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TAIL VARIANCE AND CONFIDENCE OF USING TAIL CONDITIONAL EXPECTATION: ANALYTICAL REPRESENTATION, CAPITAL ADEQUACY, AND ASYMPTOTICS

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Abstract

In this paper, we explore the applications of Tail Variance (TV) as a measure of tail riskiness and the confidence level of using Tail Conditional Expectation (TCE)-based risk capital. While TCE measures the expected loss of a risk that exceeds a certain threshold, TV measures the variability of risk along its tails. We first derive analytical formulas of TV and TCE for a large variety of probability distributions. These formulas are useful instruments for relevant research works on tail risk measures. We then propose a distribution-free approach utilizing TV to estimate the lower bounds of the confidence level of using TCE-based risk capital. In doing so, we introduce sharpened conditional probability inequalities, which halve the bounds of conventional Markov and Cantelli inequalities. Such an approach is easy to implement. We further investigate the characterization of tail risks by TV through an exploration of TV's asymptotics. A distribution-free limit formula is derived for the asymptotics of TV. To further investigate the asymptotic properties, we consider two broad distribution families defined on tails, namely, the polynomial-tailed distributions and the exponential-tailed distributions. The two distribution families are found to exhibit an asymptotic equivalence between TV and the reciprocal square of the hazard rate. We also establish asymptotic relationships between TCE and VaR for the two families. Our asymptotic analysis contributes to the existing research by unifying the asymptotic expressions and the convergence rate of TV for Student-t distributions, exponential distributions, and normal distributions, which complements the discussion on the convergence rate of univariate cases in [28]. To show the usefulness of our results, we present two case studies based on real data from the industry. We first show how to use conditional inequalities to assess the confidence of using TCE-based risk capital for different types of insurance businesses. Then, for financial data, we provide alternative evidence for the relationship between the data frequency and the tail categorization by the asymptotics of TV.

Keywords: Tail variance; tail conditional expectation; probability inequalities; capital adequacy; asymptotics

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1. Introduction

Studies on the tails of risks have always been the core research in finance and actuarial science. Regulators, investors, and other stakeholders are often interested in the risk's tail region beyond a probability level p (p is close to 1) and rely on risk measures defined on such a tail region, namely, $\left[p, 1\right]$, to assess the potential loss of the risk. For instance, Solvency II regulation accounts for the insurance risk based on Value-at-Risk (VaR) at $p = 99.5\%$ over a one-year period while Basel III tends to use Expected Shortfall (ES) as the measure of various risks for "expected losses" subject to the stressed region $\left[p, 1\right]$ [13]. These risk measures defined on tails not only suggest the amount of capital adequacy for financial institutions under potential risks but also provide insights into understanding risks' behaviors at tails. There are also other well-known risk measures defined on tails such as Conditional Value-at-Risk (CVaR), Tail Conditional Expectation (TCE, also known as CTE, i.e. Conditional Tail Expectation), etc. It has been shown that the three risk measures (ES, CVaR, and TCE), which are coinciding for continuous risks, satisfy the so-called "coherence"; see [4]. Coherence requires risk measures to satisfy a number of desirable properties such as monotonicity, subadditivity, positive homogeneity, translation invariance, etc. These properties bring mathematical superiority to coherent risk measures and make them relatively more robust in assessing risks at tails. By contrast, VaR is not coherent, but it has advantages in practical use; for instance, it is more straightforward for back-testing. We refer to [14] for a study on the comparison between VaR and ES and to [11] for more details on the coherent risk measure.

Despite the popularity of the aforementioned risk measures, they often appear to be insufficient in exploiting the information about the riskiness at the tail. [16] gave an example to illustrate such insufficiency and proposes the Tail Variance (TV) and the Tail Variance premium (TVp) to further account for the tail riskiness. In particular, in this work, TV is defined as the conditional variance of a risk on the tail region $\left[p, 1\right]$, while TVp is the well-known mean-variance premium in which the mean and the variance are replaced by TCE and TV. The authors also derive the formulas of TV and TVp for elliptically distributed risks. In a later work, by [27], the formulas of TV and a TV-based capital allocation rule were derived for log-elliptical risks. [22] conducted similar research for symmetric generalized hyperbolic risks and pointed out that these formulas of TV and TVp can estimate the loss severity at tails. Intuitively speaking, in the same manner that variance measures the dispersion of a risk from its mean, TV accounts for the magnitude of the risk deviating from its TCE on the chosen tail region. Specifically, as a measure of tail variability, TV satisfies properties such as law-invariance, standardization, and location invariance. Furthermore, [17] discussed the tail standard deviation ($\sqrt{\mathrm{TV}}$) on the measurement of tail variability. [19] defined the coherent measures on the tail variability; in fact, $\sqrt{\mathrm{TV}}$ is such a coherent measure for tail variability. [7] studied the parametric estimation on the variability. Therefore, studying TV as a risk measure, in addition to studying VaR and TCE, are of importance and interest in several aspects.

First, TV provides additional information on the riskiness of tails. Many existing results in the literature based on VaR and TCE can be extended straightforwardly by further taking TV into account. For instance, [26] considered a TCE–TV framework as an extension of the classic mean-variance framework at tails. Second, TV measures how far a risk deviates from its TCE. Therefore, when TCE is used to estimate the risk capital, TV can be used to measure the confidence level of using the TCE-based risk capital. In particular, by developing an analogue of the well-known Chebyshev inequality at the tail region, one can find an interval regarding how far the actual loss may differ from the TCE-based risk capital for a given confidence level. Third, unlike VaR and TCE, which both increase to infinity when p approaches 1, the limit of TV along $p \to 1$ shows a certain tendency that may converge to 0, a positive constant, or infinity. This phenomenon was first noticed by [22], in which the authors showed that TV $\to \infty$ for Student-t distributions while TV $\to 0$ for normal distributions. Thus, it is of interest to study the asymptotics of TV, which may characterize the tail behaviors of the risk and provide further insights into the confidence level of using TCE-based capital under extreme events.

In this paper, we investigate using TV to account for the confidence of using TCE to assess tail riskiness in the context of absolutely continuous risks. Our findings contribute to the literature in three ways. First, we provide a list of analytical TV formulas for a large variety of probability distributions, including a Pareto distribution, a Weibull distribution, a normal distribution, etc. As a necessary step in calculating TV, we also provide explicit formulas of TCE for these probability distributions. These formulas not only cover the existing results in the literature as special cases but also are useful instruments for relevant research works, such as the TV premium calculation [16], optimal capital allocation [38], loss severity estimation [22], optimal portfolio selection [12, 26], etc.

Secondly, we propose a novel approach to assessing the confidence level of using TCE-based risk capital from multiple angles. By virtue of the appropriateness of risk capital defined in [15], we introduce sharpened conditional probability inequalities based on TV, which offers a significant advancement by halving the confidence bounds of TCE-based risk capital. More specifically, we first propose a sharpened conditional Markov inequality that gives a lower bound of the confidence level of using the TCE-based risk capital. The lower bound could possibly be further improved by using a sharpened conditional Cantelli inequality. Then we present a sharpened conditional Chebyshev inequality to offer a more precise confidence interval within which possible extreme loss values may fall. Our results offer practical and accessible tools for practitioners and regulators alike to evaluate capital adequacy. To illustrate the usefulness of our approach, we present a case study based on real data from two insurance business lines. Overall, our findings demonstrate the effectiveness of our proposed method for assessing the confidence of using TCE-based risk capital and provide valuable insights for risk management practices.

Third, we study the asymptotic properties of TV. We first give a distribution-free limit of the TV when $p \to 1$ and show that, for an absolutely continuous random variable, the limit value of TV coincides with the limit value of the reciprocal square of the hazard rate. In other words, TV gives similar information as the hazard rate does but at tail regions. To derive explicit forms, we then consider two large distribution families defined by the tails, namely, the distribution family of exponential tails and the distribution family of polynomial tails. We establish explicit asymptotic formulas of TVs for the two distribution families and further show that there exist exact asymptotic equivalences between the TV and the reciprocal square of the hazard rate. In doing so, we also derive formulas of asymptotic equivalences between TCE and VaR in the context of the two distribution families. Note that the result of asymptotic equivalences is much stronger than equal limits. The asymptotic equivalence not only guarantees the same limits but also implies the same convergence rate. There are many papers discussing TV asymptotic formulas: for instance, [22] and [28]. Our work on TV not only unifies the TV asymptotic formulas discussed on specific distribution assumptions but also points out the convergence speed that previous literature failed to give. We then provide a categorization of the tails of risks based on TV's asymptotics. As an application, we show that such a categorization gives new evidence to the relationship between the financial data frequency and the tail types of risks.

We organize the rest of the paper as follows: Section 2 derives the formulas of TV and TCE for frequently used probability distributions. Section 3 presents the two applications with numerical case studies. Section 4 investigates the asymptotic behavior of TV and establishes an asymptotic relationship between the TV and the hazard rate. Section 5 concludes the paper.

2. Formulas of Tail Variance and Tail Conditional Expectation in Assessing Tail Risks

2.1. Definitions and notations

We first introduce the following notations and definitions for our paper: In this paper, X is an absolutely continuous random variable defined on a support with an unbounded supremum. We denote its probability density function (PDF) and cumulative distribution function (CDF) as $f(x)$ and $F(x)$, respectively. Furthermore, we have $\overline{F}(x) = 1 - F(x)$, the tail distribution function. In actuarial science or reliability theory, $\overline{F}(x)$ is also referred to as the survival function or the reliability function. The hazard rate of X (notation $h(x)$), defined as the ratio of density function $f(x)$ and survival function $\overline{F}(x)$ $\left(\text{i.e. } h(x) = \frac{f(x)}{\overline{F}(x)}\right)$ is also widely used as a characteristic of X. The hazard rate often refers to the force of mortality or the failure rate of the risk in actuarial science and reliability engineering. $h(x)$ is also often involved in discussing heavy-tailed and light-tailed distributions in the literature; see [24, 35] and references therein.

A risk measure is a functional mapping of X to a real number, possibly infinite, that reflects the riskiness of X. It suggests an amount of capital that should be added to the risk such that it is acceptable to stakeholders. Such an amount of risk capital should be adequate to cover the possible loss of the risk in the sense of the corresponding risk measure. In our context, the quantile function $Q_q[X] := \inf\{x, F(x) \geq q\}$, $q \in (0, 1)$, is the well-known risk measure Value-at-Risk. Then the Tail Conditional Expectation of X is defined as

$$\mathrm{TCE}_q[X] = \mathbb{E}\left[X|X > x_q\right], \tag{1}$$

where $x_q = Q_q[X]$. [16] introduces TV as follows:

$$\mathrm{TV}_q[X] = \mathrm{Var}\left[X|X > x_q\right] = \mathbb{E}\left[X^2|X > x_q\right] - \mathbb{E}^2\left[X|X > x_q\right] \tag{2}$$

and points out that $\mathrm{TV}_q[X]$ has the following property:

$$\mathrm{TV}_q[X] = \inf_c \mathbb{E}\left[(X - c)^2 \,|X\rangle\, x_q\right].$$

TV, as a risk measure, actually provides additional information on the riskiness of distribution at tails, especially when the Tail Conditional Expectation fails to distinguish the risks. In this section, we first derive the formulas of TV for frequently used distributions. Naturally, we also derive the formulas of TCE in the process, which is required when deriving TV formulas. Using these formulas, we present an example at the end of this section to illustrate the insufficiency of TCE in identifying the tail riskiness.

2.2. Explicit Tail variance and Tail Conditional Expectation formulas for frequently used distributions

Many papers have endeavored to derive the formulas of TV and TCE for specific distributions in the literature. [16] did such work in the context of the normal distribution and the elliptical distribution. In addition, the TV formula for the symmetric generalized hyperbolic distributions was derived in [22]. Furthermore, the TV formula for the generalized hyperbolic distribution has been proved in [23]. Given the increasing popularity of TV and capital allocation based on a Tail Variance premium, we provide a list of TV formulas for frequently used continuous probability distributions. Our list covers all the aforementioned results in the literature and provides many new TV formulas for other popular distributions. Thus, the list can be used as an instrumental reference for interested researchers in their relevant works. Note that according to Formula (2), we also derive the formulas of TCE for these distributions to obtain the formulas of TV. The full list of formulas is presented in the Appendix.

3. The Confidence of Using TCE-based Risk Capital

Both VaR and TCE have been suggested as risk measures to estimate the risk capital for a given risk [13]. Despite its popularity in risk management, VaR is criticized because it is defined by an extreme event, yet "thousand-year events happen on a regular basis," and it is not sub-additive; hence, unlike TCE, it is not a coherent risk measure. Using VaR to estimate the risk capital later leads to trouble when the extreme event occurs, i.e. $X > x_q$ [32]. Particularly, an estimated risk capital C is said to be appropriate at q level of confidence if

$$\mathbb{P}(X \leq C) = q \tag{3}$$

in [15]. It is then evident that using VaR as the risk capital is at q confidence level. By contrast with VaR, TCE takes all possible outcomes of the event $X > x_q$ into account and calculates the "expected loss" of the risk under $X > x_q$. Ever since the seminal work of [1], TCE has been advocated for the evaluation of risk capital from a regulatory point of view [6]. Similarly, it is of interest and importance to investigate, when the extreme event $X > x_q$ happens, whether TCE-based capital is enough to cover the loss or what the confidence level of TCE-based risk capital is; i.e. if we take C as TCE of X, how can we estimate the appropriateness in Formula (3)? Note that appropriateness here is defined by the adequacy of risk capital. As we are primarily interested in the distribution-free capital adequacy of TCE, instead of appropriateness, we say that the confidence level of using C (based on VaR or TCE) is q instead of appropriateness.

With the aid of TV, we answer this question from three aspects in a distribution-free way: First, we develop a sharpened conditional Markov inequality to give a lower bound of the confidence level of using TCE-based risk capital. Second, we introduce the conditional Cantelli inequality in conjunction with the sharpened conditional Markov inequality to give an improved lower bound of the confidence level of using TCE-based risk capital. Third, we propose a sharpened conditional Chebyshev inequality to give a confidence interval of using TCE-based capital. Such an interval is the range within which the extreme values—i.e. the losses when the event $X > x_q$ occurs—are likely to fall. Note that this interval differs from the usual confidence interval in statistical estimation. A confidence interval using TCE-based capital at a 95% level is a range that X falls into with at least 95% probability, given that $X > x_q$. Both the sharpened conditional Markov inequality and the sharpened conditional Chebyshev inequality halve the bounds compared with the original bounds, respectively.

Our approach to assessing the confidence of using TCE-based risk capital is handy and easy to implement by regulators and financial institutes. In particular, regulators aim to protect the stability of markets and financial systems from extreme events, which coincides with the nature of the tail conditional inequalities that give lower bounds to the confidence of TCE-based capital. Moreover, our approach is distribution-free. As such, it provides a fast and easy-to-understand solution that may be favorable for regulators. In fact, a similar approach has been proposed by [25], where the authors used a Chebyshev inequality to investigate bank capital regulation and bankruptcy as well as the riskiness of the bank portfolio in their models.

In the sequel of our paper, we assume that the PDF of $Y = X|X > x_q$ is non-increasing on its support and that x_q is nonnegative. These assumptions are very mild, as they hold true for the majority of distributions. As a matter of fact, when one performs asymptotic analysis, X is usually assumed to be eventually monotone in the literature, such as the monotone density theorem in regular variation theory.

3.1. The sharpened conditional Markov inequality

The well-known Markov inequality states that if X is a nonnegative random variable and if $t > 0$, then $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. Applying this inequality directly to the tail random variable $Y = X|X > x_q$ gives

$$\mathbb{P}(Y \geq t) \leq \frac{\mathrm{TCE}_q[X]}{t}.$$

If $t = \mathrm{TCE}_q[X]$, the conditional Markov inequality is trivial, as $\mathbb{P}\big(Y \geq \mathrm{TCE}_q[X]\big) \leq 1$. Therefore, we propose a sharpened conditional Markov inequality, which halves the bound of the original Markov inequality for tail risks. The proof is given in the Appendix.

Proposition 1. (Sharpened Conditional Markov Inequality) *For a random variable X, if $x_q \geq 0$ and $Y = X|X > x_q$ has a non-increasing PDF, the following inequalities hold for $t > x_q$:*

$$\mathbb{P}(Y \geq t) \leq \frac{\mathrm{TCE}_q[X]}{2t} \text{ and } \mathbb{P}(X \geq t) \leq (1-q)\,\frac{\mathrm{TCE}_q[X]}{2t}.$$

Remark 1. When $t = \mathrm{TCE}_q[X]$, the sharpened conditional Markov inequality is

$$\mathbb{P}\big(X \leq \mathrm{TCE}_q[X]\big) \geq (q+1)\,/2.$$

Hence, we can assert that the TCE-based risk capital is always over a $(q+1)\,/2$ confidence level whatever the model is. Therefore, for the same q, TCE_q is at a higher confidence level in calculating the risk capital than VaR_q. To reach the confidence level of VaR_p, one may consider $\mathrm{TCE}_{(2*p-1)}$. For example, $\mathrm{TCE}_{0.98}$ is no lower than $\mathrm{VaR}_{0.99}$. This is slightly higher than the Basel level for the TCE (0.975, [5]) bank; however, the bound here is the worst-case one.

3.2. The sharpened conditional Cantelli inequality

To further incorporate TV when assessing the confidence level of TCE, we develop a conditional Cantelli inequality. The classic Cantelli inequality, which is an improved version of the one-sided Chebyshev inequality, states that for a constant $t > 0$,

$$\mathbb{P}(X - \mathbb{E}[X] \geq t) \leq \frac{\mathrm{Var}[X]}{\mathrm{Var}[X] + t^2}.$$

Similarly, we have a sharpened conditional Cantelli inequality, which halves the bound of the classic Cantelli inequality. The random variables X and $Y = X|X > x_q$, as defined in Proposition 1. It holds for $t > 0$ that

$$\mathbb{P}\big(Y \geq \mathrm{TCE}_q[X] + t\big) \leq \frac{\mathrm{TV}_q[X]}{2\big(\mathrm{TV}_q[X] + t^2\big)}. \tag{4}$$

In particular, if we may take t as a proportion of $\mathrm{TCE}_q[X]$, i.e. $t = \lambda \mathrm{TCE}_q[X]$, where $\lambda \geq 0$, then we have

$$\mathbb{P}\big(Y \geq (1+\lambda)\,\mathrm{TCE}_q[X]\big) \leq \frac{\mathrm{TV}_q[X]}{2\Big(\mathrm{TV}_q[X] + \big(\lambda \mathrm{TCE}_q[X]\big)^2\Big)}. \tag{5}$$

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Corollary 1. (Improved lower bounds of confidence level of using TCE-based risk capital) *For a random variable X, if $x_q \geq 0$ and $Y = X|X > x_q$ has a non-increasing PDF on its support, we have the following inequality: For $\lambda \geq 0$,*

$$\mathbb{P}(Y \leq (1+\lambda)\,\mathrm{TCE}_q[X]) \geq 1 - \min\left(\frac{1}{2(1+\lambda)}, \frac{\mathrm{TV}_q[X]}{2\left(\mathrm{TV}_q[X] + \left(\lambda \mathrm{TCE}_q[X]\right)^2\right)}\right), \tag{6}$$

$$\mathbb{P}(X \leq (1+\lambda)\,\mathrm{TCE}_q[X]) \geq 1 - (1-q)\min\left(\frac{1}{2(1+\lambda)}, \frac{\mathrm{TV}_q[X]}{2\left(\mathrm{TV}_q[X] + \left(\lambda \mathrm{TCE}_q[X]\right)^2\right)}\right). \tag{7}$$

Remark 2. The sharpened conditional Cantelli inequality, Formula (4), is obtained directly from the sharpened conditional Markov inequality using the same proof method of the classic Cantelli inequality; see 6.1.e in [29] for details. Corollary 1 combines the sharpened conditional Markov and Cantelli inequalities for a more precise tail risk estimation. As mentioned by [2], while these inequalities provide rough estimates, their usefulness lies in their applicability to any random variable with finite variance. Consequently, in practical scenarios, especially for continuous random variables, these inequalities are typically strict, with equality attainable only in specific contexts.

3.3. A distribution-free confidence interval using TCE-based capital

Furthermore, the conditional Chebyshev inequality can give a confidence interval in which extreme losses are likely to fall within the tail. We also present a sharpened conditional Chebyshev inequality to give a more precise interval.

The classic Chebyshev inequality states that if X is a random variable with finite expectation and finite variance, then for any $t > 1$, we have

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq t\sqrt{\mathrm{Var}[X]}\right) \leq t^{-2}.$$

Then for tail distributions, by analogy, there is the following conditional Chebyshev inequality straightforwardly. For a random variable X, define $Y := X|X > x_q$. The following inequality holds:

$$\mathbb{P}\left(\left|Y - \mathrm{TCE}_q[X]\right| \leq t\sqrt{\mathrm{TV}_q[X]}\right) \geq 1 - t^{-2}, \tag{8}$$

where $t > 1$ and $0 < q < 1$.

The conditional Chebyshev inequality, Formula (8), implies the confidence interval that Y deviates from $\mathrm{TCE}_q[X]$. Such an interval at the confidence level $1 - t^{-2}$ is $\left[\max\left(x_q, \mathrm{TCE}_q[X] - t\sqrt{\mathrm{TV}_q[X]}\right), \mathrm{TCE}_q[X] + t\sqrt{\mathrm{TV}_q[X]}\right]$. Moreover, based on the feature of the non-increasing PDF of the tail distributions, we can give a sharpened conditional Chebyshev inequality; the proof is similar to that of Proposition 1.

Proposition 2. (sharpened conditional Chebyshev inequality) *For a random variable X, if $x_q \geq 0$ and $Y = X|X > x_q$ has a non-increasing PDF on its support, the following inequality holds for $t > 1$:*

$$\mathbb{P}\left(\left|Y - \mathrm{TCE}_q[X]\right| \leq t\sqrt{\mathrm{TV}_q[X]}\right) \geq 1 - \frac{1}{2}t^{-2}. \tag{9}$$

TABLE 1. The distribution-free confidence level of using TCE-based risk capital for two data sets

$q = 0.95$	Data set I	Data set II
TCE_q (standard errors)	3223.95 (15.17)	0.41462 (0.0008)
sharpened Markov inequality	0.975	0.975
$\text{IC}_{30\%}$	0.9808	0.9927
$\text{CI}_{95\%}$	(0.457, 6.254)	(0.777, 1.858)
sharpened $\text{CI}_{95\%}$	(0.457, 4.715)	(0.777, 1.606)
$q = 0.99$	Data set I	Data set II
TCE_q (standard errors)	7570.23 (66.12)	0.54909 (0.0011)
sharpened Markov inequality	0.995	0.995
$\text{IC}_{30\%}$	0.9962	0.9996
$\text{CI}_{95\%}$	(0.443, 5.191)	(0.818, 1.386)
sharpened $\text{CI}_{95\%}$	(0.443, 3.963)	(0.818, 1.273)

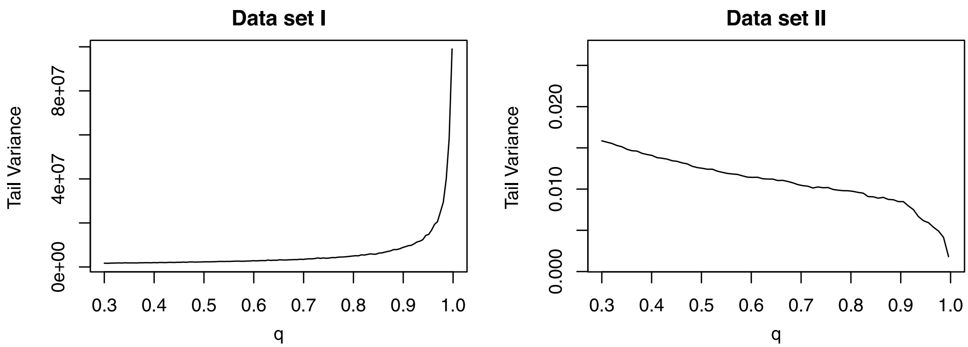

FIGURE 1. Estimated TV of the insurance and financial losses

3.4. Case study I: capital adequacy of TCE-based risk capital for insurance and financial losses

We collect two data sets of losses from the insurance and financial sectors. The first data set contains information on 2,387 business interruption losses (in 100,000 French francs) over the period 1985 to 2000 in France; the second data set includes 400 monthly loss rates of the VIX Index from February 1990 to May 2023 (the first data set is available in an R package named CASdatasets; the second data set is collected from Yahoo Finance). Assume that regulators in both sectors rely on TCE to calculate the risk capital for these business activities. We then estimate the TCE and the TV of the two sectors, respectively. In accordance with our distribution-free approach, we use non-parametric empirical estimations in this paper. To tackle the small sample in the estimation of TCE and TV at high-risk levels, we employ bootstrapping; see [18] and [36] for more details. Moreover, the highest level we can estimate is restricted to sample size; e.g. if the total sample size is 1,000, we can estimate the confidence level only up to 0.999. The lower bounds of the confidence level of using TCE-based capital for the two data sets are given in Proposition 1 and Corollary 1, respectively. Numerical results are summarized in Figure 1 and Table 1.

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From Figure 1, we can observe that Data set I (business interruption insurance) exhibits the characteristics of heavy tails, as its TV increases sharply to infinity along $q \to 1$. This is consistent with the study of [10], where the author considered the same data set and argued that log-normal distribution (heavy-tailed) outperforms gamma distribution (light-tailed) in the goodness of fit. For Data set II (VIX Index), its TV gradually decreases to 0, which is a typical light-tailed feature.

Remark 3. In Table 1, the calculations of $\mathrm{IC}_{30\%}$, $\mathrm{CI}_{95\%}$, and sharpened $\mathrm{CI}_{95\%}$ correspond to Formulas (7), (8), and (9), respectively. The estimation is conducted using bootstrapping, with 500 resamples, and the standard errors of TV_q in data sets 1 and 2 are approximately $0.02{*}\mathrm{TV}_q$. Note that $\mathrm{IC}_{30\%}$ is short for the improved lower bound of the confidence level with $\lambda = 0.3$, i.e. $\mathbb{P}\big(X \le 1.3 \times \mathrm{TCE}_q[X]\big) \ge \mathrm{IC}_{30\%}$. Moreover, the confidence interval (CI) is demonstrated as a proportion to its TCE_q. For example, $\mathrm{CI}_{95\%} = (a, b)$ implies that at a 95% confidence level, the conditional variable $Y = X|X > x_q$ falls within the interval $(a{*}\mathrm{TCE}_q[X], b{*}\mathrm{TCE}_q[X])$.

Remark 4. From Table 1, it is evident that the deviations of the loss variables X and Y from TCE are characterized by TV. Therefore, based on well-estimated TCE and TV, our results are distribution-free. Such findings underscore the inadequacy of relying solely on TCE for tail risk assessment. For insurers with risk capital exceeding TCE, IC offers a minimal confidence level, signifying a probability of at least IC that loss X remains below the prepared risk capital. Conversely, in events where $X > x_q$, to ensure that loss $Y = X|X > x_q$ falls within the CI interval with a probability of at least 95%, insurers might need to allocate a risk capital significantly higher than TCE (see the right boundary of CI). This is especially evident in data with heavy tail variance.

4. Asymptotics of Tail Variance

As we have seen in case study I, TV may exhibit a certain tendency when p approaches 1. In fact, there can be three possible circumstances for TV when $q \to 1$, i.e. 0, a positive constant, or infinity. Understanding the tendency of TV along $q \to 1$ can help us further estimate the capital adequacy of TCE. In this section, we study the asymptotics of TV. In the sequel of this paper, for two positive-valued functions f and g, we say f and g are asymptotically equivalent if $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$, and we denote it as $f(x) \sim g(x)$. We also tactically assume that $\mathbb{E}\big[X^2\big] < \infty$ holds and that the $\lim_{q\to 1}\mathrm{TV}_q$ exists on the extended real numbers in the rest of this paper.

4.1. A distribution-free limit formula of Tail Variance

We first investigate the asymptotics of TV without additional specific distribution assumptions.

Theorem 1. *Let X be an absolutely continuous random variable defined on a support with an unbounded supremum. It holds that*

$$\lim_{q\to 1} \mathrm{TV}_q[X] = \lim_{x_q\to\infty} \frac{-\overline{F}(x_q)}{f'(x_q)}$$

for differentiable $f(x)$.

Based on Theorem 1, we can establish a relationship between the TV and the hazard rate as the following corollary:

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Corollary 2. *Following Theorem 1, it holds that*

$$\lim_{q\to 1} \mathrm{TV}_q[X] = \lim_{x\to\infty} (h_X(x))^{-2}. \tag{10}$$

Corollary 2 shows an identity between the limit of the hazard rate and the TV. More specifically, if we define a tail standard deviation as the square root of TV, then the standard deviation always has the same limit as the hazard rate. We further demonstrate this with an example of Weibull distributions.

Example 1. A random variable X follows a Weibull distribution, written as $X \sim \text{Weibull}(a, b)$, if its PDF is of the form

$$f(x) = ab^{-a}x^{a-1}\exp\left[-(x/b)^a\right],$$

where $x \in [0, \infty)$, $a > 0$, and $b > 0$. Then the TCE of the Weibull distribution is

$$\mathrm{TCE}_q[X] = \frac{b}{1-q}\Gamma\left(\frac{1}{a}+1, \left(x_q/b\right)^a\right),$$

and the TV of the Weibull distribution is

$$\mathrm{TV}_q[X] = \frac{b^2}{1-q}\Gamma\left(1+2a^{-1}, \left(x_q/b\right)^a\right) - \left(\frac{b}{1-q}\Gamma\left(1+a^{-1}, \left(x_q/b\right)^a\right)\right)^2,$$

where $\Gamma(s, x)$ is the upper incomplete gamma function and is defined as

$$\Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}\mathrm{d}t.$$

The limit of $\mathrm{TV}_q[X]$ can be 0 ($a > 1$), a non-zero constant b^2 ($a = 1$), or infinity ($0 < a < 1$). For a given shape parameter a, the limits of TV and the hazard rate^{-2} converge to the same value. In fact, it can be verified that the TV and the square of the reciprocal hazard rate are asymptotically equivalent for Weibull distribution, namely,

$$\lim_{q\to 1} \frac{\mathrm{TV}_q[X]}{h^{-2}(x_q)} = 1. \tag{11}$$

A numerical illustration for such an asymptotic equivalence in the case of Weibull distribution is demonstrated in Figure 2.

The asymptotic equivalence of Equation (11) implies the same convergence rate for $\mathrm{TV}_q[X]$ and $h^{-2}(x_q)$, which is a stronger result than Corollary 2. In Theorem 1 and Corollary 2, we have only the same limit tendency, which can be 0, a positive constant, or ∞. But the convergence rate of the two sides can be different. On the other hand, we do not hold specific assumptions on the distribution in Theorem 1 and Corollary 2. The asymptotic equivalence of equation (11) actually holds more generally for distributions with certain tails. We discuss this in the next subsection.

4.2. Asymptotic equivalences of Tail Variances for distributions with polynomial and exponential tails

Theorem 1 and Corollary 2 show that the asymptotic of TV is closely related to the hazard rate. Note that Theorem 1 and Corollary 2 do not require additional distribution assumptions

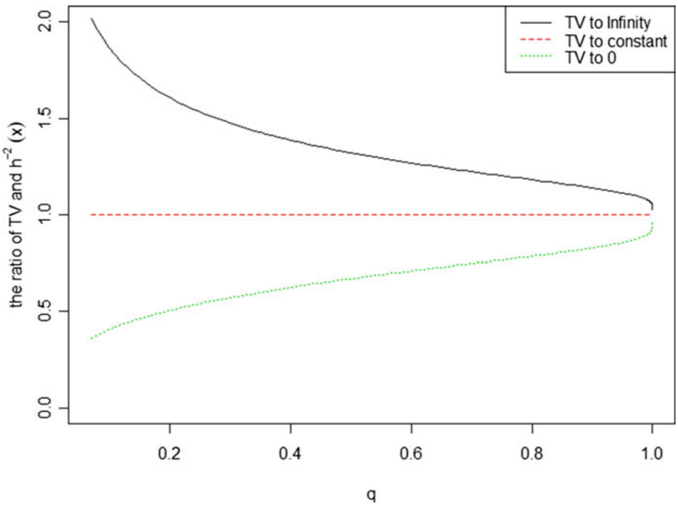

FIGURE 2. The asymptotic equivalence between $\mathrm{TV}_q[X]$ and $h^{-2}(x_q)$

on X. As such, we do not have explicit asymptotic formulas, and we cannot guarantee the asymptotic equivalence of Equation (11). To further investigate the asymptotic equivalence with more explicit results, we consider two general distribution families defined by the tails: namely, the distributions families with polynomial and exponential tails. The two distribution families cover a large variety of standard distributions; we refer to [33] for more details. To aid the proof of our results in this subsection, we introduce the following notations and prerequisite theorems in regular varying theory. For more details about regular varying, we refer to [31] and the references therein.

Let $U:\ [0, \infty) \to [0, \infty)$ be a measurable function. We say U has regular variation of order ρ at infinity if there exists a real number ρ such that for every $t > 0$,

$$\lim_{x\to\infty} \frac{U(tx)}{U(x)} = t^{\rho}.$$

Furthermore, if $\rho = 0$, the function U is said to be slowly varying at infinity. From the definition, we see that $U(x) = x^{\rho} l(x)$, where l is slowly varying.

Let l_1 and l_2 be nonnegative functions such that

$$\frac{l_1(x)}{x^{\epsilon}} \to 0 \text{ and } l_1(x)x^{\epsilon} \to \infty, \text{ as } x \to \infty, \text{ for all } \epsilon > 0,$$
$$\text{and } \frac{l_2(x)}{e^{x^{\epsilon}}} \to 0 \text{ and } l_2(x)e^{x^{\epsilon}} \to \infty, \text{ as } x \to \infty, \text{ for all } \epsilon > 0.$$

Let L_1 be the set for all l_1, and let L_2 be the set for all l_2. [33] showed that L_1 consists of all normalized slowly varying functions and that L_2 consists of all $l_2(x)$ for which $l_2\left[(\ln x)^{1/\epsilon}\right]$ is slowly varying for all $\epsilon > 0$ under the assumption that $\lim_{y\to\infty} \frac{l_2'(y)}{l_2(y)y^{\epsilon-1}}$ exists for all $\epsilon > 0$. Numerous asymptotic studies focus on regularly varying random variables, which essentially have polynomial tails, yet fail to cover a vast array of exponential-tail random variables with medium or light tails [3]. Therefore, investigating the tail asymptotic properties of these two distinct classes of random variables is of importance and interest and provides complementary results to relevant literatures.

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Equipped with the notations and theorems just presented, we are able to put forward the definitions of the polynomial tail and the exponential tail as follows:

Theorem 2. *If X's PDF f and its first derivative, f', exist, then*

- *X has a polynomial tail if and only if one of the following three holds:*

$$\overline{F}(x) \sim l_1(x)x^{-\alpha+1}, \tag{12}$$

which is equivalent to

$$f(x) \sim (\alpha - 1) \cdot l_1(x)x^{-\alpha}, \tag{13}$$

$$f'(x) \sim -(\alpha - 1)\,\alpha \cdot l_1(x)x^{-\alpha-1}, \tag{14}$$

for some $l_1 \in L_1$ and $\alpha > 1$.

- *X has an exponential tail if and only if*

$$\overline{F}(x) \sim l_2(x)e^{-x^\beta}, \tag{15}$$

which is equivalent to

$$f(x) \sim \beta \cdot l_2(x)e^{-x^\beta}x^{\beta-1}, \tag{16}$$

$$f'(x) \sim -\beta^2 \cdot l_2(x)e^{-x^\beta}x^{2\beta-2}, \tag{17}$$

for some $l_2 \in L_2$ and $\beta > 0$.

Well-known examples of distributions with polynomial tails are, for instance, Student-t distributions and Pareto distributions, while gamma distributions and Weibull distributions are typical examples of exponential-tailed distributions. Furthermore, based on Theorem 2, we have the following corollary on the asymptotics of TCE:

Corollary 3. *If a random variable X has a polynomial tail as defined in Equation (12), then*

$$\mathrm{TCE}_q[X] \sim \frac{\alpha-1}{\alpha-2}x_q,\ \alpha > 2,\ \textit{when } q \to 1;$$

if X has an exponential tail as defined in Equation (15), then

$$\mathrm{TCE}_q[X] \sim x_q,\ \textit{when } q \to 1.$$

The asymptotics of TCE have been widely discussed in the literature. For example, [20] derived similar asymptotic results for distributions of regular variation laws, which are the polynomial-tailed distributions in this paper; see also [22, 37]. Further, [30] studied the asymptotic representation of the TCE for the Generalized Pareto distribution. Since this distribution exhibits a polynomial tail (when the shape parameter $\xi > 0$) and an exponential tail (when $\xi = 0$), Corollary 3 encompasses their results.

Next, we give the asymptotics of TV for the two distribution families.

Theorem 3. *If a random variable X has a polynomial tail as defined in Equation (12), then*

$$\mathrm{TV}_q[X] \sim \frac{\alpha-1}{(\alpha-2)^2(\alpha-3)}x_q^2,\ \alpha > 3,\ \textit{when } q \to 1;$$

if X has an exponential tail as defined in Equation (15), then

$$\mathrm{TV}_q[X] \sim \beta^{-2} x_q^{2-2\beta}, \text{ when } q \to 1. \tag{18}$$

Theorem 3 characterizes the tail behavior of TV for distributions with polynomial and exponential tails. It is consistent with Theorem 4.1 in [22], which is a special case of our results. Specifically, Theorem 4.1 in [22] pointed out that if Z is a standardized Student-t distribution, then TV equals $\frac{v}{(v-1)^2(v-2)} z_q^2 \left(1 + O\left(\frac{1}{z_q^2}\right)\right)$, and if Z is the standard normal distribution, then TV equals $\frac{1}{z_q^2}\left(1 - \frac{6}{z_q^2} + O\left(\frac{1}{z_q^4}\right)\right)$. For Student-t distributions, it is straightforward to obtain two similar results by setting $\alpha = v + 1$. For the normal distribution, by substituting $x = \frac{1}{\sqrt{2}} z$ and setting $\beta = 2$ in Equation (16), Theorem 3 gives TV as being asymptotically equivalent to z_q^{-2}, which is consistent with the result mentioned before.

We give an explicit asymptotic formula for TV of exponential-tailed distributions that supplements the work on univariate cases in [28], which pointed out only $\mathrm{TV}_q[X] \sim o\left(x_q^2\right)$. [28] mentioned that there could be different converging speeds and named normal distributions and exponential distributions as instances of those speeds. Our contributions are giving a universal asymptotic formula [Equation (18)] for exponential-tailed distributions, including normal distributions and exponential distributions, and giving the converging speed in the formula. Furthermore, we can establish an asymptotic equivalence between the TV and the reciprocal square of the hazard rate for the distributions with polynomial tails and exponential tails.

Corollary 4. *If a random variable X has a polynomial tail or an exponential tail defined in Equation (12) or Equation (15), respectively, then it holds that*

$$\mathrm{TV}_q[X] \sim c \cdot h^{-2}(x_q)$$

as $q \to 1$, where

$$c = \begin{cases} \frac{(\alpha-1)^3}{(\alpha-2)^2(\alpha-3)} & \alpha > 3 \text{ if } X \text{ has a polynomial tail;} \\ 1 & \text{if } X \text{ has an exponential tail.} \end{cases}$$

4.3. Tail categorization based on TV

Risks' tail behaviors are of importance and interest in many fields, such as insurance, finance, and engineering. There is a large number of papers on the classification of risks' tails or tail ordering based on various tools and concepts in the literature. In particular, a tail categorization proposed in [34] is broadly discussed. The authors suggested using the asymptotics of hazard rates to classify distributions. Note that we have shown the asymptotic equivalence between TV and hazard rate; we can now categorize risks' tails using the TV in similar ways. The TV of each category, when $q \to 1$, can be computed straightforwardly based on formulas in Section 2.

Proposition 3. *Let X be an absolutely continuous random variable, and define* $h_0 := \lim_{x\to+\infty} \frac{1}{h(x)}$. *We have the tail variance of X when $q \to 1$ as follows:*

X	*tail behavior*	*TV*
$h_0 = 0$	*short tail*	0
$0 < h_0 < \infty$	*medium tail*	h_0^2
$h_0 = \infty$	*long tail*	∞

Proposition 3 tells us that the hazard rate essentially gives the same information as TV does when X approaches infinity. The Weibull distribution in Example 1 is a typical example illustrating such categorization. Its tail decays exponentially, and the shape parameter a indicates whether it is long-tailed ($0 < a < 1$), medium-tailed ($a = 1$), or short-tailed ($a > 1$). More generally, we present a tail categorization for distributions with polynomial tails and exponential tails.

Remark 5. Let X be an absolutely continuous random variable with a polynomial or an exponential tail as defined in Equation (12) or Equation (15), respectively; we then have the classification and the TV of X when $q \to 1$ as follows:

Distributions with what tail	$\lim_{q\to 1} \mathrm{TV}_q[X]$	Tail behavior
polynomial, $\alpha > 1$	∞	long tail
exponential, $0 < \beta < 1$		
exponential, $\beta = 1$	constant	medium tail
exponential, $\beta > 1$	0	short tail

TV categorizes the probability distributions into three classes according to tails. Despite the simplicity, it provides a brief insight into the tail behaviors of the concerned risks, which can also help us have a quick judgment on how the risks could be at the upper tails. For instance, in our case study I, we can observe a significant distinct tendency of TV along $q \to 1$ for the two sets of insurance data, which suggests that they belong to different categories and should have very different tail behaviors. Note that, from a practical application point of view, TV is much easier to estimate than the hazard rate. We further provide another practical case study based on financial data to illustrate the advantage of TV in tail categorization.

4.4. Case study II: alternative evidence for financial data categorization

Empirical studies often suggest that high-frequency financial data are usually more heavy-tailed than are low-frequency data. For example, the Gaussian models seem to fit yearly data well in many circumstances, but the high-frequency data appear to be more heavy-tailed; see [9, 39]. In addition to these empirical studies, Proposition 3 uses the asymptotics of TV to provide an alternative way to verify this conclusion.

We collect the data on the exchange rate in different frequencies between British Pound Sterling and the U.S. Dollar. We consider the minutely, daily, weekly, and yearly data and estimate the TVs along the tails, respectively, using the bootstrap method (the data encompass 8,253 minute-level points from January 4–12, 2024; 5,230 daily points from December 2003 to January 2024; 1,049 weekly points in the same interval; and 409 monthly points from January

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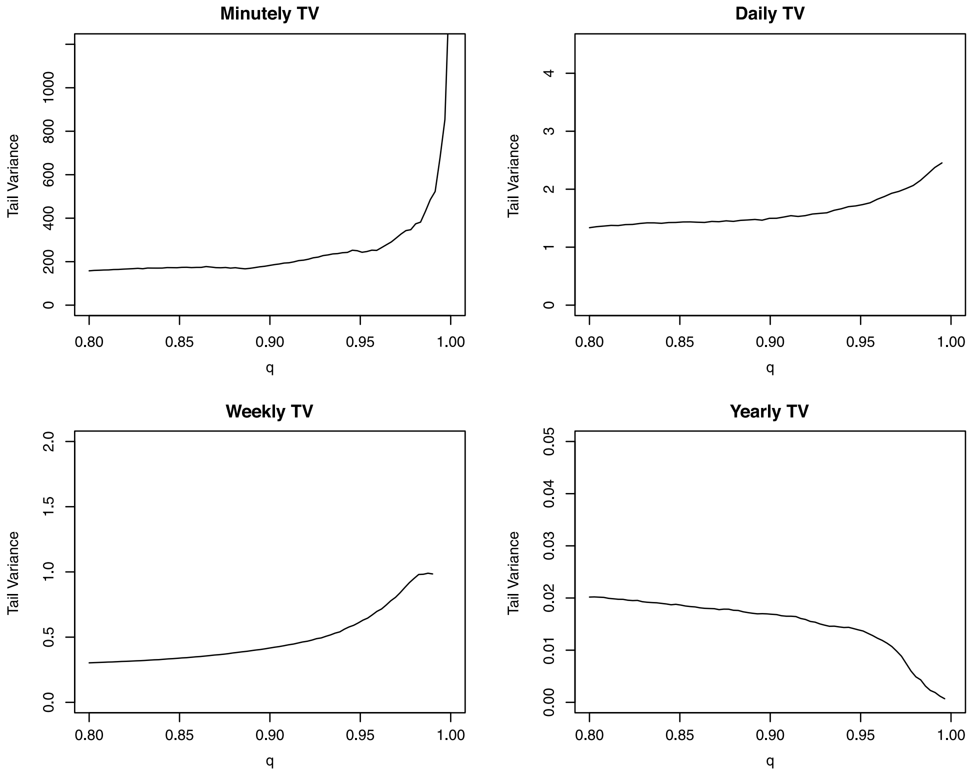

FIGURE 3. Estimated TV of losses on the exchange rate of British Pound and U.S. Dollar

1990 to January 2024, all sourced from the Wind Economic Database) in order to obtain quasi yearly losses. To facilitate comparison, the losses are converted into annualized losses, and the results are presented in Figure 3. (The standard errors of TV fluctuate around 0.015∗TV for minutely data, 0.008∗TV for daily data, 0.016∗TV for weekly data, and 0.014∗TV for yearly data.) We can easily observe that the TV increases as the frequency of data gets higher. In particular, minutely data's TV shows a tendency to increase to infinity, while the yearly data's TV converges to 0, which indicates that minutely data have heavy tails and that yearly data have light tails. For daily and weekly data, the asymptotics of TV appear to have constantcy, which implies medium tails. As such, TV gives us a quick view of the tail categorization of the data, allowing unsuitable distributions to be precluded from the calibrations. Our results also provide alternative evidence to support the empirical studies on the relationship between the frequency and the tail categorization of financial data.

5. Conclusion

Tail variance is an instrumental risk measure for the analysis of risks at the tails. In this paper, we review extant explicit formulas of TV and TCE and derive more TV and TCE formulas of frequently used distributions. These formulas can be used directly in relevant studies. When TCE is used for calculating the risk capital, TV can be taken into account for assessing how far the true loss deviates from the TCE-based risk capital under the event $X > x_q$, which

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provides valuable insights into the capital adequacy problem. We give sharpened distribution-free lower bounds of the confidence level of using the TCE-based capital adequacy. Such an approach is straightforward to implement and partially mitigates the limitations of measuring tail risk with TCE exclusivly. More specifically, using TCE solely as the risk measure plausibly does not exploit the tail information sufficiently and might result in under- or overestimation of the risk capital (see Remark 4). Moreover, because we observed that the asymptotic properties of TV are eventually determined by the risk heaviness, we studied the asymptotics of tail variance. In particular, for two large probability distribution families with certain tails (exponential tails and polynomial tails), we unified the asymptotic formulas of TCE and TV, which have been widely studied under various specific distribution assumptions. More importantly, those formulas also give more accurate convergence rates, which previous literature has failed to do generally. Two case studies based on real data are presented to support our theoretical results. Our result suggests that tail variance is indeed a useful risk measure that has potential significance to and importance in many applications.

Appendix A. Proofs

A.1. Theorems and propositions in Section 3

A.1.1. *Proposition* 1. We provide a proof of Proposition 1 that is enlightened by [21].

Assume U is a uniform random variable on $[-t, t]$. Then

$$\begin{aligned}\mathbb{P}(Y+U\geq t) &= \mathbb{P}(Y\geq t-U)\\ &= \int_{-t}^{t}\mathbb{P}(Y\geq t-u)f_U(u)\,\mathrm{d}u\end{aligned}$$

When $0\leq u\leq t$, $\mathbb{P}(Y\geq t-u)=\mathbb{P}(Y\geq t)+\mathbb{P}(t-u\leq Y<t)$. When $-t\leq u<0$, $\mathbb{P}(Y\geq t-u)=\mathbb{P}(Y\geq t)-\mathbb{P}(t\leq Y\leq t-u)$. Subsequently,

$$\begin{aligned}\mathbb{P}(Y+U\geq t) &= \int_{-t}^{t}\frac{\mathbb{P}(Y\geq t-u)}{2t}\mathrm{d}u\\ &= \int_{-t}^{t}\frac{\mathbb{P}(Y\geq t)}{2t}\mathrm{d}u+\int_{0}^{t}\frac{\mathbb{P}(t-u\leq Y<t)}{2t}\mathrm{d}u\\ &\quad -\int_{0}^{t}\frac{\mathbb{P}(t\leq Y<t+u)}{2t}\mathrm{d}u\\ &= \mathbb{P}(Y\geq t)+\frac{1}{2t}\int_{0}^{t}[\mathbb{P}(t-u\leq Y<t)-\mathbb{P}(t\leq Y<t+u)]\,\mathrm{d}u.\end{aligned}$$

Given that Y has a non-increasing PDF, we have $\mathbb{P}(t-u\leq Y<t)\geq\mathbb{P}(t\leq Y<t+u)$. Hence, $\mathbb{P}(Y\geq t)\leq\mathbb{P}(Y+U\geq t)$. Next, we show that $\mathbb{P}(Y+U\geq t)\leq\frac{\mathbb{E}[Y]}{2t}$.

$$\begin{aligned}\mathbb{P}(Y+U\geq t) &= \mathbb{E}[\mathbf{1}(Y+U\geq t)]\\ &= \mathbb{E}[\mathbb{E}[\mathbf{1}(Y+U\geq t)\,|Y]]\\ &= \mathbb{E}[\mathbb{P}(U\geq t-Y|Y)]\,,\end{aligned}$$

where $\mathbf{1}(\cdot)$ is an indicator function that equals 1 if the argument is true or equals 0 otherwise.

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When $t-Y\leq -t$, $Y\geq 2t$ and $\mathbb{P}(U\geq t-Y|Y)=1$. When $t-Y\geq t$, $Y\leq 0$ and $\mathbb{P}(U\geq t-Y|Y)=0$. When $0\leq Y\leq 2t$, $\mathbb{P}(U\geq t-Y|Y)=\frac{Y}{2t}$. Hence,

$$\mathbb{P}(Y+U\geq t)=\mathbb{E}\big[f_1(Y)\big],$$

where $f_1(y)=\mathbf{1}(y\geq 2t)+\frac{y}{2t}\mathbf{1}(0\leq y\leq 2t)$. Note that if $f_1(y)\leq \frac{y}{2t}\mathbf{1}(y\geq 0)$,we have

$$\mathbb{P}(Y+U\geq t)=\mathbb{E}\big[f_1(Y)\big]\leq \mathbb{E}\left[\frac{Y}{2t}\mathbf{1}(Y\geq 0)\right]=\frac{\mathbb{E}[Y]}{2t}.$$

Consequently, it follows that

$$\mathbb{P}(Y\geq t)\leq \mathbb{P}(Y+U\geq t)\leq \frac{\mathrm{TCE}_q[X]}{2t}.$$

A.2. Theorems and corollaries in Section 4

Before providing the proofs of theorems and corollaries in Section 4, we need to present the well-known theorems in regularly varying theory and a lemma.

Theorem 4. (Karamata's theorem) *Let $g(x)$ be slowly varying and locally bounded in $[x_0,+\infty)$ for some $x_0\geq 0$. Then*

- *for $\alpha>-1$,*

$$\int_{x_0}^{x} t^{\alpha}g(t)\,\mathrm{d}t\sim(\alpha+1)^{-1}x^{\alpha+1}g(x),\ x\to\infty;$$

- *for $\alpha<-1$,*

$$\int_{x}^{\infty} t^{\alpha}g(t)\,\mathrm{d}t\sim-(\alpha+1)^{-1}x^{\alpha+1}g(x),\ x\to\infty.$$

Based on Karamata's theorem, Corollary 5 can be deduced directly. The proof is provided in [8].

Corollary 5. *If g is slowly varying and $\alpha<-1$, then $\int_x^{\infty} t^{\alpha}g(t)\,dt$ converges and*

$$\lim_{x\to\infty}\frac{x^{\alpha+1}g(x)}{\int_x^{\infty} t^{\alpha}g(t)\,\mathrm{d}t}=-\alpha-1.$$

A.2.1. *Theorem* 1. First we can rewrite $\mathrm{TV}_q[X]$ in an integral form using $f(x)$ and x_q.

$$\begin{aligned}\mathrm{TV}_q[X]&=\mathbb{E}\Big[X^2|X\rangle\, x_q\Big]-\mathbb{E}^2\big[X|X>x_q\big]\\&=\frac{\mathbb{E}\big[X^2I_{X>x_q}\big]}{1-q}-\left(\frac{\mathbb{E}\big[XI_{X>x_q}\big]}{1-q}\right)^2\\&=\frac{\overline{F}(x_q)\int_{x_q}^{+\infty}x^2f(x)\mathrm{d}x-\left(\int_{x_q}^{+\infty}xf(x)\mathrm{d}x\right)^2}{\left(\overline{F}(x_q)\right)^2}\end{aligned}\tag{19}$$

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Note that $q \to 1$ is equivalent to $x_q \to \infty$. We compute the following:

$$\begin{aligned}\lim_{q\to 1} \mathrm{TV}_q[X] &= \lim_{x_q\to\infty} \frac{-\int_{x_q}^{+\infty} x^2 f(x)\mathrm{d}x - x_q^2\overline{F}(x_q) + 2x_q \int_{x_q}^{+\infty} xf(x)\mathrm{d}x}{-2\overline{F}(x_q)} \\ &= \lim_{x_q\to\infty} \frac{-x_q\overline{F}(x_q) + \int_{x_q}^{+\infty} xf(x)\mathrm{d}x}{f(x_q)} \\ &= \lim_{x_q\to\infty} \frac{-\overline{F}(x_q)}{f'(x_q)}.\end{aligned}$$

A.2.2. *Corollary* 2. First note that

$$\lim_{q\to 1} \mathrm{TV}_q[X] = \lim_{x_q\to\infty} \frac{-\overline{F}(x_q)}{f'(x_q)} = \lim_{x_q\to\infty} \frac{-\overline{F}(x_q)}{f(x_q)} \times \frac{f(x_q)}{f'(x_q)},$$

and $\lim_{x_q\to\infty} \frac{-\overline{F}(x_q)}{f(x_q)} = \lim_{x_q\to\infty} \frac{f(x_q)}{f'(x_q)}$. Therefore,

$$\lim_{q\to 1} \mathrm{TV}_q[X] = \lim_{x_q\to\infty} \left(\frac{f(x_q)}{f'(x_q)}\right)^2 = \lim_{q\to 1} \left(\frac{-\overline{F}(x_q)}{f(x_q)}\right)^2. \tag{20}$$

A.2.3. *Theorem* 2. We show that the three definitions of the two types of tails are equivalent, respectively. The equivalences between Equations (12) and (13) and between Equations (15) and (16) have been proved by [33]. Hence, we give a proof of the rest of the asymptotic equivalences. Moreover, we say that a function $g(x)$ is eventually monotone if there exists an x^*, for $\forall x^* < x_1 < x_2$, and if we have $g(x_1) > g(x_2)$.

(i) Suppose Equation (14) holds. Applying Corollary 3 gives us

$$\frac{x^{(-\alpha-1)+1} l_1(x)}{\int_x^\infty t^{(-\alpha-1)} l_1(t)\,\mathrm{d}t} \to \alpha.$$

Given that $\int_x^\infty t^{(-\alpha-1)} l_1(t)\,\mathrm{d}t$ converges, the derivative of the integral with respect to x gives us $-f'(x)$. Hence, we have Equation (13).

(ii) Now we suppose that Equation (13) holds. For any $0 < a < b < \infty$, and given that f is nonnegative and eventually non-increasing, we have

$$f(bx) - f(ax) = \int_{ax}^{bx} f'(t)\,\mathrm{d}t \le 0.$$

As $f'(x)$ is non-decreasing and non-positive, when x is large enough, it holds that

$$\frac{(b-a)\,xf'(ax)}{x^{-\alpha} l_1(x)} \le \frac{f(bx) - f(ax)}{x^{-\alpha} l_1(x)} \le \frac{(b-a)\,xf'(bx)}{x^{-\alpha} l_1(x)} \le 0. \tag{21}$$

We rewrite the mid-term of the inequalities in Equation (21) as

$$\frac{f(bx)}{(bx)^{-\alpha} l_1(bx)} \cdot \frac{b^{-\alpha} l_1(bx)}{l_1(x)} - \frac{f(ax)}{(ax)^{-\alpha} l_1(ax)} \cdot \frac{a^{-\alpha} l_1(ax)}{l_1(x)}, \tag{22}$$

which approaches $(\alpha-1)\left(b^{-\alpha}-a^{-\alpha}\right)$ as $x\to\infty$. Subsequently, for the first inequality of Equation (21), we have

$$\overline{\lim}_{x\to\infty}\frac{f'(ax)}{x^{-\alpha-1}l_1(x)}\leq\frac{(\alpha-1)\left(b^{-\alpha}-a^{-\alpha}\right)}{b-a}.$$

Next, taking b as 1 gives

$$\lim_{b\downarrow 1}\overline{\lim}_{x\to\infty}\frac{f'(x)}{x^{-\alpha-1}l_1(x)}\leq-(\alpha-1)\,\alpha.$$

Similarly, we have

$$\lim_{a\uparrow 1}\overline{\lim}_{x\to\infty}\frac{f'(x)}{x^{-\alpha-1}l_1(x)}\geq-(\alpha-1)\,\alpha,$$

which concludes the proof of Equation (14)'s implying Equation (13).

(iii) Suppose Equation (15) holds. Substituting x by $(\ln y)^{1/\beta}$ gives

$$\overline{F}^*(y):=\overline{F}[(\ln y)^{1/\beta}]\sim l_2[(\ln y)^{1/\beta}]y^{-1}.$$

Note that $l_2\left[(\ln y)^{1/\beta}\right]$ is slowly varying. We then define

$$f^*(y)=\frac{\mathrm{d}F^*(y)}{\mathrm{d}y}\text{ and }f^{*\prime}(y)=\frac{\mathrm{d}f^*(y)}{\mathrm{d}y}.$$

Because f^* and $f^{*\prime}$ are eventually monotone, an argument similar to that in (ii) leads to $f^*(y)\sim l_2\left[(\ln y)^{1/\beta}\right]y^{-2}$ and $f^{*\prime}(y)\sim -l_2\left[(\ln y)^{1/\beta}\right]y^{-3}$, respectively. Taking f back, we have Equation (17).

(iv) Suppose Equation (17) holds. Starting with the definition of asymptotic equivalence, for any $\epsilon>0$ there exists x_0. For all $x>x_0$, $\left|\frac{f'(x)}{-l_2(x)e^{-x^\beta}x^{2\beta-2}}-\beta^2\right|<\epsilon$. Subsequently, it holds that

$$\int_x^\infty f'(t)\,\mathrm{d}t\leq-\left(\beta^2+\epsilon\right)\int_x^\infty l_2(t)\,e^{-t^\beta}t^{2\beta-2}\mathrm{d}t.$$

Substituting t by $(\ln y)^{1/\beta}$, the right-hand side of the last inequality is $\epsilon/\beta\int_{e^{x^\beta}}^\infty l_2\left[(\ln y)^{1/\beta}\right]y^{-3}\mathrm{d}y$, which is asymptotically equivalent to $e^{-2x^\beta}l_2\left(e^{x^\beta}\right)$ by Corollary 3. Therefore, Equation (17) implies Equation (16).

A.2.4. *Corollary 3.* To derive the asymptotic relationship between TCE and VaR, we need to calculate $\lim_{q\to 1}\frac{\mathrm{TCE}_q[X]}{x_q}$.

If a random variable X has a polynomial tail, Karamata's theorem gives

$$\begin{aligned}\lim_{q\to 1}\frac{\int_{x_q}^\infty xf(x)\mathrm{d}x}{\overline{F}_X(x_q)\cdot x_q}&=\lim_{q\to 1}\frac{\int_{x_q}^\infty\ (\alpha-1)\,x^{-\alpha+1}l_1(x)\mathrm{d}x}{l_1(x_q)x_q^{-\alpha+2}}\\&=\lim_{q\to 1}\frac{\alpha-1}{l_1(x_q)x_q^{-\alpha+2}}\frac{x_q^{-\alpha+2}l_1\left(_q\right)}{\alpha-2}\\&=\lim_{q\to 1}\frac{\alpha-1}{\alpha-2}.\end{aligned}$$

Hence, $\mathrm{TCE}_q[X]\sim\frac{\alpha-1}{\alpha-2}x_q$, $\alpha>2$, *when* $q\to 1$.

If a random variable X has an exponential tail, by substituting x by $(\ln y)^{1/\beta}$, we have

$$\begin{aligned}\lim_{q\to 1}\frac{\int_{x_q}^{\infty} xf(x)\mathrm{d}x}{\overline{F}_X(x_q)\cdot x_q} &= \lim_{q\to 1}\frac{\int_{x_q}^{\infty} x\cdot \beta l_2(x)x^{\beta-1}e^{-x^{\beta}}\mathrm{d}x}{l_2(x_q)e^{-x_q^{\beta}}\cdot x_q} \\ &= \lim_{q\to 1}\frac{\int_{x_q}^{\infty} x\cdot \beta l_2(x)x^{\beta-1}e^{-x^{\beta}}\mathrm{d}x}{l_2(x_q)e^{-x_q^{\beta}}\cdot x_q} \\ &= \lim_{q\to 1}\frac{\int_{y_q}^{\infty} l_2\left[(\ln y)^{1/\beta}\right](\ln y)^{1/\beta}y^{-2}\mathrm{d}y}{l_2\left[\left(\ln y_q\right)^{1/\beta}\right]\left(\ln y_q\right)^{1/\beta}y_q^{-1}}.\end{aligned}$$

As mentioned above, $l_2\left[(\ln y)^{1/\beta}\right]$ is asymptotically equivalent to a slowly varying function. Hence, we can apply Karamata's theorem again, and the numerator will be asymptotically equivalent to $l_2\left[\left(\ln y_q\right)^{1/\beta}\right]\left(\ln y_q\right)^{1/\beta}y_q^{-1}$. Subsequently, TCE and VaR are asymptotically equivalent for a random variable with an exponential tail.

A.2.5. *Theorem* 3. To derive the asymptotic relationship between TV and VaR for a polynomial-tailed distribution, we need to calculate $\lim_{q\to 1}\frac{\mathrm{TV}_q[X]}{x_q^2}$.

Similarly to the proof of Corollary 3, if a random variable X has a polynomial tail, Karamata's theorem gives

$$\begin{aligned}\int_{x_q}^{\infty} xf(x)\mathrm{d}x &\sim (\alpha-1)\int_{x_q}^{\infty} x^{-\alpha+1}l_1(x)\mathrm{d}x \\ &= \frac{\alpha-1}{\alpha-2}x_q^{-\alpha+2}{}_1(x_q)\end{aligned}$$

and

$$\begin{aligned}\int_{x_q}^{\infty} x^2f(x)\mathrm{d}x &\sim (\alpha-1)\int_{x_q}^{\infty} x^{-\alpha+2}l_1(x)\mathrm{d}x \\ &= \frac{\alpha-1}{\alpha-3}x_q^{-\alpha+3}l_1(x_q).\end{aligned}$$

Subsequently,

$$\begin{aligned}\lim_{q\to 1}\frac{\mathrm{TV}_q[X]}{x_q^2} &= \lim_{q\to 1}\frac{\overline{F}(x_q)\int_{x_q}^{+\infty} x^2f(x)\mathrm{d}x-\left(\int_{x_q}^{+\infty} xf(x)\mathrm{d}x\right)^2}{\left(\overline{F}(x_q)\right)^2x_q^2} \\ &= \lim_{q\to 1}\frac{\frac{\alpha-1}{\alpha-3}x_q^{-2\alpha+4}l_1^2(x_q)-\left(\frac{\alpha-1}{\alpha-2}\right)^2x_q^{-2\alpha+4}l_1^2(x_q)}{l_1^2(x_q)x_q^{-2\alpha+4}} \\ &= \frac{\alpha-1}{\alpha-3}-\left(\frac{\alpha-1}{\alpha-2}\right)^2 \\ &= \frac{\alpha-1}{(\alpha-2)^2(\alpha-3)}.\end{aligned}$$

Therefore, we may conclude the proof of the case of polynomial-tailed random variables. For the case of exponential-tailed random variables, the proof is quite complicated and can be shown upon request.

A.2.6. *Corollary* 4. This is a straightforward result of Theorem 3 along with the definition of the hazard rate.

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Appendix B. A table of the summary of TV formulas and asymptotics

TABLE 2. A list of TCE and TV distributions

Distribution	$\mathrm{TCE}_q[X]$	$\mathrm{TV}_q[X]$	Tail Type	$\lim\limits_{q\to 1}\mathrm{TV}_q[X]$
Pareto, type I	$\frac{ab^a x_q^{1-a}}{(1-q)(a-1)}$, $a>1$	$\frac{ab^a x_q^{2-a}}{(1-q)(a-2)}-\mathrm{TCE}_q^2[X]$, $a>2$	polynomial, long	∞
Pareto, type II	$\frac{\lambda^a(\lambda+ax_q)}{(1-q)(a-1)}(\lambda+x_q)^{-a}$, $a>1$	$\frac{a\lambda^a(\lambda+x_q)^{-a}}{1-q}\left(\frac{(\lambda+x_q)^2}{\alpha-2}+\frac{2\lambda(\lambda+x_q)}{1-a}+\frac{\lambda^2}{a}\right)-\mathrm{TCE}_q^2[X]$, $a>2$	polynomial, long	∞
Weibull	$\frac{b}{1-q}\Gamma\left(\frac{1}{a}+1,\,(x_q/b)^a\right)$	$\frac{b^2}{1-q}\Gamma\left(1+2a^{-1},\,(x_q/b)^a\right)-\mathrm{TCE}_q^2[X]$	exponential, $\begin{cases}\text{long}, & a<1\\ \text{median}, & a=1\\ \text{short}, & a>1\end{cases}$	$\begin{cases}\infty, & \alpha<1\\ b^2, & \alpha=1\\ 0, & \alpha>1\end{cases}$
Normal	$\mu+\sigma\frac{\phi(z_q)}{1-q}$	$\sigma^2\left[1+\frac{\phi(z_q)}{1-q}\left(z_q-\frac{\phi(z_q)}{1-q}\right)\right]$	exponential, short	0
Elliptical	$\mu+\frac{f_{Z^*}(z_q)}{\overline{F}_Z(z_q)}\sigma_Z^2\sigma$	$\mathrm{Var}(X)\cdot\left[\frac{\overline{F}_{Z^*}(z_q)}{1-q}+\frac{f_{Z^*}(z_q)}{1-q}\left[z_q-\sigma_Z^2\frac{f_{Z^*}(z_q)}{1-q}\right]\right]$	Depends on density generators	
Log-normal	$\frac{\exp(\mu+\sigma^2/2)}{1-q}\Phi(\sigma-z_q)$	$\frac{\exp(2(\mu+\sigma^2))}{1-q}\Phi(2\sigma-z_q)-\mathrm{TCE}_q^2[X]$	long	∞
Log-elliptical	$\frac{e^\mu}{1-q}\psi\left(-2\sigma^2\right)\overline{F}_{Z^*}(z_q)$	$\frac{e^{2\mu}}{1-q}\psi\left(-2\sigma^2\right)\overline{F}_{Z^{**}}(z_q)-\left[\frac{1}{1-q}\psi\left(-\frac{\sigma^2}{2}\right)\overline{F}_{Z^*}(z_q)\right]^2$	long	∞
F	$\frac{k_2}{k_2-2}+2\frac{k_2+k_1x_q}{k_1(k_2-2)}\frac{f(x_q)}{1-q}x_q$	$\frac{(2+k_1)k_2}{k_1(k_2-4)}\mathrm{TCE}_q[X]+\frac{2x_q^2(k_2+k_1x_q)}{k_1(k_2-4)}\frac{f(x_q)}{1-q}-\mathrm{TCE}_q^2[X]$	polynomial, long	∞

TABLE 2. Continued

Distribution	$\mathrm{TCE}_q[X]$	$\mathrm{TV}_q[X]$	Tail Type	$\lim_{q\to 1}\mathrm{TV}_q[X]$
Beta	$\frac{\alpha}{\alpha+\beta} - \frac{x_q(x_q-1)}{\alpha+\beta}\frac{f(x_q)}{1-q}$	$\frac{\alpha+1}{\alpha+\beta+1}\mathrm{TCE}_q[X] - \frac{x_q^2(x_q-1)}{\alpha+\beta+1}\frac{f(x_q)}{1-q} - \mathrm{TCE}_q^2[X]$	polynomial, long	∞
Gamma	$\frac{1}{\beta}\left(\alpha + \frac{f(x_q)}{1-q}x_q\right)$	$\mathrm{Var}(X) + \frac{x_q f(x_q)}{\beta^2(1-q)}\left[x_q\left(\beta - \frac{(x_q)}{1-q}\right) - (\alpha-1)\right]$	exponential, medium	β^{-2}
Exponential	$\frac{1}{1-q}\left(x_q + \frac{1}{\lambda}\right)e^{-\lambda x_q}$	$\frac{1}{(1-q)\lambda^2}\left(2 + \lambda x_q\left(2 + \lambda x_q\right)\right)e^{-\lambda x_q}$	exponential, medium	λ^{-2}
Generalized inverse Gaussian	$\frac{\overline{F}(x_q\|\lambda+1,\chi,\psi)}{1-q}\cdot \frac{K_{\lambda+1}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})}\left(\frac{\psi}{\chi}\right)^{-1/2}$	$\frac{\overline{F}(x_q\|\lambda+2,\chi,\psi)}{1-q}\frac{K_{\lambda+2}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})}\left(\frac{\psi}{\chi}\right)^{-1} - \left[\frac{\overline{F}(x_q\|\lambda+1,\chi,\psi)}{1-q}\frac{K_{\lambda+1}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})}\left(\frac{\psi}{\chi}\right)^{-1/2}\right]^2$	exponential, medium	$\frac{4}{\psi^2}$
Log-generalized inverse Gaussian	$\frac{K_\lambda(\sqrt{\chi(\psi-2)})}{K_\lambda(\sqrt{\chi\psi})}\cdot\left(\frac{\psi-2}{\psi}\right)^{\lambda/2}\cdot \frac{\overline{F}(\ln x_q\|\lambda,\chi,\psi-2)}{1-q}$	$\frac{K_\lambda(\sqrt{\chi(\psi-4)})}{K_\lambda(\sqrt{\chi\psi})}\left(\frac{\psi-4}{\psi}\right)^{\lambda/2}\frac{\overline{F}(\ln x_q\|\lambda,\chi,\psi-4)}{1-q} - \left[\frac{K_\lambda(\sqrt{\chi(\psi-2)})}{K_\lambda(\sqrt{\chi\psi})}\left(\frac{\psi-2}{\psi}\right)^{\lambda/2}\frac{\overline{F}(\ln x_q\|\lambda,\chi,\psi-2)}{1-q}\right]^2$	long	∞
Generalized hyperbolic	$\mu + c_1^*\frac{\overline{F}_{X_1^*}(x_q)}{1-q}\cdot\left[\gamma + \sigma^2 h_{X_1^*}(x_q)\right]$	$c_1^*\frac{\overline{F}_{X_1^*}(x_q)}{1-q}\sigma^2\left[1 + (x_q-\mu)\,h_{X_1^*}(x_q)\right] + c_2^*\frac{\overline{F}_{X_2^*}(x_q)}{1-q}\gamma\left[\gamma + \sigma^2 h_{X_2^*}(x_q)\right] - \left\{c_1^*\frac{\overline{F}_{X_1^*}(x_q)}{1-q}\left[\gamma+\sigma^2 h_{X_1^*}(x_q)\right]\right\}^2$	exponential, medium	$\left(\frac{\sigma^2}{\sqrt{\psi\sigma^2+\gamma^2}-\gamma}\right)^2$
Log-generalized hyperbolic	$\frac{e^\mu}{1-q}c_1^*\overline{F}_{Y_1^*}(y_q)$	$e^{2\mu}\cdot\left[\frac{e^{k\mu}}{1-p}c_2^*\overline{F}_{Y_2^*}(y_q) - \left(\frac{e^{k\mu}}{1-p}c_1^*\overline{F}_{Y_1^*}(y_q)\right)^2\right]$	long	∞

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References

[1] ACERBI, C. AND TASCHE, D. (2002). On the coherence of expected shortfall. *Journal of Banking & Finance.* **26**, 1487–1503.

[2] ALSMEYER, G. (2011). Chebyshev's inequality. In *International Encyclopedia of Statistical Science*, edited by M. Lovric, Springer Berlin Heidelberg, Berlin, Heidelberg, 239–240.

[3] AROURI, M. E. H., BOUBAKER, S. AND NGUYEN, D. K. (2013). *Emerging Markets and the Global Economy: A Handbook*. Academic Press, San Diego, USA.

[4] ARTZNER, P., DELBAEN, F., EBER, J. M. AND HEATH, D. (1999). Coherent measures of risk. *Mathematical Finance.* **9**, 203–228.

[5] BANK FOR INTERNATIONAL SETTLEMENT. (2014). Consultative document: Fundamental review of the trading book *Outstanding Issues*. Available at http://www.bis.org/bcbs/publ/d305.pdf.

[6] BASEL COMMITTEE ON BANKING SUPERVISION. (2013). Fundamental review of the trading book *A Revised Market Risk Framework*.

[7] BELLINI, F., FADINA, T., WANG, R. AND WEI, Y. (2022). Parametric measures of variability induced by risk measures. *Insurance: Mathematics and Economics.* **106**, 270–284.

[8] BINGHAM, N. H., GOLDIE, C. M. AND TEUGELS, J. L. (1989). *Regular Variation* (No. 27). Cambridge University Press, Cambridge.

[9] DACOROGNA, M. M., MÜLLER, U. A., PICTET, O. V. AND DE VRIES, C. G. (2001). Extremal forex returns in extremely large data sets. *Extremes.* **4**, 105–127.

[10] DE JONG, P. AND HELLER, G. Z. (2008). *Generalized Linear Models for Insurance Data*. Cambridge Books, Cambridge, England.

[11] DENUIT, M., DHAENE, J., GOOVAERTS, M. AND KAAS, R. (2006). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. John Wiley & Sons, West Sussex, England.

[12] EINI, E. J. AND KHALOOZADEH, H. (2021). The tail mean–variance optimal portfolio selection under generalized skew-elliptical distribution. *Insurance: Mathematics and Economics.* **98**, 44–50.

[13] EIOPA. (2019). Financial Stability Report—December 2019.

[14] EMMER, S., KRATZ, M. AND TASCHE, D. (2015). What is the best risk measure in practice? A comparison of standard measures. *Journal of Risk.* **18**, 31–60.

[15] FRÖHLICH, A. AND WENG, A. (2015). Modelling parameter uncertainty for risk capital calculation. *European Actuarial Journal.* **5**, 79–112.

[16] FURMAN, E. AND LANDSMAN, Z. (2006). Tail variance premium with applications for elliptical portfolio of risks. *ASTIN Bulletin: The Journal of the IAA.* **36**, 433–462.

[17] FURMAN, E., WANG, R. AND ZITIKIS, R. (2017). Gini-type measures of risk and variability: Gini shortfall, capital allocations, and heavy-tailed risks. *Journal of Banking & Finance.* **83**, 70–84.

[18] HESTERBERG, T. (2011). Bootstrap. *Wiley Interdisciplinary Reviews: Computational Statistics.* **3**, 497–526.

[19] HU, T. AND CHEN, O. (2020). On a family of coherent measures of variability. *Insurance: Mathematics and Economics.* **95**, 173–182.

[20] HUA, L. AND JOE, H. (2011). Second order regular variation and conditional tail expectation of multiple risks. *Insurance: Mathematics and Economics.* **49**, 537–546.

[21] HUBER, M. (2019). Halving the bounds for the Markov, Chebyshev, and Chernoff inequalities using smoothing. *The American Mathematical Monthly.* **126**, 915–927.

[22] IGNATIEVA, K. AND LANDSMAN, Z. (2015). Estimating the tails of loss severity via conditional risk measures for the family of symmetric generalised hyperbolic distributions. *Insurance: Mathematics and Economics.* **65**, 172–186.

[23] KIM, J. H. AND KIM, S. Y. (2019). Tail risk measures and risk allocation for the class of multivariate normal mean–variance mixture distributions. *Insurance: Mathematics and Economics.* **86**, 145–157.

[24] KLUGMAN, S. A., PANJER, H. H. AND WILLMOT, G. E. (2012). *Loss Models: From Data to Decisions*. John Wiley & Sons, New York, USA.

[25] KOEHN, M. AND SANTOMERO, A. M., (1980). Regulation of bank capital and portfolio risk. *The Journal of Finance.* **35**, 1235–1244.

[26] LANDSMAN, Z. (2010). On the tail mean–variance optimal portfolio selection. *Insurance: Mathematics and Economics.* **46**, 547–553.

[27] LANDSMAN, Z., PAT, N. AND DHAENE, J. (2013). Tail variance premiums for log-elliptical distributions. *Insurance: Mathematics and Economics.* **52**, 441–447.

[28] LI, J. (2023). Asymptotic results on tail moment and tail central moment for dependent risks. *Advances in Applied Probability.* **55**, 1116–1143.

[29] LIN, Z. (2010). *Probability Inequalities*. Springer, Heidelberg, Germany.

[30] MCNEIL, A. J., FREY, R. AND EMBRECHTS, P. (2005). *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton Series in Finance.

[31] MIKOSCH, T. (1999). *Regular Variation, Subexponentiality and Their Applications in Probability Theory*, Vol. **99**. Eindhoven University of Technology, Eindhoven, The Netherlands.

[32] PUTNAM, B. H., SYKES WILFORD, D. AND ZECHER, P. D. (2002). A short note on the concept of risk management and VaR for asset management firms. *Review of Financial Economics.* **11**, 205–212.

[33] ROJO, J. (1988). *On the Concept of Tail-Heaviness*.Department of Statistics, University of California, Berkeley, CA.

[34] SCHUSTER, E. F. (1984). Classification of probability laws by tail behavior. *Journal of the American Statistical Association.* **79**, 936–939.

[35] SU, C. AND TANG, Q. H. (2003). Characterizations on heavy-tailed distributions by means of hazard rate. *Acta Mathematicae Applicatae Sinica.* **19**, 135–142.

[36] TIBSHIRANI, R. J. AND EFRON, B. (1993). An introduction to the bootstrap. *Monographs on Statistics and Applied Probability.* **57**, 1–436.

[37] VANDUFFEL, S. AND YAO, J. (2017). A stein type lemma for the multivariate generalized hyperbolic distribution. *European Journal of Operational Research.* **261**, 606–612.

[38] XU, M. AND MAO, T. (2013). Optimal capital allocation based on the tail mean–variance model. *Insurance: Mathematics and Economics.* **53**, 533–543.

[39] ZHOU, B. (1996). High-frequency data and volatility in foreign-exchange rates. *Journal of Business & Economic Statistics.* **14**, 45–52.