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DIMENSION AND FINITE CLOSURE

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Abstract

If $\mathfrak{M}$ is a model with dimension and finite closure, then $T(\mathfrak{M})$ is $\aleph_0$-categorical. If $\mathfrak{M}$ is atomic, has dimension and finitely many algebraic elements, then $\mathfrak{M}$ has finite closure or a finite basis. If $\mathfrak{M}$ has finite closure, satisfies the Exchange Lemma, and one-one maps between independent subsets are elementary, then $\mathfrak{M}$ has dimension.

In Crossley & Nerode (1974, p. 44), the authors assume that the theories which they treat are $\aleph_0$-categorical, but note that it is sufficient, for their purposes, to consider a complete theory T for which each $B_n(T)$ is atomistic and every model has finite closure. A large part of their work concerns models with dimension. We show, in Section 1, that for a complete theory T with an infinite model which can be covered by finitely many minimal formulae, in particular with a model with dimension, T must be $\aleph_0$-categorical for its model to have finite closure. We also show, in Section 2, that if a model is atomic, has dimension and finitely many algebraic elements, then it has either a finite basis or finite closure.

If $\mathfrak{M}$ has dimension, then $\mathfrak{M}$ satisfies the Exchange Lemma and one-one maps between independent subsets of $\mathfrak{M}$ are elementary (see Propositions 1 and 3). We show, in Section 3, that if $\mathfrak{M}$ has these two properties and finite closure, then $\mathfrak{M}$ has dimension. No form of the axiom of choice is used.

Section 0 gives the notation and conventions we follow, as well as the necessary definitions and propositions from Crossley & Nerode (1974).

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MODELS. For a model, $\mathfrak{M}$, we use $\mathfrak{M}$ to denote the domain of $\mathfrak{M}$, if no ambiguity arises. We use a, b, etc. to denote elements of $\mathfrak{M}$ and A, B etc. to

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denote subsets of $\mathfrak{M}$. $T(\mathfrak{M})$ denotes the complete theory of $\mathfrak{M}$. We do not assume that the language of a model is countable but we do assume that the language contains a symbol for equality and that $\mathfrak{M}$ is a normal model. Thus we can define in the language quantifiers $\exists^{<k}v_0\cdots$ meaning "there exist $<k$ $v_0\cdots$" and $\exists^k v_0\cdots$ meaning "there exist exactly k $v_0\cdots$". $\chi(v_0,\cdots,v_n)$ will always denote a formula of the language of $\mathfrak{M}$ with all its free variables among $v_0,\cdots,v_n$. For $a_0,\cdots,a_n\in\mathfrak{M}$ we write $\mathfrak{M}\models\chi(v_0,\cdots,v_n)[a_0,\cdots,a_n]$ or $\mathfrak{M}\models\chi[a_1,\cdots,a_n]$ if $a_0,\cdots,a_n$ satisfies χ in $\mathfrak{M}$. If T is a complete theory, $B_n(T)$ denotes the boolean algebra of equivalence classes of formulae of the language of T with all their free variables among $v_0,\cdots,v_{n-1}$, where $\chi(v_0,\cdots,v_{n-1})$, $\psi(v_0,\cdots,v_{n-1})$ are equivalent if $T\vdash\forall v_0,\cdots,v_{n-1}(\chi\leftrightarrow\psi)$. We use χ to denote the equivalence class containing χ as no ambiguity arises. A model $\mathfrak{M}$ is *atomic*, if for every n-tuple $(a_0,\cdots,a_{n-1})$ of $\mathfrak{M}$ there is an atom χ of $B_n(T(\mathfrak{M}))$ such that $\mathfrak{M}\models\chi[a_0,\cdots,a_{n-1}]$. We say $\mathfrak{M}$ *is covered by the formulae* $\chi_1(v_0),\cdots,\chi_n(v_0)$ if $\mathfrak{M}\models\forall v_0(\chi_1\vee\cdots\vee\chi_n)$. *a is a solution of* $\chi(v_0)$ if $\mathfrak{M}\models\chi[a]$.

Algebraic Closure. We follow chapters 4 and 6 of Crossley & Nerode (1974). *a is algebraic over A* if for some $a_1,\cdots,a_n\in A$, $\chi(v_0,\cdots,v_n)$ and natural number k,

$$\mathfrak{M}\models(\exists^{<k}v_0\chi(v_0,\cdots,v_n)\ \&\ \chi(v_0,\cdots,v_n))[a,a_1,\cdots,a_n].$$

a is algebraic if it is algebraic over ϕ. The *algebraic closure of* A, cl A, is the set of all elements of $\mathfrak{M}$ algebraic over A. Clearly $A\subseteq\mathrm{cl}\,A$. $\mathfrak{M}$ *has finite closure* if cl A is finite whenever A is finite. A is *independent* if for all $a\in A$, $a\notin\mathrm{cl}(A\setminus\{a\})$. We write $(a_1,\cdots,a_n)$ is *independent* if $\{a_1,\cdots,a_n\}$ is independent and the a_i are distinct. A is a *basis* of $\mathfrak{M}$ if A is independent and $\mathfrak{M}=\mathrm{cl}\,A$. $\phi(v_0)$ is a *minimal formula for* $\mathfrak{M}$ if ϕ has infinitely many solutions in $\mathfrak{M}$ and for each $\psi(v_0,\cdots,v_n)$ and $a_1,\cdots,a_n\in\mathfrak{M}$ either $\phi(v_0)$ & $\psi(v_0,a_1,\cdots,a_n)$ or $\phi(v_0)$ & $\neg\psi(v_0,a_1,\cdots,a_n)$ has finitely many solutions in $\mathfrak{M}$. Clearly if $\phi(v_0)$ is minimal and $\psi(v_0)$ has only finitely many solutions then $\phi\vee\psi$ and ϕ & $\neg\psi$ are minimal. *Min* $(\mathfrak{M})$ is the set of solutions of minimal formulae. $\mathfrak{M}$ *has dimension* if for some minimal formula ϕ, $\mathfrak{M}\models\phi[a]$ for every non-algebraic element, a, of $\mathfrak{M}$. If $\mathfrak{M}$, $\mathfrak{M}'$ have the same language $\mathscr{L}$, $A\subseteq\mathfrak{M}$, $A'\subseteq\mathfrak{M}'$ and $p\colon A\to A'$, then *p is an elementary monomorphism* if for all $a_0,\cdots,a_n\in A$ and for all $\chi(v_0,\cdots,v_n)\in\mathscr{L}$

$$\mathfrak{M}\models\chi[a_0,\cdots,a_n]\quad\text{if and only if}\quad\mathfrak{M}'\models\chi[pa_0,\cdots,pa_n]$$

(p is one-one as $\mathscr{L}$ contains equality and $\mathfrak{M}$ is a normal model).

We use the following propositions.

PROPOSITION 1. (Crossley & Nerode (1974), Lemma 6.4(ib)). (Exchange Lemma) *For any model*, $\mathfrak{M}$, *if* $\{a_1, \cdots, a_n\} \subseteq \mathfrak{M}$ *is independent but* $\{a_1, \cdots, a_{n+1}\}$ *is not, and* $a_{n+1} \in \mathrm{Min}(\mathfrak{M})$, *then* $a_{n+1} \in \mathrm{cl}\{a_1, \cdots, a_n\}$.

PROPOSITION 2. (Crossley & Nerode (1974), Lemma 6.4(ii)). *For any model* $\mathfrak{M}$, *suppose* $A, B \subseteq \mathrm{Min}(\mathfrak{M})$, $\mathrm{cl}\, A \subseteq \mathrm{cl}\, B$ *and* A *is independent. Then*

(a) *card* $A \leqq$ *card* B

(b) *there is a subset* B_0 *of* B *such that* $A \cup B_0$ *is independent and* $\mathrm{cl}(A \cup B_0) = \mathrm{cl}\, B$.

An obvious and trivial modification of the proof of Crossley & Nerode (1974), Lemma 6.9, gives:

PROPOSITION 3. *Let* $\mathfrak{M}, \mathfrak{M}'$ *be models of a complete theory* T, $A \subseteq \mathrm{Min}(\mathfrak{M})$, $B \subseteq \mathfrak{M}'$ *independent sets and* $p: A \to B$ *a one-one map such that for* $a \in A$ *there is some minimal formula,* $\phi(v_0)$ *for* $\mathfrak{M}$ *such that* $\mathfrak{M} \models \phi[a]$ *and* $\mathfrak{M}' \models \phi[p(a)]$. *Then* p *is an elementary monomorphism.*

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We can prove our first result immediately.

THEOREM 4. *Suppose a complete theory* T *has an infinite model* $\mathfrak{M}$ *with finite closure which is covered by minimal formulae* $\phi_1, \cdots, \phi_n$. *Then* $B_m(T)$ *is finite for all* m *and* T *is* $\aleph_0$*-categorical.*

PROOF. We may assume that $\mathfrak{M} \models \bigwedge_{i \neq j} \forall v_0 \neg (\phi_i(v_0) \ \& \ \phi_j(v_0))$. For if, for $i \neq j$, $\mathfrak{M} \models (\phi_i \ \& \ \phi_j)[a]$ for infinitely many a then replace ϕ_i (say) with $\phi_i \vee \phi_j$ and delete ϕ_j. Now $\mathfrak{M} \models (\phi_i \ \& \ \neg \phi_j)[a]$ for finitely many a, so $\phi_j \vee (\phi_i \ \& \ \neg \phi_j)$ (i.e. $\phi_i \vee \phi_j$) is again minimal. If $\mathfrak{M} \models (\phi_i \ \& \ \phi_j)[a]$ for finitely many a and $i < j$ replace ϕ_j with $\phi_j \ \& \ \neg \phi_i$, which is again minimal. In both cases the new ϕ's cover $\mathfrak{M}$, so a simple induction validates the assumption.

Let $D_i = \{a \in \mathfrak{M} : \mathfrak{M} \models \phi_i[a]\}$. By the definition of a minimal formula each D_i is infinite.

Suppose $m \in \omega$. Then there is an independent subset C of $\mathfrak{M}$ such that card $C \cap D_i = m$ for all i, for if not, let r be the least m for which it fails. Then $r > 0$, and there is an independent C' such that card $C' \cap D_i = r - 1$. As $\mathrm{cl}\, C'$ is finite and D_1 is infinite, there is $c_1 \in D_1$ such that $c_1 \notin \mathrm{cl}\, C'$. So by Proposition 1, $C' \cup \{c_1\}$ is independent. Thus we can construct by induction C'' such that card $C'' \cap D_i = r$ for $i = 1, \cdots, n$, contradicting the choice of r.

If $\chi \in B_m(T)$, $\chi \neq 0$, then $T \vdash \exists v_1, \cdots, v_m \chi(v_1, \cdots, v_m)$. So there are $a_1, \cdots, a_m \in \mathfrak{M}$ such that $\mathfrak{M} \models \chi[a_1, \cdots, a_m]$.

By Proposition 2 there is an independent set $A = \{a'_1, \cdots, a'_{m'}\} \subseteq$

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$\{a_1, \cdots, a_m\}$ such that $\mathrm{cl}\{a_1, \cdots, a_m\} = \mathrm{cl}\, A$. Thus $\{a_1, \cdots, a_m\} \subseteq \mathrm{cl}\, A$. As $\mathrm{card}\, D_i \cap A \leqq m$ there is a one-one map $p: A \to C$ which satisfies the hypothesis of Proposition 3 and so is elementary.

As $\{a_1, \cdots, a_m\} \subseteq \mathrm{cl}\, A$, for $i = 1, \cdots, m$, there are formulae $\psi_i(v_0, \cdots, v_{m'})$, $\sigma_i(v_0, \cdots, v_{m'})$ and natural numbers k_i such that

$$\sigma_i(v_0, \cdots, v_{m'}) = (\psi_i(v_0, \cdots, v_{m'}) \ \&\ \exists^{<k_i} v_0 \psi_i(v_0, \cdots, v_{m'}))$$

and

$$\mathfrak{M} \models \sigma_i[a_i, a'_1, \cdots, a'_{m'}].$$

Hence $\mathfrak{M} \models (\exists u_1, \cdots, u_m(\chi(u_1, \cdots, u_m) \ \&\ \bigwedge_{i=1}^m \sigma_i(u_i, v_1, \cdots, v_{m'})))[a'_1, \cdots, a'_{m'}]$ and so

$$\mathfrak{M} \models (\exists u_1, \cdots, u_m(\chi(u_1, \cdots, u_m) \ \&\ \bigwedge_{i=1}^{m} \sigma_i(u_i, v_1, \cdots, v_{m'})))[p(a'_1), \cdots, p(a'_{m'})].$$

As $p(a'_i) \in C$, there exist $c_1, \cdots, c_m \in \mathrm{cl}\, C$ such that $\mathfrak{M} \models \chi[c_1, \cdots, c_m]$.

The map $q: B_m(T) \to \mathscr{P}((\mathrm{cl}\, C)^m)$ given by

$$q(\chi) = \{(c_1, \cdots, c_m) \in (\mathrm{cl}\, C)^m : \mathfrak{M} \models \chi[c_1, \cdots, c_m]\}$$

is one-one, for suppose $\chi_1, \chi_2 \in B_m(T)$ and $\chi_1 \neq \chi_2$. Then we may assume $\chi_1 \ \&\ \neg \chi_2 \neq 0$. So by the above, there are $c_1, \cdots, c_m \in C$ such that $\mathfrak{M} \models \chi_1 \ \&\ \neg \chi_2[c_1, \cdots, c_m]$ and therefore $q(\chi_1) \neq q(\chi_2)$. But $\mathscr{P}((\mathrm{cl}\, C)^m)$ is finite as $\mathrm{cl}\, C$ is, whence $B_m(T)$ is finite.

So by Ryll-Nardzewski (1959), T is $\aleph_0$-categorical. We note that this direction of Ryll-Nardzewski's proof does not require the axiom of choice. □

Regarding the converse of Theorem 4, if $\mathfrak{M} \models T$ and $B_m(T)$ is finite for all m, indeed just for $m = 1$, then $\mathfrak{M}$ can have at most finitely many minimal formulae, as it has only finitely many inequivalent 1-place formulae. However $(Q, \leqq)$ is a model of an $\aleph_0$-categorical theory and has no minimal formulae.

COROLLARY 5. *If T is a complete theory with a model $\mathfrak{M}$ with dimension and finite closure, then $B_m(T)$ is finite for each m and T is $\aleph_0$-categorical.*

PROOF. As $\mathfrak{M}$ has finitely many algebraic elements, $v_0 = v_0$ is a minimal formula which covers $\mathfrak{M}$. □

COROLLARY 6. *If T is a complete theory with a model $\mathfrak{M}$ with dimension and finite closure then every model $\mathfrak{N}$ of T has dimension and finite closure.*

PROOF. By Corollary 5, $B_m(T)$ is finite for each m and so by Crossley & Nerode (1974), Lemma 5.9, $\mathfrak{N}$ has finite closure. Furthermore $\mathfrak{N}$ is atomic.

As $\mathfrak{M}$ has dimension and finite closure, $v_0 = v_0$ is a minimal formula for $\mathfrak{M}$. We will show $v_0 = v_0$ is a minimal formula for $\mathfrak{N}$.

Let $a_1, \cdots, a_n \in \mathfrak{N}$ and $\chi(v_0, \cdots, v_n)$ be any formula. Let $\psi(v_0, \cdots, v_{n-1})$ be the atom satisfied by $a_1, \cdots, a_n$ and let $b_1, \cdots, b_n \in \mathfrak{M}$ satisfy ψ. As $v_0 = v_0$ is a minimal formula for $\mathfrak{M}$,

$$\mathfrak{M} \models \exists^{<k} v_0 \sigma(v_0, \cdots, v_n)[b_1, \cdots, b_n],$$

for some finite k, where σ is χ or $\neg \chi$. Hence

$$T \vdash \forall v_1, \cdots, v_n(\psi(v_1, \cdots, v_n) \to \exists^{<k} v_0 \sigma(v_0, \cdots, v_n))$$

as ψ is an atom and so

$$\mathfrak{N} \models \exists^{<k} v_0 \sigma(v_0, \cdots, v_n)[a_1, \cdots, a_n].$$

So $v_0 = v_0$ is minimal for $\mathfrak{N}$ whence $\mathfrak{N}$ has dimension. □

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We first prove a theorem from which our second claim follows readily:

THEOREM 7. *Suppose $\mathfrak{M}$ is an atomic model of a complete theory T and $\phi(v_0)$ is a minimal formula for $\mathfrak{M}$. Then for all n such that $\mathfrak{M}$ contains an independent set with $\geqq n+1$ solutions of ϕ, there is a formula ρ_{n+1}, an atom of $B_{n+1}(T)$, such that for any model $\mathfrak{M}'$ of T:*

$\mathfrak{M}' \models \rho_{n+1}[a'_0, \cdots, a'_n]$ *if and only if*

$(a'_0, \cdots, a'_n)$ *is independent and* $\mathfrak{M}' \models \phi[a'_i]$ $i = 0, \cdots, n$.

PROOF. Suppose $(a_0, \cdots, a_n) \subseteq \mathfrak{M}$ is independent and $\mathfrak{M} \models \phi[a_i]$ $i = 0, \cdots, n$. Then the a_i are distinct. As $\mathfrak{M}$ is atomic, there is an atom ρ_{n+1} of $B_{n+1}(T)$ such that $\mathfrak{M} \models \rho_{n+1}[a_0, \cdots, a_n]$. We show that it has the desired property.

Suppose $(a'_0, \cdots, a'_n) \subseteq \mathfrak{M}'$ is independent and $\mathfrak{M}' \models \phi[a'_i]$ $i = 0, \cdots, n$. Then by Proposition 3, $p: a_i \mapsto a'_i$ is elementary whence $\mathfrak{M}' \models \rho_{n+1}[p(a_0), \cdots, p(a_n)]$ which is precisely $\mathfrak{M}' \models \rho_{n+1}[a'_0, \cdots, a'_n]$.

Conversely, suppose $\mathfrak{M}' \models \rho_{n+1}[a'_0, \cdots, a'_n]$ and $(a'_0, \cdots, a'_n)$ is dependent. We may assume, without loss of generality, that $a'_0 \in \mathrm{cl}\{a'_1, \cdots, a'_n\}$. So there is a formula $\psi(v_0, \cdots, v_n)$ and natural number k, such that

$$\mathfrak{M}' \models (\psi(v_0, \cdots, v_n) \;\&\; \exists^{<k} v_0 \psi(v_0, \cdots, v_n))[a'_0, \cdots, a'_n].$$

So $$T \vdash \exists v_0, \cdots, v_n(\rho_{n+1} \;\&\; \psi \;\&\; \exists^{<k} v_0 \psi).$$

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But ρ_{n+1} is an atom of $B_{n+1}(T)$.

So $$T \vdash \forall v_0, \cdots, v_n(\rho_{n+1} \rightarrow (\psi \ \& \ \exists^{<k} v_0 \psi)).$$

But $\mathfrak{M} \models \rho_{n+1}[a_0, \cdots, a_n]$, so $\mathfrak{M} \models (\psi \ \& \ \exists^{<k} v_0 \psi)[a_0, \cdots, a_n]$.

Hence $a_0 \in \mathrm{cl}\{a_1, \cdots, a_n\}$ which contradicts the independence of $(a_0, \cdots, a_n)$. So $(a'_0, \cdots, a'_n)$ is independent.

It remains to show that $\mathfrak{M}' \models \phi[a'_i]$ $i = 0, \cdots, n$.

$$\mathfrak{M} \models \left(\rho_{n+1} \ \& \ \bigwedge_{i=0}^{n} \phi(v_i)\right)[a_0, \cdots, a_n].$$

Hence $T \vdash \forall v_0, \cdots, v_n \ (\rho_{n+1} \rightarrow \bigwedge_{i=0}^{n} \phi(v_i))$ as ρ_{n+1} is an atom of $B_{n+1}(T)$, and so $\mathfrak{M}' \models \bigwedge_{i=0}^{n} \phi(v_i)[a'_0, \cdots, a'_n]$ as $\mathfrak{M}' \models \rho_{n+1}[a'_0, \cdots, a'_n]$. $\square$

Corollary 8. *Suppose $\mathfrak{M}$ is an atomic model of a complete theory T, $\phi(v_0)$ is a minimal formula and $D = \{a : \mathfrak{M} \models \phi[a]\}$. If there are arbitrarily large finite independent subsets of D, then for any finite $A \subseteq D$, $D \cap \mathrm{cl}\, A$ is finite.*

Proof. Suppose for some $A \subseteq D$, that A is finite but $D \cap \mathrm{cl}\, A$ is infinite. Let $A = \{a_1, \cdots, a_n\}$. We may assume that A is independent, for by Proposition 2, there is an independent $A' \subseteq A$ such that $\mathrm{cl}\, A' = \mathrm{cl}\, A$ (No choice is needed as A is finite). By Theorem 7 and the hypothesis there is an atom of $B_{n+1}(T)$, $\rho_{n+1}(v_0, \cdots, v_n)$, such that for $d_i \in D$, $\mathfrak{M} \models \rho_{n+1}[d_0, \cdots, d_n]$ if and only if $(d_0, \cdots, d_n)$ is independent. By Proposition 1, if $(d_1, \cdots, d_n)$ is independent, $\mathfrak{M} \models \rho_{n+1}[d_0, \cdots, d_n]$ if and only if $d_0 \notin \mathrm{cl}\{d_1, \cdots, d_n\}$.

As $D \cap \mathrm{cl}\, A$ is infinite then $\phi(v_0) \ \& \ \neg \rho_{n+1}(v_0, a_1, \cdots, a_n)$ has infinitely many solutions in $\mathfrak{M}$, and as ϕ is minimal, $\phi(v_0) \ \& \ \rho_{n+1}(v_0, a_1, \cdots, a_n)$ has finitely many solutions, $d_1, \cdots, d_k$ say.

Hence $D \subseteq \mathrm{cl}\{a_1, \cdots, a_n, d_1, \cdots, d_k\}$ and $\{a_1, \cdots, a_n, d_1, \cdots, d_k\} \subseteq \mathrm{Min}(\mathfrak{M})$. So if $\{a'_1, \cdots, a'_m\} \subseteq D$ is independent, by Proposition 2,

$$m = \mathrm{card}\{a'_1, \cdots, a'_m\} \leqq \mathrm{card}\{a_1, \cdots, a_n, d_1, \cdots, d_k\} \leqq n + k,$$

which contradicts the hypothesis of the corollary.

Hence $D \cap \mathrm{cl}\, A$ is finite for all finite $A \subseteq D$. $\square$

The main result of this section is:

Corollary 9. *If $\mathfrak{M}$ is an atomic model with dimension and finitely many algebraic elements then $\mathfrak{M}$ has a finite basis or finite closure.*

PROOF. The following are clear, as is the deduction of Corollary 9 from them and Corollary 8.

If $\mathfrak{M}$ has dimension and $\mathrm{cl}\,\phi$ is finite there is a minimal formula ϕ for which $D = \mathfrak{M}$. If $\mathfrak{M}$ does not have a finite basis then it has arbitrarily large independent subsets. $\square$

We can find atomic models with dimension and finitely many algebraic elements with a finite basis but not finite closure ((Z, S) where $S(n) = n + 1$) and with no finite basis but finite closure ($(N, =)$). Models with a finite basis and finite closure are finite and so do not have dimension, as we require a minimal formula to have infinitely many solutions.

If we do not assume that $\mathfrak{M}$ is atomic, then Corollary 9 is false. If we take $\mathfrak{M} = (V, +, f_\lambda)_{\lambda \in F}$ where V is an infinite dimensional vector space over an infinite field F, and $f_\lambda : v \mapsto \lambda v$ is a unary function, then $\mathfrak{M}$ has dimension but neither a finite basis nor finite closure, as algebraic closure is closure in the usual vector space sense. $\mathfrak{M}$ is not atomic, for if $\{a_0, \cdots, a_n\}$ is independent, $\mathfrak{M} \models a_0 \neq \lambda_1 a_1 + \cdots + \lambda_n a_n$ for all $\lambda_1, \cdots, \lambda_n \in F$, whereas $B_{n+1}(T)$ is generated by $\{\lambda_0 v_0 = \lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_i \in F\}$ as $T(\mathfrak{M})$ admits elimination of quantifiers. Hence there is no atom satisfied by $(a_0, \cdots, a_n)$.

Combining Corollaries 5 and 9 we obtain:

COROLLARY 10. *If T is a complete theory with an atomic model with dimension but no finite basis and finitely many algebraic elements then T is $\aleph_0$-categorical.*

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THEOREM 11. *Suppose $\mathfrak{M}$ has the following properties.*

(1) $\mathfrak{M}$ has finite closure.

(2) If $\{a_1, \cdots, a_n\} \subseteq \mathfrak{M}$ is independent and $\{a_1, \cdots, a_{n+1}\}$ is not, then $a_{n+1} \in \mathrm{cl}\{a_1, \cdots, a_n\}$. ($\mathfrak{M}$ satisfies the Exchange Lemma).

(3) If $A, B \subseteq \mathfrak{M}$ are independent and $p : A \to B$ is one-one, then p is elementary.

Then $\mathfrak{M}$ has dimension.

PROOF. We prove the following by induction on n:

(4) If $\{a_1, \cdots, a_m\}$ is independent, $b_1, \cdots, b_n \in \mathrm{cl}\{a_1, \cdots, a_m\}$ then there exist $c_1, \cdots, c_p$ such that $\{a_1, \cdots, a_m, c_1, \cdots, c_p\}$ is independent and for any formula $\psi(v_1, \cdots, v_{m+n+p+1})$ and for any $d_1, d_2 \notin \mathrm{cl}\{a_1, \cdots, a_m, c_1, \cdots, c_p\}$,

$$\mathfrak{M} \models \psi[a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p, d_1] \text{ if, and only if,}$$

$$\mathfrak{M} \models \psi[a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p, d_2].$$

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Suppose $n = 0$. If $d_1, d_2 \notin \mathrm{cl}\{a_1, \cdots, a_m\}$ then, by (2), $\{a_1, \cdots, a_m, d_1\}$ and $\{a_1, \cdots, a_m, d_2\}$ are independent, and (4) holds by (3).

Suppose (4) holds for some n and $b_{n+1} \in \mathrm{cl}\{a_1, \cdots, a_m\}$. Then $b_{n+1} \in \mathrm{cl}\{a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p\}$ and there is a formula $\chi_0(v_0, \cdots, v_{m+n+p})$ and a natural number $k_0 \geqq 1$, such that

$$\mathfrak{M} \models \chi_0 \ \& \ \exists^{k_0} v_0 \chi_0 [b_{n+1}, a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p].$$

We construct sequences $k_0 > k_1 > \cdots > k_q \geqq 1$, $\chi_0, \cdots, \chi_q$, $c_{p+1}, \cdots, c_{p+q}$, such that for any formula $\psi(v_1, \cdots, v_{m+n+p+q+2})$ and $d_1, d_2 \notin \mathrm{cl}\{a_1, \cdots, a_m, c_1, \cdots, c_{p+q}\}$

$$\text{(5)} \qquad \begin{array}{l} \mathfrak{M} \models \psi\,[a_1, \cdots, a_m, b_1, \cdots, b_{n+1}, c_1, \cdots, c_{p+q}, d_1] \quad \text{if, and only if,} \\ \mathfrak{M} \models \psi\,[a_1, \cdots, a_m, b_1, \cdots, b_{n+1}, c_1, \cdots, c_{p+q}, d_2]. \end{array}$$

If (5) holds with $q = 0$, we are done. If not, there is a formula $\psi(v_1, \cdots, v_{m+n+p+2})$ and $d_1, d_2 \notin \mathrm{cl}\{a_1, \cdots, a_m, c_1, \cdots, c_p\}$ such that

$$\text{(6)} \qquad \mathfrak{M} \models \psi[a_1, \cdots, a_m, b_1, \cdots, b_{n+1}, c_1, \cdots, c_p, d_1]$$

and

$$\text{(7)} \qquad \mathfrak{M} \models \neg\, \psi[a_1, \cdots, a_m, b_1, \cdots, b_{n+1}, c_1, \cdots, c_p, d_2].$$

Put $c_{p+1} = d_1$ and put

$$\chi_1(v_0, \cdots, v_{m+n+p+2}) = \chi_0(v_0, \cdots, v_{m+n+p}) \ \& \ \psi(v_1, \cdots, v_{m+n}, v_0, v_{m+n+1}, \cdots, v_{m+n+p+2}).$$

By (2), $\{a_1, \cdots, a_m, c_1, \cdots, c_{p+1}\}$ is independent. Clearly

$$\mathfrak{M} \models \chi_1[b_{n+1}, a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_{p+1}].$$

And

$$\text{(8)} \qquad \mathfrak{M} \models \exists v_0(\chi_0 \ \& \ \neg\, \chi_1)[a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_{p+1}]$$

for suppose otherwise. Then

$$\mathfrak{M} \models \forall v_0(\chi_0 \rightarrow \chi_1)[a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p, d_1]$$

whence, by (4),

$$\mathfrak{M} \models \forall v_0(\chi_0 \rightarrow \chi_1)[a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p, d_2].$$

But $\mathfrak{M} \models \chi_0[b_{n+1}, a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p]$ and so

$$\mathfrak{M} \models \chi_1[b_{n+1}, a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_p, d_2]$$

and therefore

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$$\mathfrak{M} \models \psi[a_1, \cdots, a_m, b_1, \cdots, b_{n+1}, c_1, \cdots, c_p, d_2]$$

which contradicts (7). Thus (8) holds and

$$\mathfrak{M} \models \exists^{k_1} v_0 \chi_1[v_0, a_1, \cdots, a_m, b_1, \cdots, b_n, c_1, \cdots, c_{p+1}]$$

where $1 \leqq k_1 < k_0$.

We can choose χ_i, c_i is a similar fashion until (5) holds, which is when (4) holds for $n+1$.

Thus (4) holds for all n.

Now suppose $b_1, \cdots, b_n \in \mathfrak{M}$ and $\psi(v_0, \cdots, v_n)$ is any formula. Using (2), we can choose $a_1, \cdots, a_m \in \{b_1, \cdots, b_n\}$ such that $\{a_1, \cdots, a_m\}$ is independent and $b_1, \cdots, b_n \in \mathrm{cl}\{a_1, \cdots, a_m\}$. By (4), there exist $c_1, \cdots, c_p$ such that for all $\chi(v_0, \cdots, v_n)$ and $d_1, d_2 \notin \{a_1, \cdots, a_m, c_1, \cdots, c_p\}$ $\mathfrak{M} \models \chi[d_1, b_1, \cdots, b_n]$ if, and only if $\mathfrak{M} \models \chi[d_2, b_1, \cdots, b_n]$.

If $\psi(v_0, b_1, \cdots, b_n)$ has infinitely many solutions in $\mathfrak{M}$, then $\mathfrak{M} \models \psi[d, b_1, \cdots, b_n]$ for some $d \notin \mathrm{cl}\{a_1, \cdots, a_m, c_1, \cdots, c_p\}$ as $\mathrm{cl}\{a_1, \cdots, a_m, c_1, \cdots, c_p\}$ is finite by (1). Hence $\mathfrak{M} \models \psi[d, b_1, \cdots, b_n]$ for all $d \notin \mathrm{cl}\{a_1, \cdots, a_m, c_1, \cdots, c_p\}$ and $\mathfrak{M} \models \neg\psi[d, b_1, \cdots, b_n]$ for at most $d \in \mathrm{cl}\{a_1, \cdots, a_m, c_1, \cdots, c_p\}$. Thus $\neg\psi(v_0, b_1, \cdots, b_n)$ has finitely many solutions, and so $v_0 = v_0$ is a minimal formula.

Therefore $\mathfrak{M}$ has dimension. □

Conditions (2) and (3) are not sufficient for $\mathfrak{M}$ to have dimension. Consider the model $\mathfrak{N} = (Z \times Z, <, S)$ where

$$(n_1, m_1) < (n_2, m_2) \quad \text{if} \quad n_1 = n_2 \quad \text{and} \quad m_1 < m_2$$

and

$$S((n, m)) = (n, m+1).$$

It is easy to see the following:

(a) $\mathrm{cl}\{(n_1, m_1), \cdots, (n_l, m_l)\} = \{n_1, \cdots, n_l\} \times Z$.

(b) $\{(n_1, m_1), \cdots, (n_l, m_l)\}$ is independent if and only if $n_1, \cdots, n_l$ are distinct, and therefore (2) holds.

(c) If $A, B \subseteq \mathfrak{N}$ are finite and independent and $p: A \to B$ is one-one, then p extends to an automorphism of $\mathfrak{N}$ and so is elementary. Therefore (3) holds.

(d) $\mathfrak{N}$ has no algebraic elements and $v_0 < a_1$, $\neg v_0 < a_1$ both have infinitely many solutions in $\mathfrak{N}$. Thus $\mathfrak{N}$ does not have dimension.

By Corollary 5, if $\mathfrak{M}$ satisfies (1), (2) and (3), then $B_n(T(\mathfrak{M}))$ is finite for each n, and so $\mathfrak{M}$ is atomic. However Theorem 11 does not hold if we replace (1) by "$\mathfrak{M}$ is atomic", for the model $\mathfrak{N}$ provides a counter-example.

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$$\psi(v_{11}, \cdots, v_{1k_1}, \cdots, v_{l1}, \cdots, v_{lk_l})$$
$$= \bigwedge_{i_1 \neq i_2} \Big(\neg (v_{i_1 1} < v_{i_2 1}) \ \& \ \neg (v_{i_2 1} < v_{i_1 1}) \ \& \bigwedge_{i=1,\cdots,l} \bigwedge_{j_2 < j_1} v_{ij_1} = S^{j_1 - j_2}(v_{ij_2}) \Big)$$

is a formula satisfied by

$$((n_1, m_{11}), \cdots, (n_1, m_{1k_1}), \cdots, (n_l, m_{l1}), \cdots, (n_l, m_{kl}))$$

where $n_1, \cdots, n_l$ are distinct, and is an atom, as can be seen by extending the map $a_{ij} \mapsto a'_{ij}$, where $\mathfrak{N} \models \psi(a_{11}, \cdots, a_{lk_l})$ and $\mathfrak{N} \models \psi(a'_{11}, \cdots, a'_{lk_l})$, to an automorphism of $\mathfrak{N}$.

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