(X)$, $\bar{\mathscr{L}}^a$ gives $X \to \omega^a\lambda(X)$. Suitable matrices $\lambda(X)$ are known [3] for all cases except $\mathscr{C}_n(\omega)$ and $\mathscr{C}_n(\omega^2)$; and the author has found that a suitable matrix to extend $\mathscr{C}_n(\omega)$ is $M_1 \oplus M_2$, where

$$M_1 \text{ has } (i, j) \text{ element } \omega^{i+j+1}\binom{i-2}{j-2}$$

$$M_2 \text{ has } (i, j) \text{ element } \omega^{i+j+2}\binom{i-1}{j-1},$$

[4] $1 \sim 0 \sim \infty$

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the binomial coefficients being evaluated over characteristic 2 and $\oplus$ being direct sum of matrices. We take $\mathscr{C}_0^n(\omega)$ to be the extension with

$$\lambda(X) = \omega^{2n+2}\,(M_1 \oplus M_2)$$

and otherwise follow Conlon's choice for the particular extension of $\mathscr{L}$ which will be labelled $\overline{\mathscr{L}}^0$.

If $\mathscr{L}$ is an $\mathscr{F}\mathscr{V}$-representation module with stabilizer $\mathscr{S}$ in $\mathscr{N}_k$, and $\mathscr{X}$ is as before, the following conditions hold:

$$\begin{array}{ll} \text{if } X \in \mathscr{S}, & \mathscr{V} \trianglelefteq \mathscr{A}_4 \trianglelefteq \mathscr{A}_4 \times \mathscr{X} \trianglelefteq \mathscr{S}; \\ \text{if } X \notin \mathscr{S}, & \mathscr{V} \trianglelefteq \mathscr{V} \times \mathscr{X} \trianglelefteq \mathscr{S}; \end{array}$$

in both series equality holds in the last place if and only if $T \notin \mathscr{S}$.

Suppose $\mathscr{F}\mathscr{X} \cong \oplus \mathscr{X}_i$ is a decomposition into indecomposable left ideals. Then, from (7), $\mathscr{L}^{\mathscr{V}\times\mathscr{X}} \cong \oplus \mathscr{X}_i \otimes_{\mathscr{F}\mathscr{V}} \mathscr{L}$ is a decomposition into indecomposables. If $X \in \mathscr{S}$ then in every case

$$\mathscr{L}^{\mathscr{A}_4} \cong \overline{\mathscr{L}}^0 \oplus \overline{\mathscr{L}}^1 \oplus \overline{\mathscr{L}}^2,$$

so

$$\mathscr{L}^{\mathscr{A}_4\times\mathscr{X}} \cong \bigoplus_{a=0}^{2} (\overline{\mathscr{L}}^a)^{\mathscr{A}_4\times\mathscr{X}},$$

so using (7),

$$\mathscr{L}^{\mathscr{A}_4\times\mathscr{X}} \cong \bigoplus_{a=0}^{2} \bigoplus_{i} \mathscr{X}_i \otimes_{\mathscr{F}\mathscr{A}_4} \overline{\mathscr{L}}^a.$$

Again the direct summands are indecomposable.

Let $\mathscr{M}$ be one of the direct summands in this last decomposition. $\mathscr{S}/(\mathscr{A}_4 \times \mathscr{X})$ has order 1 or 2, and so its group algebra over $\mathscr{F}$ is indecomposable. So $\mathscr{M}^{\mathscr{S}}$ is indecomposable. A similar result holds if $X \notin \mathscr{S}$. We have the following decompositions into indecomposables:

$$\text{(9)} \qquad \mathscr{L}^{\mathscr{S}} \cong \begin{cases} \bigoplus_{i} (\mathscr{X}_i \otimes_{\mathscr{F}\mathscr{V}} \mathscr{L}) & \text{if } X \notin \mathscr{S} \\ \bigoplus_{i} \bigoplus_{a=0}^{2} (\mathscr{X}_i \otimes_{\mathscr{F}\mathscr{A}_4} \overline{\mathscr{L}}^a)^{\mathscr{S}} & \text{if } X \in \mathscr{S} \end{cases}$$

The calculation of the isomorphism-classes of $\mathscr{V}$-projective $\mathscr{F}\mathscr{N}_k$-representation modules is now an easy task, using the last part of (6).

For $\mathscr{N}_7$ we find that the classes are $A_0^{a,b}$, $A_n^{a,b}$, $B_n^{a,b}$, $C_n^{a,b}(\omega)$, $C_n^{*,b}(f)$ and $D^{a,b}$, where a and b range through the integers modulo 3, n through the positive integers and f through the elements of $\mathscr{F} \cup \{\infty\}$ other than ω and ω^2. The following identities hold:

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$$A_0^{a,b} = A_0^{2a,2b}, \quad A_n^{a,b} = A_n^{2a,2b}, \quad B_n^{a,b} = B_n^{2a,2b}, \quad D^{a,b} = D^{2a,2b}$$

$$C_n^{*,b}(f) = C_n^{*,b}(g) = C_n^{*,2b}(h) \text{ when } g = 1+\frac{1}{f} \text{ or } \frac{1}{1+f},$$

$$h = \frac{1}{f} \text{ or } \frac{f}{1+f} \text{ or } 1+f$$ [5]

The classes for $\mathcal{N}_6$ are just those for $\mathcal{N}_7$ with all reference to b dropped (we write, for example, A_0^a, $C_n^*(f)$) and with $L^a = L^d$ whenever $L^{a,0} = L^{d,0}$. We use the convention that

$$(L^{a,b})_{\mathcal{N}_6} = L^a, \quad (L^a)_{\mathcal{A}_4} = L^a + L^{2a}.$$

The representation matrices for typical members of the $\mathcal{F}\mathcal{N}_7$-classes are shown in (10) in terms of the corresponding $\mathcal{F}\mathcal{A}_4$ representation with superscript 0. These corresponding representation matrices are denoted by λ. The table also shows the $\mathcal{F}\mathcal{V}$-class from which each $\mathcal{F}\mathcal{N}_7$-class is obtained and the dimension of square block-matrices involved. The $\mathcal{F}\mathcal{N}_6$-representations are found by deleting the matrix for Y.

(10)

$\mathcal{F}\mathcal{N}_7$-class	$A_n^{a,b}$, $B_n^{a,b}$	$C_n^{a,b}(\omega)$	$C_n^{*,b}(f)$
Corresponding $\mathcal{F}\mathcal{V}$-class	A_n, B_n	$C_n(\omega)$	$C_n(f)$
Block size	$2n+1$	$2n$	$6n$
Matrix for U	$\begin{bmatrix}\lambda(U) & 0\\ 0 & \lambda(V)\end{bmatrix}$	$\begin{bmatrix}\lambda(U) & 0\\ 0 & \lambda(V)\end{bmatrix}$	$\begin{bmatrix}\lambda(U) & 0\\ 0 & \lambda(V)\end{bmatrix}$
V	$\begin{bmatrix}\lambda(V) & 0\\ 0 & \lambda(U)\end{bmatrix}$	$\begin{bmatrix}\lambda(V) & 0\\ 0 & \lambda(U)\end{bmatrix}$	$\begin{bmatrix}\lambda(V) & 0\\ 0 & \lambda(U)\end{bmatrix}$
X	$\begin{bmatrix}\omega^a\lambda(X) & 0\\ 0 & \omega^{2a}\lambda(X)^2\end{bmatrix}$	$\begin{bmatrix}\omega^a\lambda(X) & 0\\ 0 & \omega^{2a}\lambda(X)^2\end{bmatrix}$	$\begin{bmatrix}\lambda(X) & 0\\ 0 & \lambda(X)^2\end{bmatrix}$
T	$\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$	$\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$	$\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$
Y	$\begin{bmatrix}\omega^b I & 0\\ 0 & \omega^{2b} I\end{bmatrix}$	$\begin{bmatrix}\omega^b I & 0\\ 0 & \omega^{2b} I\end{bmatrix}$	$\begin{bmatrix}\omega^b I & 0\\ 0 & \omega^{2b} I\end{bmatrix}$

[5] In particular $C_n^{*,b}(1) = C_n^{*,2b}(1) = C_n^{*,b}(0) = C_n^{*,b}(\infty)$

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5. Multiplication of module classes

The multiplication of module classes is defined in the usual way: If L is $\{\mathscr{L}\}$ and M is $\{\mathscr{M}\}$, then LM is $\{\mathscr{L} \otimes \mathscr{M}\}$.

We shall consider the multiplication of $\mathscr{F}\mathscr{N}_6$-classes modulo $W_{\mathscr{I}}(\mathscr{G})$. The effect is that of putting $D^a = 0$, since $W_{\mathscr{I}}(\mathscr{G})$ is a direct summand of $A_{\mathscr{H}}(\mathscr{G})$ for any $\mathscr{G}$ and any $\mathscr{H} \leqq \mathscr{G}$, [6] so there is no confusion in writing L for $L+W_{\mathscr{I}}(\mathscr{N}_6)$.

The multiplications for $\mathscr{V}$ and $\mathscr{A}_4$ are given in propositions 2—4 of [1] [7] and equations (6), (7), (9) and (11)—(16) of [3]. However we can be more specific about $\bar{C}^a_n(\omega)$. The convention of Section 4 yields upon a direct calculation the following result when $n \geqq m \geqq 1$:

$$(11) \qquad \begin{aligned} \bar{C}^0_m(\omega)\,\bar{C}^0_n(\omega) &= 2\bar{C}^0_m(\omega) &&\text{if } n \equiv 2 \pmod 3 \\ &= \bar{C}^1_m(\omega)+\bar{C}^2_m(\omega) &&\text{if } n \not\equiv 2 \pmod 3. \end{aligned}$$

(This replaces equations (13) and (14) of [3].)

From these results, the distributive law and the law

$$\mathscr{L}^{\mathscr{N}_6} \otimes \mathscr{M} \cong (\mathscr{L} \otimes \mathscr{M}_{\mathscr{A}_4})^{\mathscr{N}_6},$$

where $\mathscr{L}$ is an $\mathscr{F}\mathscr{A}_4$-representation module and $\mathscr{M}$ is an $\mathscr{F}\mathscr{N}_6$-representation module, we can calculate the products of $\mathscr{F}\mathscr{N}_6$-classes. We introduce two points of notation: we write $\bar{L}^{\mathscr{N}_6}$ for the class $\{\bar{\mathscr{L}}^{\mathscr{N}_6}\}$; and we write $f \approx g$ whenever $f \sim g$ or $f \sim 1/g$, that is whenever f and g are members of the same set of cross-ratios. Example calculations are

$$\begin{aligned} A^a_n A^b_m &= [(\bar{A}^a_n+\bar{A}^{2a}_n)\bar{A}^b_m]^{\mathscr{N}_6} \\ &= (\bar{A}^{a+b}_{n+m}+\bar{A}^{2a+b}_{n+m})^{\mathscr{N}_6} \\ &= A^{a+b}_{n+m}+A^{2a+b}_{n+m} \end{aligned}$$

and, if $m \leqq n$ and $f \not\approx 1$,

$$\begin{aligned} C^*_n(f)\,C^*_m(g) &= [(\bar{C}^*_n(f)+\bar{C}^*_n(1/f))\,\bar{C}^*_m(g)]^{\mathscr{N}_6} \\ &= \begin{cases} 2C^*_m(f) & \text{if } f \approx g, \\ 0 & \text{if } f \not\approx g, \end{cases} \end{aligned}$$

except

$$C^*_1(f)C^*_1(f) = C^*_2(f);$$

moreover

$$\begin{aligned} C^*_n(1)C^*_m(1) &= (2\bar{C}^*_n(1)\bar{C}^*_m(1))^{\mathscr{N}_6} \\ &= 4C^*_m(1). \end{aligned}$$

Similar calculations yield the following multiplication table:

[6] This follows from Corollary 5 of [3].

[7] There is an error in Bašev's work — see p. 88 of [3].

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(12)

$m \leqq n$	A^b_m	B^b_m	$C^b_m(\omega)$	$C^*_m(g),\ g \not\approx \omega$
A^a_n	$A^{a+b}_{n+m}+A^{2a+b}_{n+m}$	$A^{a+b}_{n-m}+A^{2a+b}_{n-m}$	$C^{a+b+n}_m(\omega)+C^{2a+b+n}_m(\omega)$	$2C^*_m(g)$
B^a_n	$B^{a+b}_{n-m}+B^{2a+b}_{n-m}$	$B^{a+b}_{n+m}+B^{2a+b}_{n+m}$	$C^{a+b+2n}_m(\omega)+C^{2a+b+2n}_m(\omega)$	$2C^*_m(g)$
$C^a_n(\omega)$	$C^{a+b+m}_n(\omega)$ $+C^{a+2b+m}_n(\omega)$	$C^{a+b+2m}_n(\omega)$ $+C^{a+2b+2m}_n(\omega)$	$C^{a+b}_2(\omega)$ if $m=n=1$ $2C^{a+b}_m(\omega)$ if $n \equiv 2 \pmod 3$ $C^{a+b+1}_m(\omega)+C^{a+b+2}_m(\omega)$ if $n \not\equiv 2 \pmod 3$ when $n > 1$	0
$C^*_n(f)$ $f \not\approx \omega$	$2C^*_n(f)$	$2C^*_n(f)$	0	0 if $f \not\approx g$ $4C^*_m(1)$ if $f \approx g \approx 1$ $2C^*_m(f)$ if $n \neq 1$ $C^*_2(f)$ if $m=n=1$ when $f \approx g \not\approx 1$

We see from (12) that $\frac{1}{2}A^0_0$ is an identity element for the algebra $\mathscr{R} = A_{\mathscr{V}}(\mathscr{N}_6)/W_{\mathscr{I}}(\mathscr{N}_6)$, and admits of the orthogonal idempotent decomposition $\frac{1}{2}A^0_0 = J_0+J_1$, where

$$J_0 = \tfrac{1}{6}A^0_0+\tfrac{1}{3}A^1_0,$$
$$J_1 = \tfrac{1}{3}A^0_0-\tfrac{1}{3}A^1_0.$$

Then

$$\mathscr{R} = \mathscr{R}J_0 \oplus \mathscr{R}J_1.$$

Write $A_{n\alpha} = A^0_nJ_\alpha$, $B_{n\alpha} = B^0_nJ_\alpha$, $C_{n0} = C^0_n(\omega)J_0$, $C^a_{n1} = C^a_n(\omega)J_1$. Then the set of these elements (where α is 0 or 1 and a is any integer modulo 3), J_0, J_1 and the distinct $C^*_n(f)$ together generate $\mathscr{R}$, since

$$A^0_n = A_{n0}+A_{n1}$$
$$B^0_n = B_{n0}+B_{n1}$$
$$A^1_n = \tfrac{3}{4}A_{n0}-\tfrac{3}{8}A_{n1}$$
$$B^1_n = \tfrac{3}{4}B_{n0}-\tfrac{3}{8}B_{n1}$$
$$C^a_n(\omega) = C_{n0}+C^a_{n1}.$$

Moreover the identity $C^0_{n1}+C^1_{n1}+C^2_{n1} = 0$ holds, and so we can drop C^0_{n1} from the list of generators.

Writing $X_\alpha = \frac{1}{2}A_{1\alpha}$ we see that $A_{n\alpha} = 2(X_\alpha)^n$ and $B_{n\alpha} = 2(X_\alpha)^{-n}$,

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so the set generated by all the $A_{n\alpha}$ and $B_{n\alpha}$ for a given α is isomorphic to $\mathscr{C}[X_\alpha, 1/X_\alpha]$. If we write $\mathscr{A}_0$ for the algebra generated by all the $C_n^*(f)$ and C_{n0}, and $\mathscr{A}_1$ for that generated by the C_{n1}^a, then

$$\mathscr{R}J_0 \cong \mathscr{C}[X_0, 1/X_0]+\mathscr{A}_0,$$
$$\mathscr{R}J_1 \cong \mathscr{C}[X_1, 1/X_1]+\mathscr{A}_1,$$

where $\mathscr{A}_\alpha$ is an ideal of $\mathscr{R}J_\alpha$.

We next set up orthogonal idempotents which generate $\mathscr{A}_0$ and $\mathscr{A}_1$. For $\mathscr{A}_0$ we use

$$\begin{aligned}
I_1(1) &= \tfrac{1}{4}C_1^*(1), \\
I_n(1) &= \tfrac{1}{4}(C_n^*(1)-C_{n-1}^*(1)) && \text{if } n > 1, \\
I_n(\omega) &= \tfrac{1}{4}(C_{20}+(-1)^n \sqrt{2}C_{10}) && \text{if } n = 1 \text{ or } 2, \\
I_n(\omega) &= \tfrac{1}{2}(C_{n0}-C_{n-1,0}) && \text{if } n > 2, \\
I_n(f) &= \tfrac{1}{4}(C_2^*(f)+(-1)^nC_1^*(f)) && \text{if } n = 1 \text{ or } 2, f \not\approx 1, \\
I_n(f) &= \tfrac{1}{2}(C_n^*(f)-C_{n-1}^*(f)) && \text{if } n > 2, f \not\approx 1.
\end{aligned}$$

In each case, $\mathscr{R}I_n(f) \cong \mathscr{C}I_n(f)$.

To consider $\mathscr{A}_1$ we set

$$\begin{aligned}
K_n^a &= \tfrac{1}{4}(C_{21}^a+(-1)^n \sqrt{2}C_{11}^a) && \text{if } n = 1 \text{ or } 2, \\
K_n^a &= h_nC_{n1}^a-h_{n-1}C_{n-1,1}^a && \text{if } n > 2,
\end{aligned}$$

where $h_n = \frac{1}{2}$ if $n \equiv 2 \pmod 3$ and $h_n = -1$ otherwise.

The K_n^0 are orthogonal and $\mathscr{R}K_n^0 \cong \mathscr{C}[K_n^0, K_n^1, K_n^2]$. Since $K_n^aK_n^b = K_n^{a+b}$ and $K_n^0+K_n^1+K_n^2 = 0$ we see that if we put

$$L_{n0} = \tfrac{1}{3}(K_n^0+uK_n^1+u^2K_n^2),$$
$$L_{n1} = \tfrac{1}{3}(K_n^0+u^2K_n^1+uK_n^2),$$

where u is a primitive cube root of unity in $\mathscr{C}$, then

$$\mathscr{R}K_n^0 = \mathscr{R}L_{n0} \oplus \mathscr{R}L_{n1} \text{ and } \mathscr{R}L_{n\alpha} \cong \mathscr{C}L_{n\alpha}.$$

From these we can find the structure of $\mathscr{R}$. $A_{\mathscr{V}}(\mathscr{N}_6) \cong \mathscr{R} \oplus W_{\mathscr{I}}(\mathscr{N}_6)$, and $W_{\mathscr{I}}(\mathscr{N}_6) \cong \mathscr{C} \oplus \mathscr{C}$, so $A_{\mathscr{V}}(\mathscr{N}_6)$ has the following form to within isomorphism:

$$\begin{aligned}
A_{\mathscr{V}}(\mathscr{N}_6) \cong\ & \{\mathscr{C}[X_0, 1/X_0]+ \oplus_1 \mathscr{C}I_n(f)\} \\
& \oplus \{\mathscr{C}[X_1, 1/X_1]+ \oplus_2 \mathscr{C}L_{n\alpha}\} \\
& \oplus \{\mathscr{C} \oplus \mathscr{C}\}
\end{aligned} \tag{13}$$

where $\oplus_1$ ranges over all positive integers n and all $f \in \mathscr{F} \cup \{\infty\}$ modulo the relation $\approx$, and $\oplus_2$ ranges over the positive integers n and $\alpha = 0, 1$. The $I_n(f)$ and the $L_{n\alpha}$ are sets of orthogonal idempotents, and

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$$X_0 I_n(f) = I_n(f),$$
$$X_1 L_{n\alpha} = u^{2\alpha+2} L_{n\alpha}.$$

We work similarly in the $\mathscr{N}_7$ case. The $\mathscr{V}$-projective $\mathscr{F}\mathscr{N}_7$-representation module classes are considered modulo $W_{\mathscr{I}}(\mathscr{N}_7)$, and find the following multiplication table for $\mathscr{S} = A_{\mathscr{V}}(\mathscr{N}_7)/W_{\mathscr{I}}(\mathscr{N}_7)$:

(14)

$m \leqq n$	$A_m^{b,d}$	$B_m^{b,d}$	$C_m^{b,d}(\omega)$	$C_m^{*,d}(g)$, $g \not\approx \omega$
$A_n^{a,c}$	$A_{m+n}^{a+b,c+d} + A_{m+n}^{a+2b,c+2d}$	$A_{n-m}^{a+b,c+d} + A_{n-m}^{a+2b,c+2d}$	$C_m^{n+a+b,c+d}(\omega) + C_m^{n+2a+b,2c+d}(\omega)$	$C_m^{*,c+d}(g) + C_m^{*,2c+d}(g)$
$B_n^{a,c}$	$B_{n-m}^{a+b,c+d} + B_{n-m}^{a+2b,c+2d}$	$B_{n+m}^{a+b,c+d} + B_{n+m}^{a+2b,c+2d}$	$C_m^{2n+a+b,c+d}(\omega) + C_m^{2n+2a+b,2c+d}(\omega)$	$C_m^{*,c+d}(g) + C_m^{*,2c+d}(g)$
$C_n^{a,c}(\omega)$	$C_n^{m+a+b,c+d}(\omega) + C_n^{m+a+2b,c+2d}(\omega)$	$C_n^{2m+a+b,c+d}(\omega) + C_n^{2m+a+2b,c+2d}(\omega)$	if $n = m = 1$, $C_2^{a+b,c+d}(\omega)$; if $n \neq 1$, $n \equiv 2 \pmod 3$ $2C_m^{a+b,c+d}(\omega)$; if $n \neq 1$, $n \not\equiv 2 \pmod 3$ $C_m^{a+b+1,c+d}(\omega) + C_m^{a+b+2,c+d}(\omega)$	0
$C_n^{*,c}(f)$ $f \not\approx \omega$	$C_n^{*,c+d}(f) + C_n^{*,c+2d}(f)$	$C_n^{*,c+d}(f) + C_n^{*,c+2d}(f)$	0	0 if $f \not\approx g$; $2(C_m^{*,c+d}(1) + C_m^{*,2c+d}(1))$ if $f \approx g \approx 1$; $C_2^{c+d}(f)$ if $f \approx g \not\approx 1$, $m = n = 1$; $2C_m^{c+d}(f)$ if $f \approx g \not\approx 1$, $n \neq 1$

Write $S = A_0^{0,1} + A_0^{1,0} + A_0^{1,2}$. The identity element $\frac{1}{2}A_0^{0,0}$ of $\mathscr{S}$ admits of an orthogonal idempotent decomposition:

$$\tfrac{1}{2}A_0^{0,0} = J_0 + J_{10} + J_{01} + J_{11} + J_{12},$$
$$J_0 = \tfrac{1}{18}(A_0^{0,0} + 2S)$$
$$J_{ab} = \tfrac{1}{9}(A_0^{0,0} + 3A_0^{a,b} - S)$$

We proceed much as before. It is clear that $\mathscr{S}$ is generated by the J_α and the elements

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$$A_{nx} = A_n^{0,0} J_x \qquad B_{nx} = A_n^{0,0} J_x$$
$$C_{n,0}(f) = C_n^{*,0}(f) J_0 \qquad C_{n,10}^a(f) = C_n^{*,a}(f) J_{10}$$
$$C_{n,0}(\omega) = C_n^{0,0}(\omega) J_0 \qquad C_{n,10}^a(\omega) = C_n^{0,a}(\omega) J_{10}$$
$$C_{n,x}^a(\omega) = C_n^{a,0}(\omega) J_x \ (x \neq 0, 10),$$

where a ranges through the integers modulo 3, x through $\{0, 01, 10, 11, 12\}$ and f through the non-equivalent members of $\mathscr{F}$ other than ω under the relation $\approx$.

Putting $Y_x = \frac{1}{2}A_{1x}$ we obtain as before $A_{n,x} = 2(Y_x)^n$, $B_{n,x} = 2(Y_x)^{-n}$, and

$$\mathscr{S}J_x \cong \mathscr{C}[Y_x, 1/Y_x] + \mathscr{B}_x,$$

where $\mathscr{B}_x$ is an ideal. We then write

$$\begin{aligned}
I_n(f) &= \tfrac{1}{4}(C_{2,0}(f) + (-1)^n \sqrt{2}C_{1,0}(f)) && \text{if } n = 1 \text{ or } 2,\\
I_n(f) &= \tfrac{1}{2}(C_{n,0}(f) - C_{n-1,0}(f)) && \text{if } n > 2,\\
K_{n,10}^a(f) &= \tfrac{1}{4}(C_{2,10}^a(f) + (-1)^n \sqrt{2}C_{1,10}(f)) && \text{if } n = 1 \text{ or } 2 \text{ and } f \not\approx 1,\\
K_{n,10}^a(f) &= \tfrac{1}{2}(C_{n,10}^a(f) - C_{n-1,10}^a(f)) && \text{if } n > 2 \text{ and } f \not\approx 1,\\
K_{1,10}^0(1) &= \tfrac{1}{4}C_{1,10}^0(1) && \\
K_{n,10}^0(1) &= \tfrac{1}{4}(C_{n,10}(1) - C_{n-1,10}(1)) && \text{if } n \neq 1,\\
K_{n,x}^a(\omega) &= \tfrac{1}{4}(C_{2,x}^a(\omega) + (-1)^n \sqrt{2}C_{1,x}^a(\omega)) && \text{if } n = 1 \text{ or } 2\\
K_{n,x}^a(\omega) &= h_n C_{n,x}^a(\omega) - h_{n-1}C_{n-1,x}^a(\omega) && \text{if } n > 2,
\end{aligned}$$

where $x = 01$, 11 or 12 and h_n is defined as before. It is found that $\{I_n(f)\}$, $\{K_{n,10}^0(f)\}$ and $\{K_{n,x}^0(\omega)\}$ are sets of orthogonal idempotents generating $\mathscr{B}_0$, $\mathscr{B}_{10}$ and $\mathscr{B}_x$. $\mathscr{S}I_n(f) \cong \mathscr{C}I_n(f)$ and $\mathscr{S}K_{n,10}^0(1) \cong \mathscr{C}K_{n,10}^0(1)$; in the other cases we find

$$\mathscr{S}K_{n,x}^0(f) \cong \mathscr{C}[K_{n,x}^0(f), K_{n,x}^1(f), K_{n,x}^2(f)]$$

so we put

$$L_{n,\alpha,x}(f) = K_{n,x}^0(f) + u^{\alpha+1}K_{n,x}^1(f) + u^{2\alpha+2}K_{n,x}^2(f).$$

For convenience write $L_{n,0,10}(1) = K_{n,10}^0(1)$.

Proceeding as before we find $A_{\mathscr{Y}}(\mathscr{N}_7)$ is

$$\begin{aligned}
A_{\mathscr{Y}}(\mathscr{N}_7) \cong\ & \{\mathscr{C}[Y_0, 1/Y_0] + \oplus_1 \mathscr{C}I_n(f)\}\\
& \oplus \{\mathscr{C}[Y_{10}, 1/Y_{10}] + \oplus_2 \mathscr{C}L_{n,\alpha,10}(f)\}\\
& \oplus \oplus_3 \{\mathscr{C}[Y_x, 1/Y_x] + \oplus_4 L_{n,\alpha,x}(\omega)\}\\
& \oplus \{\mathscr{C} \oplus \mathscr{C} \oplus \mathscr{C} \oplus \mathscr{C} \oplus \mathscr{C}\}
\end{aligned} \tag{15}$$

where $\oplus_1$ is over all elements f of $\mathscr{F}$, modulo $\approx$, and all positive integers n,

$\oplus_2$ is as $\oplus_1$, and also over $\alpha = 0, 1$, except for the case $f = 1$, $\alpha = 1$,

$\oplus_3$ is over $x = 01, 10, 12$,

$\oplus_4$ is over all $n \geqq 1$ and $\alpha = 0, 1$, and we have

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$$Y_0 I_n(f) = I_n(f)$$
$$Y_{10} L_{n,\alpha,10}(f) = L_{n,\alpha,10}(f)$$
$$Y_x L_{n,\alpha,x}(\omega) = u^{2\alpha+2} L_{n,\alpha,x}(\omega).$$

The classes of $\mathscr{Z}_2$-projective $\mathscr{V}$-modules are D and $C_1(1)$. Therefore

$$A_{\mathscr{Z}_2}(\mathscr{N}_6) \cong \mathscr{C}I_1(1) \oplus \mathscr{C} \oplus \mathscr{C}$$
$$A_{\mathscr{Z}_2}(\mathscr{N}_7) \cong \mathscr{C}I_{1,0}(1) \oplus \mathscr{C}L_{1,0,10}(1) \oplus \mathscr{C} \oplus \mathscr{C} \oplus \mathscr{C} \oplus \mathscr{C} \oplus \mathscr{C},$$

Thus $W_{\mathscr{V}}(\mathscr{N}_6) \cong \mathscr{R}/\mathscr{C}I_1(1)$. $I_1(1) \in \mathscr{R}J_0$, so we need only consider this factor. It can be split into two components, one of which is $\mathscr{C}I_1(1)$, by the idempotent decomposition

$$J_0 = I_1(1) + (J_0 - I_1(1))$$
$$= I_1(1) + \bar{J}_0, \text{ say.}$$

If we write $\bar{X}_0 = X_0 \bar{J}_0$, then the decomposition of $\mathscr{R}\bar{J}_0$ is just that of $\mathscr{R}J_0$ with X_0 replaced by $\bar{X}_0$, and with the case $f = 1$, $n = 1$, dropped from the summation. Notice that $\bar{X}_0 I_n(f) = X_0 I_n(f)$ except when $n = 1$ and $f = 1$. The same considerations apply to the $\mathscr{N}_7$ case. Therefore when $k = 6$ or 7 the form of $W_{\mathscr{V}}(\mathscr{N}_k)$ is just that given in (13) or (15), provided that the final term consisting of copies of $\mathscr{C}$ is omitted and that the case $n = 1, f = 1$, is dropped from all direct sums where it occurs.

6. Semisimplicity

It is now easy to see that $W_{\mathscr{V}}(\mathscr{N}_6)$ and $W_{\mathscr{V}}(\mathscr{N}_7)$ are semisimple. If $\mathscr{A}_1$ is any algebra of the form

(16) $$\mathscr{A}_1 = \mathscr{C}[X, 1/X] + \mathscr{B}$$

where $\mathscr{B}$ is an ideal of the form $\oplus\, \mathscr{C}I_r$ with r ranging through some indexing set, then $\mathscr{A}_1$ is semisimple.[8] It is clear that if $\mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \cdots$ is a finite sum of semisimple algebras then it is semisimple. But both $W_{\mathscr{V}}(\mathscr{N}_6)$ and $W_{\mathscr{V}}(\mathscr{N}_7)$ are of this form, where each $\mathscr{A}_i$ has the form of $\mathscr{A}_1$ in (16). Therefore we have the following result.

(17) THEOREM. *$W_{\mathscr{V}}(\mathscr{N}_6)$ and $W_{\mathscr{V}}(\mathscr{N}_7)$ are semisimple.*

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