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The free topological group on a cell complex

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It is proved that the free k-group on a CW-complex X is itself a CW-complex containing X as a subcomplex. It follows, as a corollary, that the free topological group on a countable CW-complex is a countable CW-complex.

1. Introduction

The work of [5] and [12] shows that if X is a k-space such that the cartesian product $X \times X$ is not a k-space, then the free topological group $F(X)$ is not a k-space. In particular, then, if X is Dowker's CW-complex [2] the free topological group $F(X)$ is *a priori* not a CW-complex, since it is not even a k-space. However, the cartesian product of two countable CW-complexes is always a countable CW-complex. So it would seem more reasonable to ask if the free topological group on a countable CW-complex is a countable CW-complex. In fact we answer a more general question here; by working wholly in the category of k-Hausdorff k-spaces we prove that the free k-group on any CW-complex is itself a CW-complex containing X as a subcomplex. As a corollary, we then obtain that the free topological group on a countable CW-complex is a countable CW-complex.

This investigation was precipitated by a question of Calder, and complements work of [10], which proved that the free product of topological groups which are countable CW-complexes is also a countable CW-complex.

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2. Results

Before stating any results, we recall the basic definitions and make some preliminary remarks on k-spaces.

A topological space X is a *k-space* if a subset A of X is closed in X whenever $f^{-1}(A)$ is closed in C, for each compact Hausdorff space C and each continuous map $f : C \to X$. There is clearly a category $k_\times$ of k-spaces and continuous maps, and a functor $k : \text{top}_\times \to k_\times$, from the category of all topological spaces to $k_\times$, which assigns to each topological space X the k-space kX obtained by giving the set X the final topology with respect to all continuous maps $f : C \to X$ from any compact Hausdorff space to X. $k_\times$ also has a product $\times_k$, which where no confusion arises will be written just $\times$. A topological space X is *k-Hausdorff* if for each compact Hausdorff space C and each continuous map $f : C \to X$, $f(C)$ is closed in X. Notice that a k-space X is k-Hausdorff if and only if the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is closed in $X \times_k X$. Throughout this paper, all spaces considered will be k-Hausdorff unless otherwise stated. In particular, a *CW-complex* is a k-Hausdorff space which is a closure finite cell-complex with the weak topology. For further information the reader is referred to [*2*, *4*, *6*, *9*, and *11*].

A *k-group* is a group object in the category $k_\times$; that is, a group G whose underlying set is a (k-Hausdorff) k-space and whose structure functions $\phi : G \times_k G \to G$, $\sigma : G \to G$ are morphisms in $k_\times$. The *(Graev) free k-group* [*3*, *4*, and *11*] on a pointed k-space (X, e) consists of a k-group $FG(X)$ together with a continuous pointed map $i : X \to FG(X)$ which is universal for continuous pointed maps from X into k-groups; that is, if $f : X \to H$ is any such map then there is a unique morphism of k-groups $f^* : FG(X) \to H$ such that $f^*i = f$. $FG(X)$ is independent of the choice of base point, contains X as a closed subset and is algebraically just the free group on the set $X\backslash\{e\}$.

Our main result is

THEOREM 1. *The (Graev) free k-group $FG(X)$ on a CW-complex X is itself a CW-complex, and contains X as a subcomplex.*

The proof is given in §3.

The *(Markov) free k-group* [4 and 8] on a k-space X is a k-group $FM(X)$ together with a continuous map $i : X \to FM(X)$ such that if $f : X \to H$ is any continuous map into a k-group H then there is a unique morphism of k-groups $f^* : FM(X) \to H$ such that $f^*i = f$. By checking universal properties, it is easy to prove that if X is any k-space, then there is an isomorphism of k-groups $FM(X) \cong FG(X \cup e)$, where $X \cup e$ is the disjoint union of X with a singleton space $\{e\}$. Thus Theorem 1 gives us

COROLLARY 2. *The (Markov) free k-group $FM(X)$ on a CW-complex X is also a CW-complex, containing X as a subcomplex.*

We can now obtain a version of Theorem 1 in the usual topological category by a standard argument. A *k_ω-space* [1, 4, 7, and 10] is a Hausdorff topological space X which has a countable covering by compact sets $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ such that X has the weak topology with respect to $\{X_n\}_{n \geq 0}$. Examples of k_ω-spaces are compact Hausdorff spaces, connected locally compact topological groups, and (most important for our purposes) countable CW-complexes. It is clear that any k_ω-space is necessarily a k-space.

The *(Graev) free topological group* [1, 3, 4, 5, 7, and 11] on a pointed topological space is, of course, defined in a similar way to the (Graev) free k-group on a pointed k-space, and it is routine to deduce from the construction of the (Graev) free k-group (*cf.* [11], Theorem 2, and [4], Chapter III, §4) that if X is a k_ω-space, then the (Graev) free k-group $FG(X)$ is also the (Graev) free topological group on X. But, in the proof of Theorem 1, we will see that if X is a *countable* CW-complex then $FG(X)$ is also a *countable* CW-complex; so that Theorem 1 again gives

COROLLARY 3. *The (Graev) free topological group $FG(X)$ on a countable CW-complex X is itself a countable CW-complex, and contains X as a subcomplex.*

A similar result for (Markov) free topological groups can be deduced from Corollary 3 in the same way as Corollary 2 was deduced from Theorem 1.

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3. Proof of Theorem 1

It is clear that we can choose the base point $e \in X$ to be a 0-cell without loss of generality. Let X^{-1} be a homeomorphic copy of X with elements x^{-1} for each $x \in X$, and let $\overline{X}$ denote the wedge product $X \vee X^{-1}$. Then by the adjunction theorem for CW-complexes ([6], p. 62, Theorem 5.11), the obvious cell-structure induced on $\overline{X}$ by the cell-structure on X makes $\overline{X}$ a CW-complex containing both X and X^{-1} as subcomplexes. Now let $\overline{X}^n$ be the product in $k_\times$ of n copies of $\overline{X}$, and let $Y_n = \overline{X}^n/R_n$, where R_n is the equivalence relation generated by

$$\left(x_1^{\varepsilon_1}, \ldots, x_{i-2}^{\varepsilon_{i-2}}, x_{i-1}^{\varepsilon_{i-1}}, x_i^{\varepsilon_i}, x_{i+1}^{\varepsilon_{i+1}}, x_{i+2}^{\varepsilon_{i+2}}, \ldots, x_n^{\varepsilon_n}\right)$$
$$\sim \left(x_1^{\varepsilon_1}, \ldots, x_{i-2}^{\varepsilon_{i-2}}, x_{i-1}^{\varepsilon_{i-1}}, e, e, x_{i+2}^{\varepsilon_{i+2}}, \ldots, x_n^{\varepsilon_n}\right)$$
$$\sim \left(x_1^{\varepsilon_1}, \ldots, x_{i-2}^{\varepsilon_{i-2}}, e, e, x_{i-1}^{\varepsilon_{i-1}}, x_{i+2}^{\varepsilon_{i+2}}, \ldots, x_n^{\varepsilon_n}\right)$$

whenever $x_{i+1}^{\varepsilon_{i+1}} = x_i^{-\varepsilon_i}$, $1 \le i < n$. Finally, let G_n be the subset of $FG(X)$ comprising all the "reduced words" of length at most n (that is, words $x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}$ in $FG(X)$ such that $x_{i+1}^{\varepsilon_{i+1}} \ne x_i^{-\varepsilon_i}$ for any $1 \le i < n$, and $x_i \ne e$ for any $1 \le i \le n$). Then it is proved in [11], Corollary 1 (*cf.* also [4], Chapter V, Theorem 3.1) that each G_n is closed in $FG(X)$, and $FG(X)$ is the iterated adjunction space $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n \subseteq \ldots$, with $G_n = G_{n-1} \cup_{f_{n-1}} Y_n$, where the attaching map $f_{n-1} : A_{n-1} \to G_{n-1}$ is given by $\left(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}\right) \to x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, and A_{n-1} is the subspace of Y_n consisting of all words $\left(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}\right)$ which have an "e" somewhere.

Thus to prove Theorem 1 it is sufficient to prove that each G_n is a CW-complex containing G_{n-1} as a subcomplex. This we do by induction.

First we observe that $G_1 = \overline{X} = X \vee X^{-1}$ is a CW-complex. Then for the inductive step, we assume that G_{n-1} is a CW-complex; so that, again by the adjunction theorem for CW-complexes, it remains only to prove

PROPOSITION 4. *For each* $n > 1$, *the space* Y_n *is a* CW-*complex containing* A_{n-1} *as a subcomplex.*

We will need the following lemma. Let I^m be the closed unit m-cube in R^m , and let S_m be the equivalence relation generated on I^m by

$$\begin{aligned}(t_1, \ldots, t_{i-2}, t_{i-1}, t_i, t_{i+1}, t_{i+2}, \ldots, t_m) \\ \sim (t_1, \ldots, t_{i-2}, t_{i-1}, 0, 0, t_{i+2}, \ldots, t_m) \\ \sim (t_1, \ldots, t_{i-2}, 0, 0, t_{i-1}, t_{i+2}, \ldots, t_m)\end{aligned}$$

whenever $t_i = t_{i+1}$, $1 \le i < n$. Then

LEMMA 5. *There is a cellular decomposition of* I^m *such that* S_m *is a cellular equivalence relation* (*cf.* [6], p. 32).

Proof. Before describing the cellular decomposition of I^m , we introduce some new notation. The intersection of a p-cell P with a q-cell Q , $p \le q$, will be called an *embeddable intersection* if $P \cap Q$ is also a p-cell, and will be called a *non-degenerate intersection* if $P \cap Q$ is a $(p-1)$-cell. Of course an embeddable intersection is necessarily degenerate.

For each $1 \le i$, $j \le m$, $i \ne j$, let L_{ij} denote the hyperplane $\{(x_1, \ldots, x_m) : x_i = x_j\}$ in I^m . Then the m-cells of I^m are the $m!$ portions into which the L_{ij} divide I^m .

Now let $M_i = \{(x_1, \ldots, x_m) : x_i = 0\}$ and $N_i = \{(x_1, \ldots, x_m) : x_i = 1\}$, $1 \le i \le m$, be the faces of I^m ; so that in the "usual" decomposition of I^m the M_i, N_i are precisely the $(m-1)$-cells. Then the $(m-1)$-cells in our "new" decomposition are all the

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embeddable intersections of the L_{ij}, M_k, N_l with the m-cells.

Notice that the $(m-2)$-cells in the usual decomposition of I^m are just the faces of the $(m-1)$-cells, namely all the non-degenerate intersections of the M_i and N_j. Similarly, in the new decomposition of I^m, the $(m-2)$-cells are all the non-degenerate intersections of the L_{ij}, M_k, N_l with the $(m-1)$-cells. We can now proceed inductively constructing the $(m-r)$-cells of the new decomposition of I^m as all the non-degenerate intersections of L_{ij}, M_k, N_l with the $(n-r+1)$-cells. The 0-cells in this decomoosition of I^m are of course the same as in the usual decomposition; that is, the "corners" of I^m.

It is obvious that I^m with the above cell-structure is a CW-complex, and routine to verify that the equivalence relation S_m is a cellular equivalence relation with respect to this cell-structure.

Proof of Proposition 4. First we construct a cell-structure for Y_n.

Let $\overline{X}$ have the cell-structure described above, so that $\phi : \overline{X} \to \overline{X}$, given by $x \to x^{-1}$ and $x^{-1} \to x$, is a regular homeomorphism. Then for any m-cell $\bar{\sigma} : I^m \to \overline{X}$, the composite $I^m \xrightarrow{\bar{\sigma}} \overline{X} \xrightarrow{\phi} \overline{X}$ is also an m-cell for $\overline{X}$, which by an abuse of notation we denote $\bar{\sigma}^{-1} : I^m \to \overline{X}$.

Let $\bar{\sigma}_i : I^{m_i} \to \overline{X}$, $1 \le i \le n$, be any m_i-cells of $\overline{X}$. If $\bar{\sigma}_{i+1} \ne \bar{\sigma}_i^{-1}$ for any $1 \le i < n$, then we have a diagram

$$\begin{array}{ccc} I^{m_1}\times\ldots\times I^{m_n} & \xrightarrow{\bar{\sigma}_1\times\ldots\times\bar{\sigma}_n} & \overline{X}\times\ldots\times\overline{X} \\ & \searrow & \downarrow P_n \\ & & Y_n \end{array}$$

in which $P_n : \overline{X}^n \to Y_n$ is the canonical quotient map associated with

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R_n ; and so $\sigma = p(\bar{\sigma}_1 \times \ldots \times \bar{\sigma}_n) : I^{m_1} \times \ldots \times I^{m_n} \to Y_n$ is an $(m_1 + \ldots + m_n)$-cell for Y_n . Conversely, if $\bar{\sigma}_{i+1} = \bar{\sigma}_i^{-1}$ for some $1 \le i < n$, then we have a diagram,

$$\begin{array}{ccc} I^{m_1} \times \ldots \times I^{m_n} & \xrightarrow{\bar{\sigma}_1 \times \ldots \times \bar{\sigma}_n} & \bar{X} \times \ldots \times \bar{X} \\ \downarrow q_\sigma & & \downarrow p_\sigma \\ \left(I^{m_1} \times \ldots \times I^{m_n}\right)/S_\sigma & \xrightarrow{\quad \sigma \quad} & Y_n \end{array}$$

where S_σ is the equivalence relation generated by

$$\begin{aligned} &(t_1, \ldots, t_{i-2}, t_{i-1}, t_i, t_{i+1}, t_{i+2}, \ldots, t_n) \\ &\qquad \sim (t_i, \ldots, t_{i-1}, t_i, 0, 0, t_{i+2}, \ldots, t_n) \\ &\qquad \sim (t_i, \ldots, t_{i-2}, 0, 0, t_{i-1}, t_{i+2}, \ldots, t_n) \end{aligned}$$

whenever $\bar{\sigma}_{i+1} = \bar{\sigma}_i^{-1}$ and $t_{i+1} = t_i$, $1 \le i < n$, and $\sigma : \left(I^{m_1} \times \ldots \times I^{m_n}\right)/S_\sigma \to Y_n$ is the unique map induced by $p_n(\bar{\sigma}_1 \times \ldots \times \bar{\sigma}_n) : I^{m_1} \times \ldots \times I^{m_n} \to Y_n$. But by Lemma 5, S_σ is a cellular equivalence relation, so that $\left(I^{m_1} \times \ldots \times I^{m_n}\right)/S_\sigma$ is a (finite) cell-complex, and $\sigma : \left(I^{m_1} \times \ldots \times I^{m_n}\right)/S_\sigma \to Y_n$ determines a (finite) number of cells for Y_n . It is straightforward to check that the set of all cells σ constructed as above defines a closure finite cell structure for Y_n . Thus the proof is completed by

LEMMA 6. *Let X be any k-Hausdorff k-space having the weak topology with respect to some cover $\{X_\alpha\}$, and let R be an equivalence relation on X such that the graph of R is closed in $X \times X$. Then the quotient space X/R is a k-Hausdorff k-space having the weak topology with respect to $\{X_\alpha/R\}$.*

The proof is routine (*cf.* [9], Proposition 2.4).

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