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Two properties of Bochner integrals

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Two theorems for Lebesgue integrals, namely the Gauss-Green Theorem relating surface and volume integrals, and the integration-by-parts formula, are shown to possess generalizations where the integrands take values in a Banach space, the integrals are Bochner integrals, and derivatives are Fréchet derivatives. For integration-by-parts, the integrand consists of a continuous linear map applied to a vector-valued function. These results were required for a generalization of the calculus of variations, given in another paper.

This paper assumes the definition, and standard properties, of Bochner integrals, as given in Hille and Phillips [2] and in Yosida [3]. Neither of these books gives the theorems proved in this note. Let V denote a Banach space, over the real field, and let $[V]$ denote the Banach space of all bounded linear maps from V into V, with the usual norm. Let $I = [a, b]$ denote a compact real interval; let $\chi_E(\cdot)$ denote the characteristic function of E, where E is a measurable subset of I.

Let G be a bounded open subset of Euclidean p-space R^p, with boundary ∂G; let $\mu_p(x)$ denote p-dimensional Lebesgue measure, where $x = (x_1, \ldots, x_p) \in \mathrm{R}^p$; for $x \in \partial G$, let $\nu(x)$ and $\Phi(x)$ denote suitable defined unit exterior normal and surface area on the "surface" ∂G, as defined in Craven [1].

If $g(x) = (g_1(x), \ldots, g_p(x))$ is a p-vector valued function of

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x ,

(1) $$\operatorname{div} g(x) = \sum_{i=1}^{p} \frac{\partial g_i(x)}{\partial x_i} \quad \text{and} \quad g(x).\nu(x) = \sum_{i=1}^{p} g_i(x)\nu_i(x)$$

then the Gauss-Green Theorem states that, if G and g satisfy suitable conditions, then

(2) $$\int_G \operatorname{div} g(x) d\mu_p(x) = \int_{\partial G} g(x).\nu(x) d\Phi(x) .$$

If g maps the closure $\bar{G}$ of G into V^p , where V is a Banach space, instead of into $\mathbb{R}^p$, and, for $i = 1, 2, \ldots, p$, g_i is Fréchet-differentiable with respect to x_i , for fixed x_j $(j \neq i)$, then (1) and (2) remain meaningful in terms of Fréchet derivative and Bochner integrals; both $\operatorname{div} g(.)$ and $g(.).\nu(.)$ are maps of G into V . The Gauss-Green Theorem then holds in the following form:

THEOREM A. *Let G be a bounded open subset of $\mathbb{R}^p$, such that ∂G is a countable union of disjoint continuous images of the unit sphere in $\mathbb{R}^p$, and $\Phi(\partial G) < \infty$; let $E \subset G$ satisfy the same conditions as ∂G . Let $g : \bar{G} \to V^p$ be a continuous map, such that $\operatorname{div} g(x)$ exists at each point of $G - E$, and $\|\operatorname{div} g(x)\|$ is Lebesgue integrable on G . Then* (2) *holds for G , ∂G , and g .*

Proof. Since the proof differs only in a few key details from the proof for $V = \mathbb{R}$ (Theorems 1, 2 and 3 of [1]), only the changes need be stated. By Bochner's Theorem ([3], p. 133), integrability of $\|\operatorname{div} g(x)\|$ implies that the left side of (2) exists as a Bochner integral. The proof of Theorem 1 of [1] remains applicable, with the norm $\|.\|$ of V replacing absolute value $|.|$ where appropriate. In the proof of Theorem 2, equation (11) applies with $\|.\|$ replacing $|.|$. The definition of the function ψ requires modification. Let

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$$z = g_i(x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_p) - g_i(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_p) - \int_{T_i} \frac{\partial g_i}{\partial x_i}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p)dx_i \;;$$

thus $z \in V$. Denote by z'' the canonical image of z in the second dual space V'' . Then Lemma 5 of [1] applies to

$$w(x) = f\left(g_i(x_1, \ldots, x_i, x, x_{i+1}, \ldots, x_p)\right)$$

for each f in the dual space V' such that $\|f\| = 1$. Then

$$|f(z)| \le K = N\left(b_i - a_i - \mu_1(T_i)\right) ,$$

so that

$$\|z\| = \|z''\| = \sup\{|f(z)| : \|f\| = 1\} \le K .$$

From this, equations (13) and (16) of the proof of Theorem 2 follow, with $\|.\|$ replacing $|.|$. Then Theorem 3, with $g : \bar{G} \to V^p$ replacing $g : \bar{G} \to \mathrm{R}^p$, is an immediate consequence of the modified Theorems 1 and 2.

Integration by parts for the Bochner integral depends on the following lemma (for integration with respect to Lebesgue measure).

LEMMA 1. *Let* $f : I \to V$ *be Bochner-integrable on* I *; let* $T_0 \in [V]$ *; let* $a \le \alpha < \beta < b$ *; then*

$$\int_a^b \chi_{(\alpha,\beta]}(t) T_0\left(\int_a^t f(s)ds\right)dt = -\int_a^b \left(\int_a^s \chi_{(\alpha,\beta]}(t)dt\right) T_0 f(s)ds + (\beta-\alpha)T_0 \int_a^b f(s)ds .$$

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Proof.

$$\int_\beta^\alpha T_0\left(\int_a^t f(s)ds\right)dt = \int_\alpha^\beta \left(\int_a^t \left(T_0f\right)(s)ds\right)dt \text{ by [2], Theorem 3.7.12,}$$

$$= \int_a^\alpha \left(\int_\alpha^\beta T_0f(s)dt\right)ds + \int_\alpha^\beta \left(\int_s^\beta T_0f(s)dt\right)ds \text{ by [2], Theorem 3.7.13,}$$

$$= \int_a^\alpha (\beta-\alpha)T_0f(s)ds + \int_\alpha^\beta (\beta-s)T_0f(s)ds \ ;$$

which yields the right side of the stated result by rearrangement.

THEOREM B. *If* $f : I \to V$ *and* $T(.) : I \to [V]$ *are Bochner-integrable on* I *, then*

$$\int_I T(t)\left(\int_a^t f(s)ds\right)dt = -\int_I \left(\int_a^s T(t)dt\right)f(s)ds + \left(\int_I T(t)dt\right)\left(\int_I f(s)ds\right) .$$

REMARK. For each $t \in I$, $T(t)$ is a bounded linear map from V into V . If, in particular, $V = \mathrm{R}$, then $T(t)\int^t f(s)ds$ is of the form $\varphi(t)\int^t f(s)ds$, for some function $\varphi(.)$; and the result reduces to the usual integration-by-parts formula. But in general, each integral is a Bochner integral on $I = [a, b]$ or a subinterval.

Proof. Lemma 1 gives the result, in case $T(t) = \chi_{(\alpha,\beta]}(t)T_0$ and $T_0 \in [V]$; hence Theorem B holds for any Bochner-integrable f and any step-function $T(.)$, that is, any function $T(.)$ which assumes only finitely many values in $[V]$, each on a subinterval of I . In terms of the norm $|||T||| = \int_I \|T(t)\|dt$, the step-functions are a dense subspace of the Bochner-integrable functions; so Theorem B follows, from the definition of Bochner integral.

Theorem B has a variant in terms of line integrals in a (real) Banach space V , taken by convention along straight segments. If $a, b \in V$, β is a real variable, $x = a + \beta b$, $\|b\| = 1$, denote also $\int \ldots d|x|$ to

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mean $\int \dots d\beta$ and $\int \dots dx$ to mean $\int \dots (d\beta)b$.

THEOREM C. *Let* A *and* V *be real Banach spaces;* U *a convex open subset of* A *;* $T(a) \in [V]$ *for each* $a \in U$ *;* $h : U \to V$ *a continuous Fréchet-differentiable map such that, for* $a, b \in A$ *,* $\|b\| = 1$ *, and* $a, c = a + \lambda b \in U$ *,* $h(a+\beta b)b$ *is an absolutely continuous function of* $\beta \in [0, \lambda]$ *. Then*

$$\int_a^c T(x)h(x)d|x| = -\int_a^c \left(\int_a^z T(x)d|x|\right)h'(z)dz + \left(\int_a^c T(x)d|x|\right)h(c) .$$

Proof. From Theorem B,

$$\int_0^\lambda T(a+\beta b)\left(\int_0^\alpha h'(a+\alpha b)bd\alpha\right)d\beta =$$

$$-\int_0^\lambda \left(\int_0^\alpha T(a+\beta b)d\beta\right)h'(a+\alpha b)bd\alpha + \left(\int_0^\lambda T(a+\beta b)d\beta\right)\left(\int_0^\lambda h'(a+\alpha b)bd\alpha\right) .$$

Define $f : [0, \lambda] \to V$ by $f(\beta) = h(a+\beta b)b$; let $e \in V$; let P be the projector of V onto the one-dimensional subspace spanned by e ; then Pf is absolutely continuous, mapping $[0, \beta]$ into Re ; therefore

$$(Pf)(\alpha) - (Pf)(0) = \int_0^\alpha (Pf)'(\beta)d\beta = P\int_0^\alpha f'(\beta)d\beta \quad (0 < \alpha < \lambda)$$

since h , and therefore f , is Fréchet-differentiable. Therefore, since e is arbitrary,

$$h(a+\alpha b) - h(a) = f(\alpha) - f(0) = \int_0^\alpha h'(a+\beta b)bd\beta .$$

Substitution of this expression into the result from Theorem B proves the theorem.

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