JACOBIANS FOR MEASURES IN COSET SPACES

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Let G be a locally compact topological group, let H be a closed subgroup and let G/H be the space of left cosets $\bar{x} = xH$ with the natural topology. We denote by μ a non-negative measure in G/H defined on the ring of Baire sets. G acts by left multiplication as a transitive group of homeomorphisms on G/H: Every $t \in G$ defines the homeomorphism $\bar{x} \to t\bar{x} = \overline{tx}$. We write, for $E \subset G/H$, $tE = \{t\bar{x} : \bar{x} \in E\}$. The measure μ is called *stable* (cf. [**3**], [**4**]) if from $t \in G$, $E \subset G/H$ and $\mu(E) = 0$ follows $\mu(tE) = 0$. We say that μ is *locally finite* [**3**], [**5**] if every set of positive measure contains a subset of positive finite measure.

Let us write ξ for a typical element of H and let $d\xi$ indicate integration with respect to a left invariant Haar measure in H. Evidently, if $f(x)$ is a non-negative Baire function on G, then $\int f(x\xi)\, d\xi$ is everywhere defined and, considered as a function of x, constant on cosets xH; thus it is really a function of $\bar{x}$ (see [**2**]). We write

$$\bar{f}(\bar{x}) = \int \bar{f}(x\xi)\, d\xi.$$

Then $\bar{f}(\bar{x})$ is a non-negative Baire function on G/H [**2**], [**3**].

We call μ *inherited* if there is a measure $\tilde{\mu}$ in G defined on Baire sets which is absolutely continuous with respect to the Haar measure and such that, for every non-negative Baire function $f(x)$ on G,

$$\int f(x)\, d\tilde{\mu}(x) = \int f(\bar{x})\, d\mu(\bar{x}). \qquad \text{......................................(1)}$$

Such measures always do exist (cf. [**2**]). In particular, every stable measure μ which is finite on some open set is inherited [**3**], [**4**].

Given a topological space X, we shall denote by $B_+(X)$ the class of all non-negative extended real-valued Baire functions on X (extended real numbers include $\pm\infty$). $LB_+(X)$ will stand for the class of all functions $f(x)$ such that $f(x)g(x) \in B_+(X)$ whenever $g(x) \in B_+(X)$ (locally Baire functions). A set $Q \subset X$ will be called an LB-set if there is a function $f(x) \in LB_+(x)$ with $f(x) = 0$ or 1 and $f(x) = 1$ if and only if $x \in Q$. Given a measure μ on Baire subsets of X, we generalize the notion of a set of measure zero to LB-sets as follows: $\mu(Q) = 0$ if and only if $\mu(Q \cap E) = 0$ for every Baire set E. (We note that $Q \cap E$ is a Baire set.) In this generalized sense we use the phrase " almost everywhere with respect to μ ", which will be denoted also by $[\mu]$.

A function $J(t, \bar{x}) \in LB_+(G \times G/H)$ is called a Jacobian for μ (or μ-Jacobian) if, for every $f(\bar{x}) \in B_+(G/H)$ and $t \in G$,

$$\int f(t^{-1}\bar{x})\, d\mu(\bar{x}) = \int f(\bar{x}) J(t, \bar{x})\, d\mu(\bar{x}). \qquad \text{..............................(2)}$$

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It is not difficult to check (see [2], [4]) that, for each $s, t \in G$, a μ-Jacobian satisfies

$$J(st, \bar{x}) = J(s, t\bar{x})J(t, \bar{x}) \qquad \text{.....................................(3)}$$

for almost all $\bar{x}$ with respect to the measure μ. A Jacobian is called *exact* if the above equality holds identically for every $\bar{x}$. It is known that, for a stable locally finite inherited measure μ, there always exists an exact positive and finite-valued Jacobian (see [3], and Lemma 3 below). The class of exact Jacobians was characterized in [3] by functional equations. We give in this note an analogous characterization of the class of all Jacobians for these measures.

We denote by m a left-invariant Haar measure in G. The phrases " almost all ", " almost every " etc. will be used, unless stated otherwise, with respect to the Haar measures m, $m^2 = m \times m$ and $m^3 = m \times m \times m$ in G, $G^2 = G \times G$ and $G^3 = G \times G \times G$ respectively. These phrases will occasionally be replaced by the symbols $[m]$, $[m^2]$ and $[m^3]$. For convenience we shall sometimes write $J(t, x)$ instead of $J(t, \bar{x})$, assuming always that $J(t, x) = J(t, x\xi)$ for all $\xi \in H$. Let $\Delta(x)$ and $\delta(\xi)$ denote the modular functions for m and for the left invariant Haar measure in H.

THEOREM. *A function $J(t, \bar{x}) \in LB_+(G \times G/H)$ is a Jacobian for some stable locally finite inherited measure in G/H if and only if it satisfies the following conditions :*

(α) *$0 < J(t, x) < \infty$ for almost all pairs $\langle t, x \rangle \in G^2$,*

(β) *for every $s, t \in G$, $J(st, x) = J(s, tx)J(t, x)$ holds for almost all $x \in G$,*

(γ) *for every $\xi \in H$, $J(t\xi x^{-1}, x)\Delta(\xi) = J(tx^{-1}, x)\delta(\xi)$ holds for almost all pairs $\langle t, x \rangle \in G^2$.*

In the proof we shall use some earlier results of A. M. Macbeath and the present author. We list them here in Lemmas 1–4.

LEMMA 1. *A measure $\tilde{\mu}$ is associated by* (1) *with a stable locally finite inherited measure μ if and only if the Radon-Nikodym derivative* [1, § 32, p. 133] *$d\tilde{\mu}/dm = h(x) \in LB_+(G)$ is such that*

(i) $0 < h(x) < \infty \quad [m]$,

(ii) *for every $\xi \in H$, $h(x\xi)\Delta(\xi) = h(x)\delta(\xi)$ holds for almost every $x \in G$.*

The function $h(x)$ is called a *μ-factor function* [2], [3].

LEMMA 2. *For every stable locally finite inherited measure μ there is a μ-factor function $h_0(x)$ such that both* (i) *and* (ii) *in Lemma* 1 *hold identically for all $x \in G$, $\xi \in H$* (see [3]).

LEMMA 3. *Given a μ-factor function $h_0(x)$ satisfying the condition of Lemma* 2, *the formula*

$$J_0(t, x) = h_0(tx)/h_0(x)$$

defines an exact μ-Jacobian. We note that, for each t, $h_0(tx)/h_0(x)$ is constant on the coset xH since (ii) *holds identically also for tx in place of x and for all $\xi \in H$* (see [3]).

LEMMA 4. *The mapping $f \to \bar{f}$ maps $B_+(G)$ onto $B_+(G/H)$* (see [2], [3]).

We denote by ϕ the canonical mapping of G onto G/H, $(\phi(x) = \bar{x})$. Then we obtain from Lemmas 1 and 4 the

COROLLARY. *For any LB-set $Q \subset G/H$, $\mu(Q) = 0$ is equivalent to $m(\phi^{-1}(Q)) = 0$.*

To see this we note that $m(\phi^{-1}(Q)) = 0$ means, by condition (i) of Lemma 1, that the $\tilde{\mu}$-integral of every $f \in B_+(\phi^{-1}(Q))$ is equal to zero. Similarly, $\mu(Q) = 0$ means that the μ-integral of every $g \in B_+(Q)$ is equal to zero. Thus, by formula (1) and Lemma 4, both conditions mean the same.

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LEMMA 5. *Each of the transformations* $S\langle t, x\rangle = \langle tx, x\rangle$, $T_v\langle t, x\rangle = \langle tv, x\rangle$, ($v \in G$ *fixed*), $R_1\langle s, t, x\rangle = \langle stx, tx, x\rangle$, $R_2\langle s, t, x\rangle = \langle s, tx, x\rangle$ *of* G^2 *or* G^3 *onto itself maps sets of measure zero on sets of measure zero.*

Proof. Since the right and left Haar measures are absolutely continuous one with respect to the other [**1**], we may assume in this proof that the considered measures are right invariant. Then the transformations are all measure-preserving. This is well known for S [**1**, § 59, Theorem B] and it is trivial for T_v. To prove that the R_i are also measure preserving, we note that, if we have two spaces U, W with measures m_1 and m_2 and $S(u)$ is a measure preserving transformation of U into itself, then the transformation $\langle u, w\rangle \to \langle S(u), w\rangle$ preserves the measure $m_1 \times m_2$ in $U \times W$ (because every rectangle is transformed into a rectangle of the same measure). Now each R_i can be obtained by combining transformations of the above kind where S is the transformation $S\langle t, x\rangle = \langle tx, x\rangle$. This proves the lemma.

We shall frequently apply a consequence of Fubini's theorem stating that a subset of a product space has measure zero if and only if every section has measure zero [**1**, § 36, Theorem A].

Necessity of (α), (β) *and* (γ). Suppose that $J(t, \bar{x}) \in LB_+(G \times G/H)$ is a μ-Jacobian. By Lemma 3, there is an everywhere finite and positive exact μ-Jacobian $J_0(t, \bar{x})$. For every t, $J_0(t, \bar{x}) = J(t, \bar{x})$ [μ], by (2), and hence $J_0(t, x) = J(t, x)$ for almost all x, by the corollary to Lemma 4. Thus, by Fubini's theorem, we have (α).

Condition (β) is an immediate consequence of (3) and of the same corollary. To prove (γ) observe that, for the μ-Jacobian $J_0(t, x)$, (γ) follows from condition (ii) of Lemma 1. Now, for each t, $J_0(t, x) = J(t, x)$ for almost all x; hence $J_0(t, x) = J(t, x)$ [m^2]. Using the transformations S and T_v of Lemma 5, we have, for each $v \in G$, $J_0(tv^{-1}x^{-1}, x) = J(tv^{-1}x^{-1}, x)$ for almost all pairs $\langle t, x\rangle$. Putting $v = \xi^{-1}$ and e in turn we derive (γ) from the analogous equality for $J_0(t, x)$.

Sufficiency of (α), (β) *and* (γ). We assume (α), (β) and (γ), and we shall prove that $J(t, \bar{x}) \in LB_+(G \times G/H)$ is a Jacobian for a certain inherited stable locally finite measure μ.

LEMMA 6. *There is a function* $h(s) \in LB_+(G)$ *such that, for every* σ*-compact open subgroup* $\Gamma \subset G$,

(a) $0 < h(s) < \infty$ *for almost all* $s \in \Gamma$,

(b) *for almost all* $t \in \Gamma$, *there is a number* $0 < c_t < \infty$ *such that*

$$h(s) = c_t J(st^{-1}, t) \quad \text{for almost all} \quad s \in \Gamma. \qquad \text{.........................(4)}$$

Denote by $\Phi(\Gamma)$ the class of Baire functions which are defined on Γ and have the properties (a) and (b). We establish first some properties of this class. (In the proofs of A and B below we assume that all transformations and functions are restricted to Γ.)

A. *The class* $\Phi(\Gamma)$ *is not empty.*

Using the transformation $S\langle t, x\rangle = \langle tx, x\rangle$ we derive from (α), by Lemma 5, that $0 < J(tx^{-1}, x) < \infty$ [m^2]. Hence the set M_0 of all $x \in \Gamma$ such that $0 < J(tx^{-1}, x) < \infty$ holds for almost all t, satisfies $m(\Gamma - M_0) = 0$.

Applying Fubini's theorem and the transformation $R_1\langle s, t, x\rangle = \langle stx, tx, x\rangle$, we obtain from ($\beta$) that

$$J(sx^{-1}, x) = J(st^{-1}, t) J(tx^{-1}, x) \quad [m^3].$$

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Hence, if M_1 denotes the set of all $x \in \Gamma$, such that the above equality holds for almost all $\langle s, t\rangle \in \Gamma \times \Gamma$, then $m(\Gamma - M_1) = 0$. Thus $M_0 \cap M_1 \neq 0$. Let $x_0 \in M_0 \cap M_1$. Then, for almost all $t \in \Gamma$, $0 < J(tx_0^{-1}, x_0) < \infty$ and

$$J(sx_0^{-1}, x_0) = J(st^{-1}, t)J(tx_0^{-1}, x_0) \quad \text{for almost all} \quad s \in \Gamma. \qquad \text{...............(5)}$$

Define $c_t = J(tx_0^{-1}, x_0)$, $h(s) = J(sx_0^{-1}, x_0)$. Then $0 < c_t < \infty\ [m]$ and $0 < h(s) < \infty\ [m]$, and (4) holds by (5). This proves A.

B. *If $g, h \in \Phi(\Gamma)$, then there is exactly one constant c such that $g(s) = h(s) \cdot c$ for almost all $s \in \Gamma$.*

Suppose that $\{c_t^0\}$ is the set of numbers which appear in condition (*b*) for g in place of h. There is a $t \in \Gamma$, such that (4) holds for both g and h, and then, by (*a*), $g(s) = h(s) \cdot c$, where $c = c_t^0/c_t$. The uniqueness of c follows from condition (*a*).

We observe that $\Gamma_0 \subset \Gamma_1$ implies $\Phi(\Gamma_1) \subset \Phi(\Gamma_0)$. Hence from B we have

C. *If $\Gamma_0 \subset \Gamma_1$ are σ-compact open subgroups of G and $h_i(s) \in \Phi(\Gamma_i)$, $(i = 0, 1)$, then there is a unique constant c_0 such that $h_1(s) \cdot c_0 = h_0(s)\ [m]$ on Γ_0. Obviously $h_1(s) \cdot c_0$ belongs to $\Phi(\Gamma_1)$.*

Proof of Lemma 6. Let Γ_0 be a fixed σ-compact open subgroup of G and let $h_0^* \in \Phi(\Gamma_0)$. For every σ-compact group $\Gamma \subset G$ which contains Γ_0 we denote by h^* a function in $\Phi(\Gamma)$ which satisfies $h^*(s) = h_0^*(s)\ [m]$ on Γ_0. Such a function exists by C. If we have Γ_i in place of Γ, the corresponding function h^* will be denoted by h_i^*. We now define $h(s)$ on G, defining it on each Γ_0-coset. First we put $h(s) = h_0^*(s)$ on Γ_0. For each coset $x_i\Gamma_0$, denote by Γ_i the group generated by the set $\Gamma_0 \cup x_i\Gamma_0$. This is a σ-compact group and we define $h(s) = h_i^*(s)$ on $x_i\Gamma_0$.

It remains to verify that conditions (*a*) and (*b*) hold. Obviously we may assume that Γ contains Γ_0, so that it is a union of Γ_0-cosets. We shall prove that, for every coset $x_i\Gamma_0 \subset \Gamma$ $h(s) = h^*(s)\ [m]$ on $x_i\Gamma_0$. Since Γ is σ-compact, it contains at most a countable number of Γ_0-cosets; hence $h(s) = h^*(s)\ [m]$ on Γ. This will prove the lemma, because h^* satisfies (*a*) and (*b*) by definition. Since Γ_i is a subgroup of Γ, there is, by C, a unique constant c_i such that $h^*(s) \cdot c_i = h_i^*(s)\ [m]$ on Γ_i. We have, on Γ_0, $h^*(s) \cdot c_i = h_i^*(s) = h_0^*(s) = h^*(s)\ [m]$. Thus $c_i = 1$ and $h^*(s) = h_i^*(s)\ [m]$ on Γ_i. In particular this is true on $x_i\Gamma_0$, where $h_i^*(s) = h(s)$. This proves the lemma.

We show that the function $h(s)$ constructed above satisfies condition (ii) of Lemma 1. Obviously it is enough to verify that (ii) holds on every σ-compact open subgroup $\Gamma \subset G$. Let $v \in \Gamma$. We have, from Lemma 6, for almost all $t \in \Gamma$,

$$h(s) = c_t J(st^{-1}, t) \quad \text{for almost all} \quad s \in \Gamma;$$

hence $h(sv) = c_t J(svt^{-1}, t)$ for almost all $s \in \Gamma$. Consequently

$$h(sv)\, J(st^{-1}, t) = h(s) J(svt^{-1}, t) \quad \text{for almost all pairs } \langle s, t\rangle,$$

and if we take $v = \xi \in H$, then the above condition and (γ) imply (ii).

Since h is almost everywhere finite and positive, it defines, by Lemma 1, a stable locally finite inherited measure μ on G/H. It remains to show that $J(t\ x)$ is a μ-Jacobian. This will follow if we prove

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LEMMA 7. *On every open σ-compact group $\Gamma \subset G$ we have, for all $s \in \Gamma$, $J(s, v) = J_0(s, v)$ for almost all $v \in \Gamma$, where $J_0(s, v)$ is the μ-Jacobian given by Lemma* 3.

Proof. (We restrict all functions and transformations to Γ.) Condition (β) implies that $J(sv, t) = J(s, vt)J(v, t)$ $[m^3]$. Applying the transformation $R_2\langle s, v, t\rangle = \langle s, vt, t\rangle$, we have, by Lemma 5, $J(svt^{-1}, t) = J(s, v)J(vt^{-1}, t)$ $[m^3]$. Thus, by property (*b*) of $h(s)$,

$$h(sv) = J(s, v)h(v) \quad [m^2].$$

Now, if h_0 is a μ-factor function which satisfies the condition of Lemma 2, then $h_0(s) = h(s)$ $[m]$ (because each of these functions is a Radon-Nikodym derivative $d\tilde{\mu}/dm$). Consequently $h_0(sv) = J(s, v)h_0(v)$ $[m^2]$. This means that, for almost all $s \in \Gamma$, $J(s, v) = J_0(s, v)$ holds for almost all $v \in \Gamma$, where J_0 is the exact μ-Jacobian given by Lemma 3. If $M \subset \Gamma$ is the set of all s such that $J(s, v) = J_0(s, v)$ for almost all $v \in \Gamma$, then $m(\Gamma - M) = 0$ and we have to show that $M = \Gamma$. Now, if $s, t \in M$, then, by (β) and since J_0 is an exact Jacobian, we have $st \in M$. This proves that $M = \Gamma$, because every element $x \in \Gamma$ can be represented in the form $x = st$, where $s, t \in M$; it is enough to take $s \in M \cap xM^{-1}$.

REFERENCES

1. P. R. Halmos, *Measure theory*, New York 1951.

2. A. M. Macbeath and S. Świerczkowski, Measures in homogeneous spaces, *Fundamenta Math.*, **49** (1960), 15–24.

3. A. M. Macbeath and S. Świerczkowski, Inherited measures ; to appear in *Proc. Roy. Soc. Edinburgh.*

4. S. Świerczkowski, Measures equivalent to the Haar measure, *Proc. Glasgow Math. Assoc.*, **4** (1960), 157–162.

5. A. C. Zaanen, A note on measure theory, *Nieuw Arch. Wisk.* (3) **6** (1958), 58–65.

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