# ON A SINGULAR MEASURE OF D. M. CONNOLLY AND J. H. WILLIAMSON

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(Received 5th August 1974)

In (**1**) a measure $\lambda \in M(\mathbb{R})$ is constructed and shown to satisfy the following:

(i) $\lambda * \tilde{\lambda}$ is absolutely continuous, where $\tilde{\lambda}$ denotes the measure with $\tilde{\lambda}(E) = \overline{\lambda(E^{-1})}$ for all Borel sets $E$,

(ii) $\lambda * \lambda = \lambda^2$ is singular,

(iii) $\lambda^k = \lambda^{k-1} * \lambda$ is absolutely continuous for $k \geqq 6$.

The purpose of this note is to show that (iii) can be sharpened to read "for $k \geqq 3$".

Let us now fix some notation and show how the measure $\lambda$ has been constructed. Denote by $G$ the complete direct product $\prod_{r=1}^{\infty} G_r$, where $G_r$ are finite cyclic groups $\simeq \mathbb{Z}(q_r)$. There is a natural continuous map from $G$ into $\mathbb{R}$ (or $T$) given by $g \to \sum_{r=1}^{\infty} g_r \,.\, d_r^{-1}$, where $d_r = q_1 q_2, \ldots, q_r$. This map, $\phi$ say, gives rise to a map between $M(G)$ and $M(\mathbb{R})$ ($M(T)$) by $\phi(\mu)[f] = \mu[f \circ \phi]$ for $f \in C_0(\mathbb{R})$, $\mu \in M(G)$.

In (**1**) the authors have chosen $q_r = a_r^2 + a_r + 1$, $a_r$ is a power of a prime $p_r$, and Singer's theorem shows the existence of sets $X_r \subset G_r$ such that

(i) card $X_r = a_r + 1$,

(ii) $X_r - X_r = \{x_1 - x_2 \mid x_1 \in X_r, x_2 \in X_r\} = G_r$,

(iii) $\sum_{r=1}^{\infty} a_r^{-1} < \infty$.

If we now take $\mu_r$ to be the uniform probability measure supported on $X_r$, i.e.

$$\mu_r\{x\} = (a_r + 1)^{-1} \quad \text{for} \quad x \in X_r$$
$$\mu_r\{x\} = 0 \qquad\qquad\quad x \notin X_r$$

and take $\mu \in M(G)$ to be $\mu = \bigotimes_{r=1}^{\infty} \mu_r$, the unique product measure corresponding to $\{\mu_r\}_{r=1}^{\infty}$, then $\lambda$ is just the image $\phi(\mu)$ of $\mu$.

G. Brown and W. Moran have established the following remarkable result (**2**, lemma 5, p. 12):

Let $\mu_r = m_r + \rho_r$, $m_r$ is the normalised Haar measure on $G_r$, $\mu_r$ a probability measure on $G_r$. Let $\mu = \bigotimes_{r=1}^{\infty} \mu_r$ and let $\phi(\mu) = \nu \in M(T)$. Write

$$\alpha_r(k)^2 = \int_{G_r} \left| \frac{d\rho_r^k}{dm_r} \right|^2 dm_r = \int_{\hat{G}_r} |(\rho_r^k)\hat{\ }(\gamma)|^2 d\gamma = \sum_{\gamma \in \hat{G}_r} |(\rho_r^k)\hat{\ }(\gamma)|^2.$$

Then, if $\sum_{r=1}^{\infty} \alpha_r(k)^2 < \infty$, $\nu^k$ is absolutely continuous, i.e. $\nu^k \in L^1(T)$.

Let us now come back to the results of (1). Then $\lambda = \phi(\mu)$, where $\mu = \otimes \mu_r$ and $\mu_r$ are as specified above. Despite the fact that we do not know much about the sets $X_r$ we can give a very precise description of $\mu_r * \tilde{\mu}_r$. It is trivial to see that

$$\mu_r * \tilde{\mu}_r = \frac{1}{(a_r+1)^2} \{(a_r+1)\delta_0 + \sum_{g \neq 0} \delta_g\},$$

where $\delta_g$ is the unit mass of point $g$. So

$$\mu_r * \tilde{\mu}_r = \frac{1}{(a_r+1)^2} \{a_r \delta_0 + \sum_{g \in G_r} \delta_g\}$$

$$= \frac{1}{(a_r+1)^2} \{a_r \delta_0 + (a_r^2 + a_r + 1) m_r\}$$

$$= m_r + \left\{ \frac{a_r}{(a_r+1)^2} \delta_0 - \frac{a_r}{(a_r+1)^2} m_r \right\}$$

$$= m_r + \rho_r * \tilde{\rho}_r$$

when $\mu_r = m_r + \rho_r$.

Thus

$$\rho_r * \tilde{\rho}_r = \frac{a_r}{(a_r+1)^2} \delta_0 - \frac{a_r}{(a_r+1)^2} m_r$$

and so

$$(\rho_r * \tilde{\rho}_r)\hat{\ }(\gamma) = |\hat{\rho}_r(\gamma)|^2 = \frac{a_r}{(a_r+1)^2} \quad \text{if } \gamma \neq 0$$

$$= 0 \qquad \text{if } \gamma = 0.$$

Hence the criterion of Brown and Moran takes the form

$$\alpha_r(k)^2 = \sum_{\gamma \in \hat{G}_r} |(\rho_r^k)\hat{\ }(\gamma)|^2 = \sum_{\gamma \in \hat{G}_r} (|\hat{\rho}_r(\gamma)|^2)^k = (a_r^2 + a_r) \,.\, a_r^k/(a_r+1)^{2k}$$

since card $\hat{G}_r$ = card $G_r$. So $\alpha_r(k)^2 \sim a_r^{2-k}$ and since by (iii) $\sum_{r=1}^{\infty} a_r^{-1} < \infty$ we have that $\sum_{r=1}^{\infty} \alpha_r(k)^2 < \infty$ for $k \geqq 3$, which implies that $\lambda^k$ is absolutely continuous for $k \geqq 3$.

## REFERENCES

(1) D. M. Connolly and J. H. Williamson, An application of a theorem of Singer, *Proc. Edinburgh Math. Soc.* **19** (1974), 119-123.

(2) G. Brown and W. Moran, Coin tossing and powers of singular measures, *Proc. Cambridge Philos. Soc.*, to appear.

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