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A NOTE ON THE LATTICE OF DENSITY PRESERVING MAPS

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We study here the poset $DP(X)$ of density preserving continuous maps defined on a Hausdorff sapce X and show that it is a complete lattice for a compact Hausdorff space without isolated points. We further show that for countably compact T_3 spaces X and Y without isolated points, $DP(X)$ and $DP(Y)$ are order isomorphic if and only if X and Y are homeomorphic. Finally, Magill's result on the remainder of a locally compact Hausdorff space is deduced from the relation of $DP(X)$ with posets $IP(X)$ of covering maps and $E_K(X)$ of compactifications respectively.

0. INTRODUCTION

Throughout the spaces considered (usually denoted by symbols X, Y) are Hausdorff and the maps are continuous. A map $f : X \to Y$ is called a *density preserving map* if $\operatorname{Int}\operatorname{Cl} f(A) \neq \phi$, whenever $\operatorname{Int} A \neq \phi$, $A \subseteq X$ ([**1**]). Two density preserving maps f and g with domain X and range Rf and Rg respectively are said to be *equivalent* ($f \approx g$) if there exists a homeomorphism $h : Rf \to Rg$ satisfying $h \circ f = g$. We identify equivalent density preserving maps on a fixed domain X, and denote by $DP(X)$ the set of all such equivalent classes of density preserving maps. The relation '$\leqslant$' defined on $DP(X)$ by $g \leqslant f$ if there exists a continuous map $h : Rf \to Rg$ such that $h \circ f = g$ turns out to be a partial order relation. Recall that a perfect irreducible continuous surjection is called a *covering map*. In Section 1 we prove that if X is a compact space without isolated points, then $DP(X)$ is a complete lattice. In Section 2, we determine the order structure of $DP(X)$ by proving that for countably compact T_3 spaces X and Y without isolated points, $DP(X)$ and $DP(Y)$ are order isomprphic if and only if X and Y are homeomorphic. Section 3 is devoted to the natural relation of $DP(X)$ with the poset $IP(X)$ of covering maps on X ([**3**]) and the poset $E_K(X)$ of compactifications of a locally compact space X ([**2**]). We show that if U is an open dense set in a compact space X then $DP(X,U) = IP(X,U)$, where $IP(X,U)$ (respctively $DP(X,U)$) is the poset of all covering (respectively density preserving-) maps f on X satisfying $\left|f^{-1}(f(x))\right| = 1$ for each x in U. Using this result we deduce Magill's result which states that for locally compact spaces X and Y, $E_K(X)$ and $E_K(Y)$ are order isomorphic if and only if $\beta X - X$ and $\beta Y - Y$ are homeomorphic ([**2**]).

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1. Lattice $DP(X)$

We immediately have the following lemmas.

Lemma 1.1. *$DP(X)$ is a partially ordered set.*

Lemma 1.2. *Let f, $g \in DP(X)$ be such that $g \leqslant f$. Then the map $h : Rf \to Rg$ satisfying $h \circ f = g$ is a density preserving map.*

Proof: Let $A \subseteq Rf$ be such that $\operatorname{Int} A \neq \phi$. Then by setting $f^{-1}(A) = A^*$, we get $\phi \neq \operatorname{Int}\operatorname{Cl} g(A^*) = \operatorname{Int}\operatorname{Cl}(h \circ f)(A^*) \subseteq \operatorname{Int}\operatorname{Cl} h(A)$. Hence h is a density preserving map. ∎

Remark 1.3. Fibres of a surjective density preserving map $f : X \to Y$ are closed nowhere dense subsets of X, where X is a space without isolated points.

Definition 1.4: For $f \in DP(X)$, define $\wp(f) = \{f^{-1}(y) \mid y \in Rf\}$.

From here onwards we assume that members of $DP(X)$ are quotient maps. If X is compact, this condition is automatically satisfied.

Lemma 1.5. *Let f, $g \in DP(X)$. Then $f \leqslant g$ if and only if $\wp(g) \subseteq \wp(f)$.*

Proof: Let $f \leqslant g$ then there exists $h : Rg \to Rf$ satisfying $h \circ g = f$. If $g^{-1}(y) = A \in \wp(g)$ and if $h(y) = x$, then $A \subseteq (h \circ g)^{-1}(x) = f^{-1}(x)$. Conversely, suppose $\wp(g) \subseteq \wp(f)$, then for $z \in Rg$ take the unique $y \in Rf$ for which $g^{-1}(z) \subseteq f^{-1}(y)$ and define $h : Rg \to Rf$ by $h(z) = y$. Clearly h is continuous, $h \circ g = f$ and hence $f \leqslant g$. ∎

Note 1.6. Two maps f and g are equivalent if and only if $\wp(f) = \wp(g)$.

Lemma 1.7. *Let X be a compact space without isolated points. Then $DP(X)$ is a complete upper semi-lattice.*

Proof: Let S be a non-empty subset of $DP(X)$ and let $Z = \prod\{Rf \mid f \in S\}$. Consider the natural evaluation map $g : X \to Z$ such that $\pi_f(g(p)) = f(p)$, where $\pi_f : Z \to Rf$ is the f^{th} projection map. Set $T = g(X)$, $\pi'_f = \pi_f|_T$ and define $g' : X \to T$ by $g'(p) = g(p)$, $p \in X$. It is easy to verify that g' is the least upper bound of S. ∎

Theorem 1.8. *Let X be a compact space without isolated points. Then $DP(X)$ is a complete lattice.*

Proof: Since a constant map onto its image is a density preserving map and any two such maps are equivalent, $DP(X)$ has the minimum element. The required result now follows from Lemma 1.7 and the fact that a complete upper semilattice with minimum element is a complete lattice. ∎

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2. Order structure of $DP(X)$

The order structure of the poset $DP(X)$ is always determined by the topology on X, that is, if spaces X and Y are homeomorphic then $DP(X)$ and $DP(Y)$ are order isomorphic. We show here that the converse is true when X and Y are countably compact T_3 spaces without isolated points. The following terms and results are along the lines of [**2**, Lemmas 6, 9 and 10]. Throughout this section, our spaces are without isolated points.

Definition 2.1: A Map $f \in DP(X)$ is said to be

(i) *primary* if $\wp(f)$ has at most one non-singleton member.

(ii) *dual* if it is primary and $\wp(f)$ contains exactly one doubleton.

Notation. If for some $f \in DP(X)$, $\wp(f)$ contains n non-singleton members, say $K_1, K_2, \dots, K_n$, then f is denoted by $(f, K_1, K_2, \dots, K_n)$. In particular, if K is a non-singleton closed nowhere dense set in X, then (f, K) denotes the natural density preserving map defined on X obtained by collapsing K to a point.

Lemma 2.2.

I *A map $f \in DP(X)$, $f \neq id_X$ is primary (respectively dual) if and only if there do not exist dual points g, $h \in DP(X)$ (respectively $g \in DP(X)$) such that $f \wedge g = f \wedge h \neq f$ and the only dual points greater than $g \wedge h$ are g and h (respctively $f < g < id_X$).*

II *For two closed nowhere dense subsets K_1 and K_2 of X,*

$$(f, K_1) \wedge (g, K_2) = \begin{cases} (h, K_1, K_2), & \text{if } K_1 \cap K_2 = \phi \\ (h, K_1 \cup K_2), & \text{if } K_1 \cap K_2 \neq \phi\,. \end{cases}$$

III *An oder isomorphism $\varphi : DP(X) \to DP(Y)$ maps dual points to dual points.*

Definition 2.3: A bijection $f : X \to Y$ is called a *cln-bijection* if $\{f(A) \mid A$ is a closed nowhere dense subset of $X\} = \{B \mid B$ is closed nowhere dense subset of $Y\}$.

Lemma 2.4. *Let $\varphi : DP(X) \to DP(Y)$ be an order isomorphism. Then there exists a cln-bijection $F : X \to Y$ such that $f \in DP(X)$ implies $\wp(\varphi(f)) = \{F(A) \mid A \in \wp(f)\}$.*

Proof: Take $p \in X$ and choose distinct points q, $r \in X - \{p\}$. By Lemma 2.2(III), $\varphi(f, \{p,q\})$, $\varphi(g, \{p,r\})$ are dual points of $DP(Y)$ say $(\overline{f}, \{a,b\})$ and $(\overline{g}, \{c,d\})$ respectively. Clearly $(\overline{f}, \{a,b\}) \wedge (\overline{g}, \{c,d\}) = \varphi(f \wedge g, \{p,q,r\})$. If $\{a,b\} \cap \{c,d\} = \phi$, then $(\overline{f}, \{a,b\}) \wedge (\overline{g}, \{c,d\}) = (\overline{f} \wedge \overline{g}, \{a,b\}, \{c,d\})$; $(f, \{p,q\}), (g, \{p,r\})$, $(h\{q,r\})$ are three dual points greater than $(f \wedge g, \{p,q,r\})$ and $(\overline{f}, \{a,b\})$, $(\overline{g}, \{c,d\})$

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are two dual points greater than $(\overline{f} \wedge \overline{g}, \{a,b\}, \{c,d\})$ which is not possible. Therefore $\{a,b\} \cap \{c,d\} \neq \phi$, in fact it is a singleton, say $\{a\}$. Define $F : X \to Y$ by $F(p) = a$. Note that the choice of a does not depend on the choice of r and q. In general, if $f \in DP(X)$ is of the form (f,H) and if $\varphi(f,H) = \overline{f}$, then it is easy to verify that $\overline{f} = (\overline{f}, K)$ for some closed nowhere dense subset K of Y. Further, if $p,q \in H$, $p \neq q$ then $(g,\{p,q\}) \geqslant (f,H)$ which implies $(\overline{g},\{a,b\}) \geqslant (\overline{f},K)$ therefore $F(\{p,q\}) = \{a,b\} \subseteq K$ and hence $F(H) \subseteq K$. Similarly we can use φ^{-1} to define $\overline{F} : Y \to X$ and obtain $\overline{F}(K) \subseteq H$. Observe that $\overline{F} \circ F$ is identity on X. In fact, if $p \in X$ and $q \in X - \{p\}$, then $\varphi(f,\{p,q\})$ is dual point say $(\overline{f},\{a,b\})$ and $F(p) \in \{a,b\}$. Assume $F(p) = a$. Suppose $\overline{F}(a) \neq p$. Then $\overline{F}(a) = q$. Choose $r \in X - \{q,p\}$ then there exists $c \in Y$ such that $\varphi(g,\{p,r\})$ is a dual point say $(\overline{g},\{a,c\})$. Since $\overline{F}(a) \in \{p,r\}$ and $\overline{F}(a) \neq p$, therefore $\overline{F}(a) = r$, a contradiction. Similaraly, $F \circ \overline{F}$ is identity on Y. We have also shown in the process that if $\varphi(f,H) = (\overline{f},K)$, then $F(H) = K$. ∎

Recall that a subset A of countably compact T_3 space X without isolated points is closed if and only if whenever $B \subseteq A$ and $\mathrm{Cl}_X B$ is nowhere dense in X then $\mathrm{Cl}_X B \subseteq A$. Using this fact, Lemma 2.4 and the technique of [**3**, Theorem 1.1], we have the following.

Theorem 2.5. *Let X and Y be countably compact T_3 spaces without isolated points. Then $DP(X)$ and $DP(Y)$ are order isomorphic if and only if X and Y are homeomorphic.*

Note 2.6. The map $f : Q \cup \{p\} \to Q \cup \{q\}$ in [**3**, example 3.9] defined by $f(x) = x$ if $x \in Q$ and $f(p) = q$, where p and q are remote points of Q such that Stone's extension of no self-homeomorphism of Q maps p to q, is a cln-bijection between non countably compact spaces which is not a homeomorphism.

3. $DP(X)$ and $IP(X)$

Definition 3.1: For a subset A in X we define

$$DP(X,A) = \left\{ f \in DP(X) \mid \left| f^{-1}(f(x)) \right| = 1, \text{ for all } x \in A \right\}.$$

Note 3.2.

(i) $DP(X,A)$ is a poset with respect to the order defined on $DP(X)$.

(ii) If $g \in DP(X,A)$, $f \in DP(X)$ and $g \leqslant f$, then $f \in DP(X,A)$.

Theorem 3.3. *Let A be a subset of a compact space X containing all isolated points of X. The $DP(X,A)$ is a complete upper semilattice.*

Proof: Follows from Lemma 1.7 and Note 3.2(ii). ∎

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THEOREM 3.4. *Let A_i be any subset of X_i containing all isolated points of X_i, $i = 1, 2$ and $\varphi : DP(X_1, A_1) \to DP(X_2, A_2)$ be an order isomorphism. Then there is a cln-bijection $F : X_1 - A_1 \to X_2 - A_2$.*

PROOF: Follows along the lines of Lemma 2.4. ∎

THEOREM 3.5. *Let A be a dense subspace of a space X. Then every f in $DP(X, A)$ is ireducible.*

PROOF: Let $f \in DP(X, A)$. F be a proper closed subset of X and $f(F) = Rf$. Then for every $y \in (X - F) \cap A$, $\left|f^{-1}(f(y))\right| \neq 1$ which contradicts the choice of f. ∎

COROLLARY 3.6. *If X is compact and A is dense in X then $DP(X, A) = IP(X, A)$. In particular, if X is locally compact the $DP(\alpha X, X) = IP(\alpha X, X)$, where αX is a compactification of X.*

PROOF: Set $D_C(X, A) = \{f \in DP(X, A) \mid f \text{ is closed}\}$. Observe that $D_C(X, A) \subseteq IP(X)$ and $D_C(X, A) = DP(X, A)$. ∎

NOTE 3.7. In general, if A is not dense then $D_C(X, A) \subseteq IP(X)$ need not be true. For example take $X = [0, 1]$, $A = [0, 1/2)$ and define $f : X \to X$ by

$$f(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \dfrac{1}{2} \\ \dfrac{3}{2} - x, & \dfrac{1}{2} \leqslant x \leqslant 1 \end{cases}. \quad \text{Clearly } f \in D_C(X, A) - IP(X).$$

We recall the following result [**3,** Lemma 3.11].

LEMMA 3.8. *Let X be a locally compact space. The function $\psi : IP(\beta X, X) \to E_K(X)$ defined by $\psi(f) = \beta X \mid \wp(f)$ is an order isomorphism, where $\beta X \mid \wp(f)$ is the natural compactification of X obtained by collapsing each fibre in $\wp(f)$ to a point.*

We now deduce following result due to Magill [**2,** Theorem 12].

THEOREM 3.9. *Let X and Y be locally compact spaces. Then $E_K(X)$ and $E_K(Y)$ are order isomorphic if and only if $\beta X - X$ and $\beta Y - Y$ are homeomorphic.*

PROOF: If $E_K(X)$ and $E_K(Y)$ are ordered isomorphic, then by Corollary 3.6 and Lemma 3.8, $DP(\beta X, X)$ and $DP(\beta Y, Y)$ are order isomorphic and hence Theorem 3.4 gives a cln-bijection $F : \beta X - X \to \beta Y - Y$. Since all closed subsets in $\beta X - X$ are nowhere dense, F is a closed map. Similarly F^{-1} is also a closed map. ∎

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