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Camina Triples

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Abstract. In this paper, we study Camina triples. Camina triples are a generalization of Camina pairs, first introduced in 1978 by A. R. Camina. Camina's work was inspired by the study of Frobenius groups. We show that if (G, N, M) is a Camina triple, then either G/N is a p-group, or M is abelian, or M has a non-trivial nilpotent or Frobenius quotient.

1 Introduction

In this paper, we study Camina triples. Camina triples are a generalization of Camina pairs, first introduced in 1978 by A. R. Camina in [1]. Camina's work in [1] was inspired by the study of Frobenius groups.

Throughout this paper, we say that (G, N) is a Camina pair when N is a normal subgroup of a group G, and for all $x \in G \setminus N$, x is conjugate to all of xN. Chillag and Macdonald proved in [2] two equivalent conditions of a pair (G, N) to be a Camina pair. They showed that if (G, N) is a Camina pair, then for every $x \in G \setminus N$ we have $|\mathbf{C}_G(x)| = |\mathbf{C}_{G/N}(xN)|$. Also, they proved that if (G, N) is a Camina pair, then for all $x \in G \setminus N$ and $z \in N$, there exists an element $y \in G$ so that $[x, y] = z$. In [9], MacDonald showed that if (G, N) is a Camina pair where G is a p-group, then N is a term in both the lower and the upper central series. As was proved in [2], if $\chi \in \mathrm{Irr}(G)$ where $N \nleq \ker(\chi)$, then χ vanishes on $G \setminus N$. Camina proved in [1] that if (G, N) is a Camina pair, then either N is a p-group or G/N is a p-group for some prime p, or G is Frobenius group with kernel N. In our first theorem, we prove some facts about the subgroup M when (G, N, M) is a Camina triple. In this paper, we use the same notations as in [4].

First, define $\mathrm{Irr}(G \mid M) = \{\chi \in \mathrm{Irr}(G) \mid M \nleq \ker(\chi)\}$.

Definition 1.1 Let $1 < M \leq N$ be two nontrivial normal subgroups of a finite group G. We say that (G, N, M) is a *Camina Triple* if for every $g \in G \setminus N$, g is a conjugate to all of gM.

Notice that Camina pairs are special cases of Camina triples when $M = N$.

Theorem 1 *If (G, N, M) is a Camina triple then the following are true.*

(i) *M is solvable.*
(ii) *M has a normal π-complement Q with M/Q is nilpotent, where π is the set of primes that divide $|G : N|$.*

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(iii) *If $x \in M$, then there exists $\chi \in \mathrm{Irr}(G \mid M)$ such that $\chi(x) \neq 0$.*
(iv) *If $x \in G \setminus N$, then $\chi(x) = 0$ for all $\chi \in \mathrm{Irr}(G \mid M)$.*

The following collorary is an immediate consequence of Theorem 1.

Corollary 1.2 *If (G, N) is a Camina pair and $x \in N$, then there exist $\chi \in \mathrm{Irr}(G \mid N)$ such that $\chi(x) \neq 0$.*

Lewis showed in [8] that if $V(G) < G$, then for every $x \in G \setminus V(G)$ we have $\mathrm{cl}(x) = xG'$. Thus, if $V(G) < G$, then the triple $\big(G, V(G), G'\big)$ is a Camina triple. So by Theorem 1, we deduce that G' is solvable. Therefore, G is solvable.

Our second theorem, which we consider the main result of this paper.

Theorem 2 *If (G, N, M) is a Camina triple, then at least one of the following holds:*

(i) *G/N is a p-group.*
(ii) *M has a non-trivial nilpotent quotient.*
(iii) *M has a non-trivial Frobenius quotient with an Frobenius complement that is an elementary abelain p-group.*
(iv) *M is abelian.*

In closing, we prove some facts about Camina pairs using Camina triples results, and given the fact that they are special cases of Camina triples. In [1], Camina defined a different hypothesis that is equivalent to Camina pairs. Let G be a finite group with a proper normal subgroup $N \neq 1$ and a set of irreducible non-trivial characters of G, $A = \{\chi_1, \dots, \chi_n\}$, where n is a natural number, such that

(1) χ_i vanishes on $G \setminus N$ and
(2) there exist natural numbers $\alpha_1, \dots, \alpha_n > 0$ such that $\sum_{i=1}^{n} \alpha_i \chi_i$ is constant on $N \setminus \{1\}$.

We are able to identify the characters in Camina hypothesis in [1]. First, let N be a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$. The inertia group of θ in G denoted by T and defined by $\{g \in G \mid \theta^g = \theta\}$.

Theorem 3 *Let (G, N) be a Camina Pair then, $A = \mathrm{Irr}(G \mid N)$.*

Our last theorem states some new conditions for a pair (G, N) to be a Camina pair.

Theorem 4 *Let G be a finite group and $N \lhd G$, then the following are equivalent:*

(i) *(G, N) is a Camina pair.*
(ii) *$V(G \mid N) = N$.*
(iii) *There is no x in N such that $\chi(x) = 0$ for all χ's in $\mathrm{Irr}(G \mid N)$, and if $x \in G \setminus N$, then $\chi(x) = 0$ for all χ in $\mathrm{Irr}(G \mid N)$.*

2 Camina Triples

In this section, we prove Theorems 1 and 2 along with some facts about Camina triples. First, we prove some equivalent conditions for a triple (G, N, M) to be a Camina triple.

Theorem 2.1 *If $1 \neq M < N$ are two normal subgroups of a finite group G, then the following are equivalent:*

(i) (G, N, M) *is a Camina triple.*
(ii) $|C_G(g)| = |C_{G/M}(Mg)|$ *for every* $g \in G \setminus N$.
(iii) *For every* $g \in G \setminus N$, *we have* $\chi(g) = 0$ *for all* $\chi \in \mathrm{Irr}(G \mid M)$.
(iv) $V(G \mid M) \leq N$.
(v) *For all* $g \in G \setminus N$ *and* $z \in M$, *there exists* $y \in G$ *such that* $[g, y] = z$.

Proof First, we show that (i) implies (ii). Assume that (G, N, M) is a Camina triple and let $g \in G \setminus N$. Notice that $\mathrm{cl}(g) = \bigcup_{x \in G} (Mg)^x$. Hence, $|G : C_G(g)| = |G/M : C_{G/M}(Mg)|\,|M|$, and so $|C_G(g)| = |C_{G/M}(Mg)|$ as desired. We now show that (ii) implies (iii). Assume (ii) and let $g \in G \setminus N$. By the Second Orthogonality Relation, we have

$$|\mathbf{C}_G(g)| = \sum_{\chi \in \mathrm{Irr}(G)} |\chi(g)|^2 = \sum_{\chi \in \mathrm{Irr}(G|M)} |\chi(g)|^2 + \sum_{\chi \in \mathrm{Irr}(G/M)} |\chi(g)|^2.$$

But we know by (ii) that $|C_G(g)| = |C_{G/M}(Mg)| = \sum_{\chi \in \mathrm{Irr}(G/M)} |\chi(g)|^2$. Hence, we obtain $\sum_{\chi \in \mathrm{Irr}(G|M)} |\chi(g)|^2 = 0$. Since $|\chi(g)|^2 \geq 0$ for all $\chi \in \mathrm{Irr}(G \mid M)$, we deduce that $\chi(g) = 0$ for all $\chi \in \mathrm{Irr}(G \mid M)$. Next, we prove (iii) implies (iv). Assume that for every $g \in G \setminus N$, $\chi(g) = 0$ for all $\chi \in \mathrm{Irr}(G \mid M)$. Hence, all the generators of $V(G \mid M)$ are contained in N. Thus, $V(G \mid M) \leq N$ as desired. Now, we show that (iv) implies (i). Assume that $V(G \mid M) \leq N$ and let $x \in G \setminus N$ and $y \in M$. Hence, $yx \notin N$. Thus $yx \notin V(G \mid M)$. So for any $\chi_i \in \mathrm{Irr}(G \mid M)$, $\chi_i(x) = \chi_i(yx) = 0$. Recall that $\mathrm{Irr}(G) \setminus \mathrm{Irr}(G \mid M) = \mathrm{Irr}(G/M)$. Write $\mathrm{Irr}(G/M) = \{\overline{\phi}_1, \dots, \overline{\phi}_r\}$. For each $\overline{\phi}_i$ there exists $\phi_i \in \mathrm{Irr}(G) \setminus \mathrm{Irr}(G \mid M)$ such that $\phi_i(x) = \overline{\phi}_i(Mx) = \phi_i(yx)$. Hence, x and yx have the same character values for all irreducible characters of G. Since the irreducible characters form a basis for the class functions, all class functions have the same value on x and xy. This implies that x and xy are in the same class. Hence, x is conjugate to all of xM. We conclude that (G, N, M) is a Camina triple. Thus, (iv) implies (i).

To finish the proof of the theorem, it is enough to show that (i) is equivalent to (v). First assume that (G, N, M) is a Camina triple; that is, if $g \in G \setminus N$, then g is conjugate to all of gM. Hence, if $z \in M$, then there exists $y \in G$ such that $y^{-1}gy = gz$. It follows that $g^{-1}y^{-1}gy = z$. Conversely, suppose that for all $g \in G \setminus N$ and $z \in M$ there exists $y \in G$ such that $[g, y] = z$. Fix $g \in G \setminus N$ and $z \in M$. We need to show that g is conjugate to gz. we know there exists y such that $g^{-1}y^{-1}gy = z$. This implies that $y^{-1}gy = gz$. Hence, g is conjugate to every element in gM, and (G, N, M) is a Camina triple as required. ■

The following lemma describes the relationship between two Camina triples in the same group.

Lemma 2.2 *If (G, N_1, M) and (G, N_2, M) are Camina triples, then $(G, N_1 \cap N_2, M)$ is a Camina triple.*

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Proof Notice that $1 < M \leq N_1 \cap N_2$. If $g \in G \setminus N_1 \cap N_2$, then either $g \in G \setminus N_1$ or $g \in G \setminus N_2$. In either case, g is conjugate to all of gM. Hence, $(G, N_1 \cap N_2, M)$ is a Camina triple as desired. ■

We now show that Camina pairs are special cases of Camina triples.

Lemma 2.3 *The triple (G, N, N) is a Camina triple if and only if (G, N) is a Camina pair.*

Proof Observe that (G, N) is a Camina pair if and only if for every $g \in G \setminus N$, we have $\mathrm{cl}(g) = gN$. This occurs if and only if (G, N, N) is a Camina triple. ■

We now prove a fact about the center of a group G in the case when (G, N, M) is a Camina triple. Note that it is not difficult to see that the intersection of $\mathbf{Z}(G)$ and the set of elements in $G \setminus N$ has to be the empty set.

Lemma 2.4 *If (G, N, M) is a Camina triple, then the following are true.*

(i) $\mathbf{Z}(G) \leq N$.
(ii) *If $K \lhd G$ and $K < M$, then $(G/K, N/K, M/K)$ is a Camina triple.*

Proof If $g \in \mathbf{Z}(G)$, then g is only conjugate to itself. Hence g is not conjugate to all of gM, and so $g \in N$. Therefore $Z(G) \leq N$. Now, let $K \lhd G$, with $K < M$. Hence, $1 < M/K \leq N/K < G/K$. Since every $\chi \in \mathrm{Irr}(G \mid M)$ vanishes on $G \setminus N$, every $\chi \in \mathrm{Irr}(G/K \mid M/K)$ vanishes on $G/K \setminus N/K$. It follows that $(G/K, N/K, M/K)$ is a Camina triple as desired. ■

Next, consider the terms of the upper central series of G when (G, N, M) is a Camina triple. Let $Z_1 = \mathbf{Z}(G)$ and $Z_i/Z_{i-1} = \mathbf{Z}(G/Z_{i-1})$ for $i > 1$.

Lemma 2.5 *If (G, N, M) is a Camina triple and $Z_m < M$, then $Z_{m+1} \leq N$.*

Proof By Lemma 2.4(ii), $(G/Z_m, N/Z_m, M/Z_m)$ is a Camina triple. So applying Lemma 2.4(i) to G/Z_m, we get $\mathbf{Z}(G/Z_m) \leq N/Z_m$. Hence $Z_{m+1} \leq N$ as desired. ■

Now we need to state the following very useful theorem, which is Theorem D in [7], due to Berkovich.

Theorem 2.6 *Let N be a normal subgroup of G and suppose that every member of $\mathrm{cd}(G \mid N')$ is divisible by some fixed prime p. Then N is solvable and has a normal p-complement.*

We need the next lemma to prove the remaining parts of our first theorem.

Lemma 2.7 *If (G, N, M) is a Camina triple, then M is solvable and has a normal p-complement for every prime p that divides $|G : N|$.*

Proof Let $\chi \in \mathrm{Irr}(G \mid M)$. We know by Lemma 2.1 that $\chi(g) = 0$ for all $g \in G \setminus N$. By the discussion in [4, p. 200]. we deduce that for every prime p divisor of $|G : N|$, p divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(G \mid M')$. So by Berkovich's theorem, M is solvable and M has a normal p-complement. ■

Now we show that if (G, N, M) is a Camina triple, then M has a normal π-complement, where π is the set of primes that divide $|G : N|$. This proves the remaining parts of Theorem 1.

Lemma 2.8 *If (G, N, M) is a Camina triple, and $\pi = \{p \text{ prime} \mid p \text{ divides } |G : N|\}$, then M has a normal π-complement Q such that M/Q is nilpotent.*

Proof Since (G, N, M) is a Camina triple, by Lemma 2.7, we know that M has a normal p-complement for every $p \in \pi$. Now let Q be the intersection of these normal p-complements. Hence, Q is a normal π-complement of M. To prove that M/Q is nilpotent, it will be enough to show that any finite group having a normal p-complement for every prime p is nilpotent. Let G be a finite group that has a normal p-complement for every prime p. We work by induction on $|G|$. If $|G| = 1$, then the result is trivial, so we may assume $G > 1$. Let p be a prime, and we show that G has a normal Sylow p-subgroup. If G is a p-group, then this is trivial. Thus, we may assume that $|G|$ is divisible by some prime q which is not p. By hypothesis, G has a normal q-complement N. Observe that $N < G$, and so the induction hypothesis implies that N is nilpotent. Hence, N has a normal Sylow p-subgroup P and thus, P is characteristic in N. But G/N is a q-group, so P is a Sylow p-subgroup of G. It follows that G has a normal Sylow p-subgroup as required. ■

The following lemma is very useful.

Lemma 2.9 *Let (G, N, M) be a Camina triple. If $x \in G \setminus N$, $o(x) = m$, and $y \in C_M(x)$, then the order of y divides m.*

Proof Since $xy \in xM$, we know that xy is a conjugate to x. Thus, xy has order m. Hence $x^m y^m = 1$, and so, $y^m = 1$ as desired. ■

We show in the next lemma that if G/N is not a p-group for any prime p, then $M \cap Z(G) = \{1\}$.

Lemma 2.10 *Let (G, N, M) be a Camina Triple, and G/N is not a p-group for any prime p, then $M \cap Z(G) = \{1\}$.*

Proof If G/N is not a p-group, then we can find $x \in G \setminus N$ such that $o(Nx) = p^a$, and $y \in G \setminus N$ such that $o(Ny) = q^b$, where p, q are two distinct primes. Let n be the p'-part of the order of x, and m the q'-part of the order of y. Notice that $x^n \notin N$ and $y^m \notin N$. Also, the order of x^n is p^α and the order of y^m is q^β. Hence, $C_M(x^n)$ is a p-group and $C_M(y^m)$ is a q-group. We know that $M \cap Z(G) \subseteq C_M(x^n) \cap C_M(y^m) = \{1\}$ as desired. ■

In the next result, we prove that if G is nilpotent, then G/N and M are p-groups for the same prime p.

Lemma 2.11 *Let (G, N, M) be a Camina Triple, if G is nilpotent then M and G/N are p-groups for some prime p.*

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Proof Since G is nilpotent, $Z(G)$ cannot intersect with M trivially. Hence, by Lemma 2.10, G/N is p-group for some prime p. Now let $x \in G \backslash N$ where $o(Nx) = p^a$ and let n be the p'-part of the order of x. Notice that $x^n \notin N$ and the order of x^n is p^α. Hence, $C_M(x^n)$ is a p-group. Thus, $M \cap Z(G)$ is a p-group. Now suppose that there exists a prime $q \neq p$ such that q divides $|M|$. Hence, there exists $y \in M$ where the order of y is q^m. Since G is nilpotent and $\big(o(x^n), o(y)\big) = 1$, we have $y \in C_G(x^n)$. But, by Lemma 2.9, we know that $o(y)$ divides the order of x^n. Which leads to a contradiction, and M is a p-group. ■

We are now ready to prove Theorem 2.

Proof of Theorem 2 If G is nilpotent, then by Lemma 2.11 we have (i) and (ii) hold. So we may assume that G is not nilpotent. If G/N is a p-group, then (i) holds. Assume that G/N is not a p-group, and let $\pi = \{p : \text{prime such that } p \text{ divides } |G/N|\}$, by Lemma 2.8, M has a normal π-complement Q such that M/Q is nilpotent. If $Q \neq \{1\}$ and proper in M, then (ii) holds. Also, if $Q = \{1\}$, then M is nilpotent and (ii) holds. If $M = Q$, then $(|M|, |G/N|) = 1$. In this case, we know that maybe M does not have a nilpotent quotient. If M is abelian, then (iv) holds. So we may assume that M is not abelian. By Theorem 1, we know that if (G, N, M) is a Camina triple, then M is solvable. Hence, if M is not abelian, then we can consider K to a maximal normal subgroup of M such that M/K is not abelian. Note that $(M/K)'$ is the unique minimal normal subgroup of M/K. Therefore, the group M/K satisfies the hypothesis of Theorem 12.3 of [4]. So either M/K is a p-group, and hence M/K is nilpotent and (iii) holds, or M/K is a Frobenius group with an abelian Frobenius complement and $(M/K)'$ is the Frobenius kernel and is an elementary abelian p-group, and (iv) holds as desired. ■

3 Camina Pairs

We now prove some results about Camina pairs using Camina triples results, given the fact that they are special cases of Camina triples. In [1], Camina defined a different hypothesis that is equivalent to Camina pairs. Let G be a finite group with a proper normal subgroup $N \neq 1$ and a set of irreducible non-trivial characters of G, $A = \{\chi_1, \dots, \chi_n\}$, where n is a natural number, such that

(1) χ_i vanishes on $G \setminus N$ and
(2) there exist natural numbers $\alpha_1, \dots, \alpha_n > 0$ such that $\sum_{i=1}^{n} \alpha_i \chi_i$ is constant on $N \setminus \{1\}$.

We are able to identify the characters in the Camina hypothesis in [1]. First, let N be a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$. The inertia group of θ in G denoted by T and defined by $\{g \in G \mid \theta^g = \theta\}$.

Theorem 3.1 *Let (G, N) be a Camina Pair. Then $A = \mathrm{Irr}(G \mid N)$.*

Proof First, we show that $A \subseteq \mathrm{Irr}(G \mid N)$. To see this, suppose $\chi_j \in A \setminus \mathrm{Irr}(G \mid N)$. This implies that $\chi_j \in \mathrm{Irr}(G/N)$. On the other hand, since $\chi_j \in A$, we have that $\chi_j(x) = 0$ for all $x \in G \setminus N$. This implies $\chi_j(xN) = 0$ for all $xN \in G/N \setminus \{N\}$, and hence, χ_j is a multiple of the regular character of G/N. Since $N < G$, we know that the regular character of G/N is not irreducible, and so we have a contradiction, since it is not possible for an irreducible character to be a multiple of a reducible character. Thus, no such χ_j exist in A. Therefore, $A \subseteq \mathrm{Irr}(G \mid N)$. On the other hand, for every $1_N \neq \theta \in \mathrm{Irr}(N)$ and by Theorem 6.11 in [4], there exist $\chi_i \in A$ such that $\chi_i \in \mathrm{Irr}(G \mid \theta)$. Notice that $\theta^G(g) = 0$ if $g \notin N$, and if $g \in N$, then $\theta^G(g) = \frac{1}{|N|}\sum_{x\in G}\theta^x(g)$, hence

$$\theta^G(g) = \frac{1}{|N|}|T|\big(\theta_1(g) + \cdots + \theta_n(g)\big) = \theta^G(g) = |T : N|\big(\theta_1(g) + \cdots + \theta_n(g)\big)$$

where θ_i, $i = 1, \ldots, n$, are the distinct conjugates of θ in G. Note that $\chi_i(g) = 0$ if $g \notin N$, and if $g \in N$, then $\chi_i(g) = a\big(\theta_1(g) + \cdots + \theta_n(g)\big)$, where a is a non-negative integer. Hence $\chi_i = c\theta^G$. Thus, χ_i is the unique irreducible constituent of θ^G. Thus, $|\mathrm{Irr}(G \mid \theta)| = 1$, and $\mathrm{Irr}(G \mid \theta) \subseteq A$. Since $\mathrm{Irr}(G \mid N) = \bigcup_{1\neq\theta\in\mathrm{Irr}(N)} \mathrm{Irr}(G|_\theta)$, we have $|A| = |\mathrm{Irr}(G \mid N)|$. And since $A \leq \mathrm{Irr}(G \mid N)$, we have $A = \mathrm{Irr}(G \mid N)$ as desired. ■

Our last result in this section states some new conditions for a pair (G, N) to be a Camina pair.

Theorem 3.2 *Let G be a finite group and $N \lhd G$, then the following are equivalent:*

(i) (G, N) *is a* CP.

(ii) $V(G \mid N) = N$.

(iii) *There is no x in N such that $\chi(x) = 0$ for all χ's in* $\mathrm{Irr}(G \mid N)$ *and if $x \in G \setminus N$, then $\chi(x) = 0$ for all χ in* $\mathrm{Irr}(G \mid N)$.

Proof Notice that (iii) implies (ii) is trivial. To prove (ii) implies (i), assume that $V(G \mid N) = N$, by Theorem 2.1, (G, N, N) is a Camina triple. Thus, by Lemma 2.3, (G, N) is a Camina pair. To prove (i) is equivalent to (iii), by Lemma 2.3 and Theorem 2.1, (G, N) is a Camina pair if and only if $V(G \mid N) \leq N$. ■

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