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LOCAL UNIQUE FACTORIZATION IN THE SEMIGROUP OF PATHS IN $\mathbb{R}^n$

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ABSTRACT. Let S denote the semigroup of all rectifiable, piecewise continuously differentiable paths in $\mathbb{R}^n$ under concatenation. We prove a theorem to the effect that every finite collection of paths is contained in a subsemigroup of S which has the unique factorization property with respect to certain primes and straight lines. We also determine an abstract necessary sufficient condition for a subsemigroup of S to have this unique factorization property.

Throughout this paper, $\mathbb{R}$, $\mathbb{R}^+$, Z^+ will denote the sets of all real numbers, positive reals, positive integers, respectively. n will denote a fixed positive integer and $\mathbb{R}^n$ the Euclidean n-space. Let $\mathcal{M}$ denote the set of all rectifiable, piecewise continuously differentiable functions f from $[0, 1]$ into $\mathbb{R}^n$ such that $f(0)=0$ and f is not constant on any subinterval of $[0, 1]$. If $f, g \in \mathcal{M}$, then let $fg \in \mathcal{M}$ be defined by

$$fg(x)=\begin{cases} f(2x) & 0 \le x \le \frac{1}{2} \\ f(1)+g(2x-1), & \frac{1}{2} \le x \le 1 \end{cases}$$

If $f, g \in \mathcal{M}$, then define $f \equiv g$ if $g = f \circ \phi$ for some strictly increasing, continuous self-map ϕ of $[0, 1]$ with $\phi(0)=0$, $\phi(1)=1$. Then $S=\mathcal{M}/\equiv$ is a cancellative semigroup (see [2, 3] where $\mathcal{D}_1^*$ was used to denote this semigroup). Let $\mathcal{L}$ denote the set of all lines in S. If $u \in S$, then $l(u)$ denotes the length of u. If $a \in \mathcal{L}$, $\alpha \in \mathbb{R}^+$, then let a^α denote the line parallel to a having length $\alpha l(a)$. If $a, b \in \mathcal{L}$, then we will write $a \sim b$ if a is parallel to be (i.e., $b = a^\alpha$ for some $\alpha \in \mathbb{R}^+$). If $A \subseteq S$, then let $\langle A \rangle$ denote the semigroup generated by A. If Γ is a set, then let $\mathcal{F}=\mathcal{F}(\Gamma)$ denote the free semigroup on Γ. If $\Lambda \subseteq \Gamma$, then let $\mathcal{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$ denote the semigroup of words in alphabet Γ such that for $A \in \Gamma \backslash \Lambda$ the exponents of A in the word are allowed to be positve real numbers. So $\mathcal{F}_{\mathbb{R}}(\Gamma \mid \Gamma)=\mathcal{F}(\Gamma)$ and $\mathcal{F}_{\mathbb{R}}=\mathcal{F}_{\mathbb{R}}(\Gamma)=\mathcal{F}_{\mathbb{R}}(\Gamma \mid \varnothing)$ is the free product of $|\Gamma|$ copies of positive reals under addition (see [3]). Let S^1 be the semigroup $S \cup \{1\}$, $1 \notin S$ such that 1 is the identity element of S^1. If $X \subseteq S$, then $X^1 = X \cup \{1\}$.

If $X \subseteq S$, then the *power closure* of X, $\bar{X}=\{u^i \mid u \in X, i \in \mathbb{Z}^+\} \cup \{u^\alpha \mid u \in X \cap \mathcal{L}, \alpha \in \mathbb{R}^+\}$. Let $a, b \in S$, X, $Y \subseteq S$. Then $a <_{X,Y} b$ is $b = xay$ for some $x, y \in S^1$ such that $(x, y) \in (X^1 \times X^1) \backslash (Y^1 \times Y^1)$. Let T be a subsemigroup of S. Then T satisfies the *descending chain condition* if there is

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no sequence in T of the following type

$$\cdots \underset{T,\mathscr{L}}{<} a_3 \underset{T,\mathscr{L}}{<} a_2 \underset{T,\mathscr{L}}{<} a_1$$

T is a *weakly unitary* subsemigroup of S if for all $a \in S$, the conditions $aT \cap T \neq \varnothing$ and $Ta \cap T \neq \varnothing$, together imply $a \in T$. This condition, due to Schützenberger, comes up naturally in the study of free semigroups [1; p. 119]. T is *power closed* if $\bar{T} = T$. T is *free-like* if T is weakly unitary, is power closed and satisfies the descending chain condition.

REMARK. Intersection of free-like subsemigroups of S is again free-like.

THEOREM 1. *Let T be a free-like subsemigroup of S. Then $T \cong \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$ for some Γ, Λ.*

Proof. Let $\mathscr{L}_t = T \cap \mathscr{L}$. Then $T \backslash \mathscr{L}_t$ is a subsemigroup of T. Let $D = \{a \mid a \in T\backslash\mathscr{L}_t,\ a \neq bc$ for any $b, c \in T\backslash\mathscr{L}_t\}$. We first show that $T\backslash\mathscr{L}_t = \langle D\rangle$. For suppose $a \in T\backslash\mathscr{L}_t$, $a \notin \langle D\rangle$. Then $a = bc$ for some b, $c \in T\backslash\mathscr{L}_t$. So $b <_{T,\mathscr{L}} a$, $c <_{T,\mathscr{L}} a$. Either $b \notin \langle D\rangle$ or $c \notin \langle D\rangle$. Thus there exists $a_1 \in T\backslash\mathscr{L}_t$ such that $a_1 \notin \langle D\rangle$, $a_1 <_{T,\mathscr{L}} a$. Continuing, we find a sequence $\{a_i\}_{i\in\mathbb{Z}^+}$ in $T\backslash(\mathscr{L}_t \cup \langle D\rangle)$ such that

$$\cdots \underset{T,\mathscr{L}}{<} a_2 \underset{T,\mathscr{L}}{<} a_1 \underset{T,\mathscr{L}}{<} a.$$

This violates the descending chain condition of T. So $T\backslash\mathscr{L}_t = \langle D\rangle$. Let $P_1 = \{a \mid a \in D,\ a \notin bT$ for any $b \in \mathscr{L}_t\}$, $P_2 = \{a \mid a \in D,\ a \notin Tb$ for any $b \in \mathscr{L}_t\}$, $P = P_1 \cap P_2$. We claim that $D \subseteq \langle\mathscr{L}_t\rangle^1 P_1^1$. For suppose $a \in D$, $a \notin \langle\mathscr{L}_t\rangle^1 P_1^1$. Then there exists $b_1 \in \mathscr{L}_t$ such that $a = b_1 a_1$, $a_1 \in T\backslash\mathscr{L}_t$, $a_1 \notin b_1^\alpha T$ for any $\alpha \in \mathbb{R}^+$. Then clearly $a_1 \in D$ and so $a_1 \notin \langle\mathscr{L}_t\rangle^1 P_1^1$. Continuing we find a sequence $\{b_i\}_{i\in\mathbb{Z}^+}$ in $\mathscr{L}_t$, $\{a_i\}_{i\in\mathbb{Z}^+}$ in D such that $b_i x b_{i+1}$ for any i and for any $i \in Z^+$, $a_i = b_{i+1}a_{i+1}$. So $a_i = b_{i+1}b_{i+2}a_{i+2}$ and $a_{i+2} <_{T,\mathscr{L}} a_i$. So

$$\cdots \underset{T,\mathscr{L}}{<} a_6 \underset{T,\mathscr{L}}{<} a_4 \underset{T,\mathscr{L}}{<} a_2 \underset{T,\mathscr{L}}{<} a.$$

This violates the descending chain condition of T. Hence, $D \subseteq \langle\mathscr{L}_t\rangle^1 P_1^1$. Similarly $D \subseteq P_2^1\langle\mathscr{L}_t\rangle^1$. Let $a \in D$. Then $a = bc$ for some $b \in \langle\mathscr{L}_t\rangle^1$, $c \in P_1^1$. Now $c = dh$ for some $d \in P_2^1$, $h \in \langle\mathscr{L}_t\rangle^1$. Since $c \in P_1^1$, $d \in P_1^1 \cap P_2^1 = P^1$. So $a = bdh$, b, $h \in \langle\mathscr{L}_t\rangle^1$, $d \in P^1$. Thus $D \subseteq \langle\mathscr{L}_t\rangle^1 P^1 \langle\mathscr{L}_t\rangle^1$. Since $T\backslash\mathscr{L}_t = \langle D\rangle$ we see that $T = \langle P \cup \mathscr{L}_t\rangle$. Let $\mathscr{L}_u = \{a \mid a \in \mathscr{L}_t,\ l(a) = 1\}$. If $a, b \in \mathscr{L}_u$, then $a \sim b$ implies $a = b$. Also $\bar{\mathscr{L}}_u = \mathscr{L}_t$. So clearly $\langle\mathscr{L}_t\rangle \cong \mathscr{F}_{\mathbb{R}}(\mathscr{L}_u)$. Also it is clear that $T = \langle P \cup \bar{\mathscr{L}}_u\rangle$. Let $\Gamma = P \cup \mathscr{L}_u$, $\Lambda = P$. We claim that $T \cong \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$. To see this, let $a \in T$, $a = a_1 \cdots a_m = b_1 \cdots b_p$ where $a_1, \ldots, a_m$, $b_1, \ldots, b_p \in P \cup \mathscr{L}_t$ such that if a_i, $a_{i+1} \in \mathscr{L}_t$, then $a_i x a_{i+1}$ and if b_j, $b_{j+1} \in \mathscr{L}_t$, then $b_i x b_{j+1}$. We must show that $m = p$ and $a_i = b_i$ for all i. Let $u = a_2 \cdots a_m$, $v = b_2 \cdots b_p$. Then $a_1 u = b_1 v$. First suppose $a_1 \in \mathscr{L}_t$. We claim that $b_1 \in \mathscr{L}_t$. For suppose $b_1 \in P$. Since $b_1 \notin \mathscr{L}_t$,

$l(b_1) > l(a_1)$. So $b_1 = a_1c$ for some $c \in S$. Then a_1, v, a_1c, $cv = u \in T$. Since T is weakly unitary, $c \in T$. This contradicts the fact that $b_1 \in P$. So $b_1 \in \mathscr{L}_t$. Then clearly $a_1 \sim b_1$. We claim that $a_1 = b_1$. Otherwise by symmetry assume $l(a_1) < l(b_1)$. Then $b_1 = a_1c$ for some $c \in \mathscr{L}$. Since $c \sim a_1$, T is power closed, $c \in \mathscr{L}_t$. Also,

$$a_2 \cdots a_m = cv.$$

As above, $a_2 \in \mathscr{L}_t$, $a_2 \sim c \sim a_1$, a contradiction. So $a_1 = b_1$. Next assume a_1, $b_1 \in P$. Suppose $l(a_1) < l(b_1)$. Then $b_1 = a_1c$ for some $c \in S$. So a_1c, a_1, $cv = u$, $v \in T$ and so $c \in T$. If $c \in \mathscr{L}_t$, we get a contradiction to the fact that $b_1 \in P$. Otherwise we get a contradiction to the fact that $b_1 \in D$. Thus $a_1 = b_1$ in all cases. We are now done by induction.

REMARK. Let T be a free-like subsemigroup of S, $T \cong \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$. If the elements of Λ are thought of as primes, then T has the unique factorization property with respect to primes and lines.

COROLLARY 2. *Let T be a subsemigroup of S such that $T \cap \mathscr{L} = \varnothing$. Then T is free if and only if T is free-like.*

REMARK. The converse of Theorem 1 is false for the following reason. Let K be a proper subsemigroup of $(\mathbb{R}^+, +)$ such that $K \cong (\mathbb{R}^+, +)$. Let $u \in \mathscr{L}$ and set $T = \{u^\alpha \mid \alpha \in \mathbb{R}^+\}$. Then clearly $T \cong (\mathbb{R}^+, +)$ but T is not free-like.

THEOREM 3. *Let T be a power-closed subsemigroup of S. If $T \cong \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$ for some Γ, Λ, then T is free-like.*

Proof. Let $\mathscr{L}_t = T \cap \mathscr{L}$. Let $\phi : T \to \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$ be the given isomorphism. Let $a \in S$, $b, c \in T$ such that $ab, ca \in T$. Then $(ca)b = c(ab)$. So $\phi(ca)\phi(b) = \phi(c)\phi(ab)$. There exists $u \in \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$ such that either $\phi(ca) = \phi(c)u$ or $\phi(c) = \phi(ca)u$. Let $a_1 = \phi^{-1}(u) \in T$. Then $caa_1 = c$ or $ca = ca_1$. First case being ruled out, $a = a_1 \in T$. So T is weakly unitary in S. Let $\mathscr{K} = \{A^\alpha \mid A \in \Gamma \backslash \Lambda, \alpha \in \mathbb{R}^+\}$. If $a, b \in \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$, then define $a < b$ if $b = xay$ for some $x, y \in \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)^1$ such that $(x, y) \notin \mathscr{K}^1 \times \mathscr{K}^1$. Clearly $(\mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda), <)$ satisfies the descending chain condition. Thus to show that T satisfies the descending chain condition, we must show that $\phi(\mathscr{L}_1) = \mathscr{K}$. If $a \in \mathscr{L}_t$, then for each $i \in Z^+$, there exists $a_i \in T$, such that $a_i^i = a$. Moreover by the author [2], the elements of $\mathscr{L}_t$ are characterized by this property. The same holds true for $\mathscr{K}$ in $\mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$. This proves that $\phi(\mathscr{L}_1) = \mathscr{K}$, completing the proof.

THEOREM 4. *Every finite subset of S is contained in a free-like subsemigroup of S.*

Proof. Let $a_1, \ldots, a_m \in S$. Let k be the smallest non-negative integer such that there exist $b_1, \ldots, b_k \in S$ such that $a_1, \ldots, a_m \in \langle b_1, \ldots, b_k, \mathscr{L} \rangle$ $(k = 0$

means $a_1, \ldots, a_m \in \langle \mathscr{L} \rangle$). Let $\mathscr{L}_u = \{a \mid a \in \mathscr{L}, l(a)=1\}$, $T = \langle b_1, \ldots, b_k, \bar{\mathscr{L}}_u \rangle = \langle b_1, \ldots, b_k, \mathscr{L} \rangle$. We will show that T is free-like. Let $\Gamma = \{b_1, \ldots, b_k\} \cup \mathscr{L}_u$, $\Lambda = \{b_1, \ldots, b_k\}$. We claim that $T \cong \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$. Suppose not. Then it is easily seen that there exist $A_1, \ldots, A_r, B_1, \ldots, B_s \in \Lambda \cup \mathscr{L}$ such that

$$A_1 \cdots A_r = B_1 \cdots B_s \tag{1}$$

and either A_1 or $B_1 \in \Lambda$ and so that if both $A_1, B_1 \in \Lambda$, then $A_1 \neq B_1$. Let $\{A_1, \ldots, A_r, \ B_1, \ldots, B_s\} \cap \mathscr{L} = \{c_1, \ldots, c_t\}$. Introduce variables $x_1, \ldots, x_k$, $y_1, \ldots, y_t$, and words $w_1 = w_1(x_1, \ldots, x_k, \ y_1, \ldots, y_t)$, $w_2 = w_2(x_1, \ldots, x_k, y_1, \ldots, y_t)$ such that $A_1 \cdots A_r$ is formally equal to $w_1(b_1, \ldots, b_k, \ c_1, \ldots, c_t)$ and $B_1 \cdots B_s$ is formally equal to $w_2(b_1, \ldots, b_k, \ c_1, \ldots, c_t)$. We can express $\{1, \ldots, t\}$ as a disjoint union of $T_1, \ldots, T_p$ such that for $\alpha, \beta \in \{1, \ldots, t\} c_\alpha \sim c_\beta$ if and only if α, β lie in same T_j. For $j = 1, \ldots, p$, let $M_j = \{(y_j, l(c_j)) \mid j \in T_j\}$. In the notation of [3], consider the constrained word equation $\mathscr{A} = \{w_1, w_2; M_1, \ldots, M_p\}$ in free variables $x_1, \ldots, x_k$ and constrained variables $y_1, \ldots, y_t$. Then $\mu = (b_1, \ldots, b_k, \ c_1, \ldots, c_t)$ is a solution of $\mathscr{A}$. By [3; Theorem 5.2], μ follows from a solution ν of $\mathscr{A}$ in some $\mathscr{F}_{\mathbb{R}}(\Gamma' \mid \Lambda')$. Moreover a close examination of the proof of [3; Lemma 3.13], shows that in fact we can choose Λ' such that $|\Lambda'| < k$ (this is because of the non-triviality of (1)). There exists $\phi : \Gamma' \to S$, $\phi(\Gamma' \backslash \Lambda') \subseteq \mathscr{L}$, such that the natural extension $\hat{\phi} : \mathscr{F}_{\mathbb{R}}(\Gamma' \mid \Lambda') \to S$ has the property that if $\nu = (u_1, \ldots, u_k, \ v_1, \ldots, v_t)$, then $\hat{\phi}(u_i) = b_i$, $\hat{\phi}(v_j) = c_j$. Let $\phi(\Lambda') = \{d_1, \ldots, d_\theta\}$. So $\theta < k$. Also $b_1, \ldots, b_k \in \hat{\phi}(\mathscr{F}_{\mathbb{R}}(\Gamma' \mid \Lambda')) = \langle \phi(\Lambda') \cup \overline{\phi(\Gamma' \backslash \Lambda')} \rangle$. So $b_1, \ldots, b_k \in \langle d_1, \ldots, d_\theta, \mathscr{L} \rangle$. Hence $a_1, \ldots, a_m \in \langle d_1, \ldots, d_\theta, \mathscr{L} \rangle$ contradicting the minimality of k. This contradiction shows that $T \cong \mathscr{F}_{\mathbb{R}}(\Gamma \mid \Lambda)$. We are done by Theorem 3.

Remark. Let $a_1, \ldots, a_m \in S$. Then by the above theorem and the remark preceding Theorem 1, there is a unique minimal free-like subsemigroup of S containing $a_1, \ldots, a_m$.

$\mathscr{F}_{\mathbb{R}}(\Gamma)$ is clearly embeddable in a group (in fact in the free product of $|\Gamma|$ copies of reals under addition). So by [1, Theorem 12.6], we have,

Theorem 5. *S is embeddable in a group.*

Conjecture. Let T be a subsemigroup of S. Then T can be embedded in $\mathscr{F}_{\mathbb{R}}(\Gamma)$ for some Γ if and only if T satisfies the descending chain condition.

Example. We give an example of a subsemigroup T of S such that T is embeddable in a free semigroup but T is not contained in a free-like subsemigroup of S. We can choose sequences $a_1, a_2, \ldots, b_1, b_2, \ldots, \ c_1, c_2, \ldots,$ in $S \backslash \mathscr{L}$ such that the following properties are true: (1)$a_{i+1} b_i = a_i$, $i = 1, 2, \ldots,$ (2) $l(c_i) \geq 3l(a_i)$, $i = 1, 2, \ldots,$ (3) no segment of b_i is a segment of b_j for $i \neq j$, (4) no segment of c_i is a segment of c_j for $i \neq j$, and (5) b_i is not an initial segment of a_j for any i, j. Let T be the subsemigroup of S generated by c_i, $c_i a_i$, $a_i c_i$, $c_{i+1} b_i$,

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b_ic_{i+1}, $i=1, 2, \ldots$. We claim that T is not contained in any free-like subsemigroup of S. For suppose $T \subseteq R \subseteq S$ and R is free-like. Then clearly $a_i, b_i \in T$ for all i. So

$$\cdots \underset{T,\mathcal{L}}{<} a_3 \underset{T,\mathcal{L}}{<} a_2 \underset{T,\mathcal{L}}{<} a_1,$$

a contradiction. On the other hand T can be embedded in a free semigroup. To see this, let $\mathcal{F}$ be the free semigroup on the letters $A_1, A_2, \ldots, B_1, B_2, \ldots, C_1, C_2, \ldots$. Let K be the subsemigroup of $\mathcal{F}$ generated by C_i, C_iA_i, A_iC_I, $C_{i+1}B_i$, B_iC_{i+1}, $i=1,2,\ldots$. Then it can be shown that $T \cong K$ with c_i, c_ia_i, a_ic_i, $c_{i+1}b_i$, b_ic_{i+1}, corresponding to C_i, C_iA_i, A_iC_i, $C_{i+1}B_i$, B_iC_{i+1}, respectively.

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