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ON THE COMPLEX OSCILLATION FOR A CLASS OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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Abstract Using a combined dominant condition, we obtain general results concerning the complex oscillation for a class of homogeneous linear differential equations $w^{(k)} + A_{k-2}w^{(k-2)} + \cdots + A_1w' + (A_0 + A)w = 0$ with $k \geqslant 2$, which has been investigated by many authors. In particular, we discover that there exists a unique case that possesses k linearly independent zero-free solutions for these equations, and we resolve an open problem and simultaneously answer a question of Bank.

Keywords: differential equation; entire coefficient; complex oscillation

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1. Introduction

For convenience in our statement, we first explain the notations used in this paper. We denote the order of growth of a function $g(z)$ meromorphic in the plane by $\sigma(g)$, the exponent of convergence of the zero sequence of $g(z)$ by $\lambda(g)$, and the exponent of convergence of the sequence of distinct zeros of $g(z)$ by $\bar{\lambda}(g)$. Other notations of function theory are standard (see, for example, [**11**, **13**]). In addition, following Hayman, the abbreviation 'n.e.' means 'everywhere except in a set of r with finite linear measure'.

In [**3**], Bank and Laine proved the following theorem.

Theorem 1.1. *Suppose that $A(z)$ is entire and satisfies $\lambda(A) < \sigma(A)$. Then, for any solution $f \not\equiv 0$ of the equation*

$$w'' + Aw = 0,$$

we have $\lambda(f) \geqslant \sigma(A)$.

Afterwards, they improved the above result in [**4**]. If we exchange the condition $\lambda(A) < \sigma(A)$ for $\bar{\lambda}(A) < \sigma(A)$, then the conclusion in Theorem 1.1 still holds. They resorted to Hayman's inequality (see [**11**, p. 60]) in their proof.

For the equations with order $k > 2$, Bank, Frank and Laine proved the following theorem in [**2**].

Theorem 1.2. *Let $k \geqslant 2$, $A(z)$ be entire, and satisfy $\overline{N}(r,1/A) = S(r,A)$. Then, for any solution $f \not\equiv 0$ of the equation*

$$w^{(k)} + Aw = 0, \tag{1.1}$$

we have

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{N(r,1/f)}{T(r,A)} > 0,$$

where E is a set of r with finite linear measure.

We can easily obtain the following theorem from Theorem 1.2.

Theorem 1.3. *Let $k > 2$, $A(z)$ be entire, and satisfy $\lambda(A) < \sigma(A)$. Then, for any solution $f \not\equiv 0$ of equation (1.1), we have $\lambda(f) \geqslant \sigma(A)$.*

If $A(z)$ only satisfies $\bar{\lambda}(A) < \sigma(A)$, does Theorem 1.3 still hold? This has remained open.

In this paper, using a combined dominant condition, we obtain general results concerning the complex oscillation for a class of homogeneous linear differential equations of the form

$$w^{(k)} + A_{k-2}w^{(k-2)} + \cdots + A_1w' + (A_0 + A)w = 0, \tag{1.2}$$

with order $k \geqslant 2$, which has been investigated by many authors, and, as a corollary, we resolve the above open problem under the broader conditions. Let $k \geqslant 2$, $A(z)$ be entire, and satisfy $\bar{\lambda}(A) < \sigma(A)$ or $\overline{N}(r,1/A) = S(r,A)$. Then, for any solution $f \not\equiv 0$ of equation (1.1), we have $\bar{\lambda}(f) \geqslant \sigma(A)$ and

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{\overline{N}(r,1/f)}{T(r,A)} > 0,$$

where E is a set of r with finite linear measure.

What is the combined dominant condition mentioned above?

To characterize the dominance of a meromorphic function $A(z)$ in the plane with respect to a non-negative and increasing function $R(r)$ on $(0,\infty)$, two conditions have been used frequently. Either $R(r) = S(r,A) = o\{T(r,A)\}$ n.e. as $r \to \infty$, or $\sigma_R < \sigma(A)$, σ_R standing for the order of growth of $R(r)$ (see [**11**, p. 16]). However, they are not equivalent in general. To unify these practices, we apply the following combined dominant condition. There exists a constant $d < \sigma(A)$ such that $R(r) = S(r,A) + o(r^d)$ as $r \to \infty$. For using this dominance, we require the following fact, which is easily checked. If at least one of $R(r) = S(r,A)$ and $R(r) = o(r^d)$ as $r \to \infty$ holds, then $R(r) = S(r,A) + o(r^d)$ as $r \to \infty$ must hold; equivalently, if $R(r) \neq S(r,A) + o(r^d)$ as $r \to \infty$, then $R(r) \neq S(r,A)$ and $R(r) \neq o(r^d)$ as $r \to \infty$ must simultaneously hold.

Moreover, it is specially worth mentioning the following discovery as one of our results. For equation (1.2), there exists a unique case that possesses k linearly independent zero-free solutions for each $A \neq 0$.

To understand the meaning of this result, we need to go back to the works of some other authors.

It was shown in [**3**, p. 356] that for any non-constant polynomial $P(z)$ there is a polynomial $Q(z)$ such that the equation

$$w'' + (Q + \mathrm{e}^P)w = 0$$

possesses two linearly independent zero-free solutions. Bank and Langley [**8**] and Bank [**1**] again showed that this had led to investigations of the more general equation

$$w^{(k)} + (Q + R\mathrm{e}^P)w = 0, \tag{1.3}$$

where $k \geqslant 2$, $R \not\equiv 0$ is also a polynomial (see, for example, [**1**–**10**,**12**]). However, as Bank said (see [**1**, p. 166]):

> To the author's knowledge, no examples have been found of an equation (1.3) of order $k > 2$ which possesses a solution $f \not\equiv 0$ for which $\lambda(f) < \infty$.

However, if we consider the more general equations obtained by allowing middle terms in (1.3), the situation is far different. In fact, Bank gave a general example of a third-order equation in [**1**, § 9]: for any non-constant polynomial $P(z)$, there are polynomials $Q_0(z)$ and $Q_1(z)$ such that the equation

$$w''' + Q_1 w' + (Q_0 + \mathrm{e}^P)w = 0$$

possesses three linearly independent zero-free solutions.

It is a problem whether such an example exists for this class of equations with any order $k > 3$. In the present paper we reveal its law, and thus answer this question. We prove that for any non-constant zero-free entire function A, there is a unique equation (1.2) with $k \geqslant 2$ that possesses k linearly independent zero-free solutions under the condition that A is dominant. We also simultaneously give the method such that this equation can be more easily deduced for given k and $A \neq 0$, and examples in [**3**, p. 356] and [**1**, § 9] are only the special cases of this result. In addition, from this fact and other results of equation (1.2), we can see that there exist no examples of equation (1.3) of order $k > 2$ that possess a solution $f \not\equiv 0$ with $\lambda(f) < \sigma(\mathrm{e}^P)$. Thus, we also answer, essentially, the above question of Bank.

2. Main results

Theorem 2.1. *Let $k \geqslant 2$, $A, A_0, \dots, A_{k-2}$ be entire functions with A non-constant, and suppose that there exists a constant $d < \sigma(A)$ such that $T(r, A_j) = S(r, A) + o(r^d)$, $j = 0, \dots, k-2, \overline{N}(r, 1/A) = S(r, A) + o(r^d)$ as $r \to \infty$. The following assertions hold.*

(i) *If A has at least one zero, then, for any solution $f \not\equiv 0$ of the equation*

$$w^{(k)} + A_{k-2}w^{(k-2)} + \cdots + A_1 w' + (A_0 + A)w = 0, \tag{2.1}$$

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we have $\bar{\lambda}(f) \geqslant \sigma(A)$ *and*

$$\overline{\lim_{\substack{r\to\infty\\ r\notin E}}} \frac{\overline{N}(r,1/f)}{T(r,A)} > 0,$$

where E is a set of r with finite linear measure.

(ii) *If A has no zeros and equation (2.1) possesses a solution $f \not\equiv 0$ such that $\overline{N}(r,1/f) = S(r,A) + o(r_1^d)$ as $r \to \infty$ for a constant $d_1 < \sigma(A)$, then (2.1) must be of the form*

$$w^{(k)} + D_{k-2}(h')w^{(k-2)} + \cdots + D_1(h')w' + (D_0(h') - e^{kh})w = 0, \tag{2.2}$$

which possesses k linearly independent zero-free solutions

$$f_j = \exp\left\{\omega_j \int_0^z e^{h(t)}\,dt - \tfrac{1}{2}(k-1)h(z) + c\right\}, \tag{2.3}$$

where $e^{kh} = -A$, $D_l(h')$ are differential polynomials in h' with constant coefficients, $l = 0, \ldots, k-2$, in particular,

$$D_{k-2} = -\tfrac{1}{24}[k(k+1)(k-1)](h'^2 - 2h''),$$

$\omega_j^k = 1$ (i.e. ω_j are the kth roots of unity), c is constant. For all solutions $f \not\equiv 0$ of (2.2) that are not constant multiples of the solutions f_j, and for all solutions $f \not\equiv 0$ in case (2.1) not of the type (2.2), the conclusions in (i) remain valid.

Corollary 2.2. *With the hypotheses of Theorem 2.1 for k, $A, A_0, \ldots, A_{k-2}$, if $\sigma(A_{k-2}) \neq \sigma(A'/A)$, or $A_{k-2} \equiv 0$, or A_{k-2} and A'/A are not both polynomials, then, for any solution $f \not\equiv 0$ of equation (2.1), we have the conclusions of Theorem 2.1 (i).*

Corollary 2.3. *With the hypotheses of Theorem 2.1 for A and A_0, but $k > 2$, for any solution $f \not\equiv 0$ of the equation*

$$w^{(k)} + (A_0 + A)w = 0,$$

we have the conclusions of Theorem 2.1 (i). These conclusions still hold if $k = 2$ and $A_0 \equiv 0$.

Remark 2.4. Corollary 2.3 generalizes the result for $k = 2$ in [4], and Corollary 2.3 resolves the open problem for $k \geqslant 3$ mentioned in § 1 under broader conditions. Moreover, Corollary 2.3 also essentially answers the question of Bank mentioned in § 1.

Remark 2.5. For any solution $f \not\equiv 0$ of the equation

$$w^{(k)} + \sum_{k=0}^{k-1} B_j w^{(j)} = 0,$$

with $B_j (j = 0, \ldots, k-1)$ entire and $k \geqslant 2$, it is not difficult to check that $N(r,1/f) \leqslant$ (k-1)$\overline{N}(r,1/f)$. Thus, $\bar{\lambda}(f) = \lambda(f)$, and the estimate of $\overline{N}(r,1/f)/T(r,A)$ in the conclusions of Theorem 2.1 (i) is, in fact, equivalent to the estimate of $N(r,1/f)/T(r,A)$.

3. Lemmas

From Lemma 3.3 (Clunie) and its proof in [**11**], it is easy to get the following lemma.

Lemma 3.1. *Let $f(z)$ be a function meromorphic in the plane and satisfying $f^n P(f) = Q(f)$, where $P(f)$ and $Q(f)$ are differential polynomials in f with coefficients b_j meromorphic in the plane, and the degree of $Q(f)$ is at most n. Then*

$$m(r, P(f)) = O\Big\{\sum_j m(r, b_j) + S(r, f)\Big\}, \quad \text{as } r \to \infty,$$

where $S(r, f) = O\{\log(rT(r, f))\}$ n.e. as $r \to \infty$.

From Theorem 3.9 and its proof in [**11**], it is not difficult to check that the following lemma holds.

Lemma 3.2. *Let $\xi(z)$ be a function meromorphic and non-constant in the plane, and $g(z) = \xi(z)^n + P_{n-1}(\xi)$, providing that there exists a constant $d < \sigma(\xi)$ such that $N(r, \xi) + \overline{N}(r, 1/g) = S(r, \xi) + o(r^d)$ as $r \to \infty$, where $P_{n-1}(\xi)$ is a differential polynomial in ξ with degree at most $n-1$, its coefficients are b_j meromorphic in the plane and satisfying $T(r, b_j) = S(r, \xi) + o(r^d)$ as $r \to \infty$. Then $g(z) = \eta(z)^n$, where $\eta(z) = \xi(z) + a(z)$, $a(z)$ is a function meromorphic in the plane and satisfying $T(r, a) = S(r, \xi) + o(r^d)$ as $r \to \infty$, and $na(z)\eta(z)^{n-1}$ can be obtained by the following method: it is equal to the part with degree $n-1$ in $P_{n-1}(\xi)$, but needing to substitute η for ξ, η' for ξ', etc., in this part.*

The following lemma is a generalization of Theorem 3.10 in [**11**].

Lemma 3.3. *Suppose that $f(z)$ is a function meromorphic and non-constant in the plane, and that $k \geqslant 2$, $A_0, \dots, A_{k-2}$ are entire functions, providing that there exists a constant $d < \sigma(f'/f)$ such that $T(r, A_j) = S(r, f'/f) + o(r^d)$ as $r \to \infty$, $j = 0, \dots, k-2$, and*

$$\overline{N}(r, f) + \overline{N}(r, 1/f) + \overline{N}(r, 1/L_k(f)) = S(r, f'/f) + o(r^d), \quad \text{as } r \to \infty, \tag{3.1}$$

where

$$L_k(f) = f^{(k)} + A_{k-2} f^{(k-2)} + \dots + A_0 f. \tag{3.2}$$

Then

(i) *if f'/f is constant, then $f = \mathrm{e}^{az+b}$ with a, b constants;*

(ii) *if f'/f is not constant, then we must have*

$$f = \exp\Big\{\int_0^z \mathrm{e}^{h(t)+c_1}\,\mathrm{d}t - \tfrac{1}{2}(k-1)h(z) + c_2\Big\}, \tag{3.3}$$

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with c_1, c_2 constants, and

$$\left.\begin{aligned} B_{k-2} &= \tfrac{1}{24}[k(k+1)(k-1)](h'^2 - 2h'') + A_{k-2} \equiv 0,\\ B_{k-3} &= b_{k-3}(h', A_{k-2}) + A_{k-3} \equiv 0,\\ B_{k-4} &= b_{k-4}(h', A_{k-2}, A_{k-3}) + A_{k-4} \equiv 0,\\ &\vdots\\ B_0 &= b_0(h', A_{k-2}, \ldots, A_1) + A_0 \equiv 0, \end{aligned}\right\} \tag{3.4}$$

where b_{k-j} are polynomials in $A_{k-2}, \ldots, A_{k-j+1}$, h' and derivatives of h' with constant coefficients, and are linear in $A_{k-2}, \ldots, A_{k-j+1}$ ($j = 3, \ldots, k$), B_{k-j} ($j = 2, \ldots, k$) are determined by the following relation:

$$\left.\begin{aligned} L_k(f)/f &= \xi^k + P_{k-1}(\xi)\\ &= \eta^k + B_{k-2}\eta^{k-2} + \cdots + B_0 \equiv \eta^k,\\ \xi &= f'/f,\\ \eta &= \xi + \tfrac{1}{2}(k-1)h' = e^{h+c_1}, \end{aligned}\right\} \tag{3.5}$$

$P_{k-1}(\xi)$ is a differential polynomial in ξ with degree $k-1$, each coefficient of which is a constant multiple of one of $A_0, \ldots, A_{k-2}$.

In addition, h is entire, $\sigma(h') = \sigma(A_{k-2})$, h' is a polynomial if and only if A_{k-2} is a polynomial, h' is constant if and only if A_{k-2} is constant, and $h' \equiv 0$ if and only if $A_{k-2} \equiv 0$.

Proof. Set $\xi(z) = f'(z)/f(z)$.
(i) It is evident.
(ii) $\xi(z)$ is not constant, so, from lemma 3.5 in [**11**], we get

$$\begin{aligned} L_k(f)/f = \xi^k + \tfrac{1}{2}(k(k-1))\xi^{k-2}\xi' + (\alpha_k\xi^{k-3}\xi'' + \beta_k\xi^{k-4}\xi'^2 + A_{k-2}\xi^{k-2})\\ + (A_{k-2}\tfrac{1}{2}[(k-2)(k-3)]\xi^{k-4}\xi' + A_{k-3}\xi^{k-3}) + P_{k-3}(\xi) = g(z), \end{aligned} \tag{3.6}$$

where $P_{k-3}(\xi)$ is a differential polynomial in ξ with degree $k-3$, each coefficient of which is a constant multiple of one of $A_0, \ldots, A_{k-2}$. It follows from (3.1) that

$$\begin{aligned} \overline{N}(r, 1/g) = \overline{N}(r, f/L_k(f)) \leqslant \overline{N}(r, f) + \overline{N}(r, 1/L_k(f)) = S(r, \xi) + o(r^d), \quad &\text{as } r \to \infty,\\ N(r, \xi) = N(r, f'/f) = \overline{N}(r, f) + \overline{N}(r, 1/f) = S(r, \xi) + o(r^d), \quad &\text{as } r \to \infty. \end{aligned}$$

Hence, adding the hypotheses for $A_0, \ldots, A_{k-2}$, the conditions of Lemma 3.2 are satisfied. Therefore, (3.6) becomes $g(z) = \eta(z)^k$, where

$$\eta(z) = \xi(z) + a(z), \qquad ka(z)\eta(z)^{k-1} = \tfrac{1}{2}[k(k-1)]\eta(z)^{k-2}\eta'(z),$$

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i.e.

$$a(z) = \tfrac{1}{2}(k-1)\eta'(z)/\eta(z), \quad T(r,a) = S(r,\xi) + o(r^d), \quad \text{as } r \to \infty.$$

From

$$\begin{aligned}\eta' &= \frac{2a}{k-1}\eta,\\ \eta'' &= \frac{2}{k-1}(a\eta' + a'\eta) = \left\{\frac{4a^2}{(k-1)^2} + \frac{2a'}{k-1}\right\}\eta,\\ &\vdots\end{aligned}$$

we have

$$\begin{aligned}\xi &= \eta - a,\\ \xi' &= \eta' - a' = \frac{2a}{k-1}\eta - a',\\ \xi'' &= \eta'' - a'' = \left\{\frac{4a^2}{(k-1)^2} + \frac{2a'}{k-1}\right\}\eta - a'',\\ &\vdots\end{aligned}$$

Substituting them into (3.6) gives

$$(\eta-a)^k + \tfrac{1}{2}[k(k-1)](\eta-a)^{k-2}\left(\frac{2a}{k-1}\eta - a'\right) + \alpha_k(\eta-a)^{k-3}\left\{\left(\frac{4a^2}{(k-1)^2} + \frac{2a'}{k-1}\right)\eta - a''\right\}$$
$$+ \beta_k(\eta-a)^{k-4}\left(\frac{2a}{k-1}\eta - a'\right)^2 + A_{k-2}(\eta-a)^{k-2} + P_{k-3}(\eta-a) \equiv \eta^k. \quad (3.7)$$

Expanding the left-hand side of (3.7) and gathering the terms according to the degree of η, we get

$$B_{k-2}\eta^{k-2} + B_{k-3}\eta^{k-3} + \cdots + B_0 \equiv 0, \quad (3.8)$$

where

$$\left.\begin{aligned}B_{k-2} &= \tilde{b}_{k-2}(a) + A_{k-2},\\ B_{k-3} &= \tilde{b}_{k-3}(a, A_{k-2}) + A_{k-3},\\ &\vdots\\ B_0 &= \tilde{b}_0(a, A_{k-2}, \ldots, A_1) + A_0,\end{aligned}\right\} \quad (3.9)$$

where $\tilde{b}_{k-j}$ are polynomials in $A_{k-2}, \ldots, A_{k-j+1}$, a and derivatives of a with constant coefficients, and are linear in $A_{k-2}, \ldots, A_{k-j+1}$, $j = 2, \ldots, k$. It is easy to see that

$T(r, B_{k-j}) = S(r,\xi) + o(r^d)$ as $r \to \infty$. If $k = 2$, then $B_{k-2} = B_0 \equiv 0$ from (3.8). If $k > 2$, applying Lemma 3.1 to (3.8) gives

$$\begin{aligned} m(r, B_{k-2}\eta) &= O\left\{\sum_{j=0}^{k-3} m(r, B_j) + S(r,\eta)\right\} \\ &= S(r,\xi) + o(r^d), \quad \text{as } r \to \infty. \end{aligned}$$

Since

$$\begin{aligned} N(r, B_{k-2}\eta) &\leqslant N(r, B_{k-2}) + N(r,\eta) = N(r, B_{k-2}) + O\{N(r,g)\} \\ &= N(r, B_{k-2}) + O\{N(r,\xi)\} = S(r,\xi) + o(r^d), \quad \text{as } r \to \infty, \end{aligned}$$

we get

$$T(r, B_{k-2}\eta) = S(r,\xi) + o(r^d), \quad \text{as } r \to \infty.$$

If $B_{k-2} \not\equiv 0$, then

$$\begin{aligned} T(r,\eta) &= T(r, B_{k-2}\eta/B_{k-2}) \leqslant T(r, B_{k-2}\eta) + T(r, B_{k-2}) + O(1) \\ &= S(r,\xi) + o(r^d), \quad \text{as } r \to \infty. \end{aligned}$$

Thus

$$T(r,\xi) = T(r,\eta - a) = S(r,\xi) + o(r^d), \quad \text{as } r \to \infty.$$

This gives

$$T(r,\xi) = o(r^d) \text{ n.e.}, \quad \text{as } r \to \infty.$$

From fact (A) in [**3**, §2, p. 353], this implies that $\sigma(\xi) \leqslant d$, and the contradiction with assumption occurs. Hence, we must have $B_{k-2} \equiv 0$. We can successively prove $B_{k-3} \equiv \cdots \equiv B_0 \equiv 0$ using the same reasoning. Calculating (3.7) directly gives

$$B_{k-2} = \tfrac{1}{6}[k(k+1)]\left(\frac{a^2}{k-1} - a'\right) + A_{k-2}.$$

It can be changed into the form

$$B_{k-2} = \tfrac{1}{24}[k(k+1)(k-1)]\left\{\left(\frac{2a}{k-1}\right)^2 - 2\left(\frac{2a}{k-1}\right)'\right\} + A_{k-2}.$$

Setting $h' = 2a/(k-1)$, noting that $B_{k-2} \equiv 0$, gives

$$B_{k-2} = \tfrac{1}{24}[k(k+1)(k-1)](h'^2 - 2h'') + A_{k-2} \equiv 0. \tag{3.10}$$

Firstly, from (3.10) we can see that h' is entire, and so is h. Otherwise, assume z_0 is a pole of h'. It is easy to check from (3.10) that z_0 must be a simple pole of h', and the principal

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part of Laurent expansion of h' at z_0 is $-2/(z-z_0)$. From this, $\eta'/\eta = 2a/(k-1) = h'$ and $f'/f = \eta - \frac{1}{2}(k-1)h'$, we can obtain

$$f = \exp\left\{\frac{d_0}{z-z_0}\right\}(z-z_0)^{k-1}\mathrm{e}^{B(z)},$$

where $B(z)$ is holomorphic in the neighbourhood of z_0, d_0 is a non-zero constant. Thus, f has an essential singularity at z_0. This contradicts that f is meromorphic. Secondly, it follows from (3.10) that $\sigma(h') = \sigma(A_{k-2})$. In fact, on the one hand, we can easily get $\sigma(A_{k-2}) \leqslant \sigma(h')$ from (3.10) and fact (A) in [**3**, § 2, p. 353]. On the other hand, applying Lemma 3.1 to (3.10) gives $m(r,h') = O\{m(r,A_{k-2}) + S(r,h')\}$. Thus, $T(r,h') = O\{T(r,A_{k-2})\}$ n.e. as $r \to \infty$. We get $\sigma(h') \leqslant \sigma(A_{k-2})$ from this and fact (A) in [**3**, § 2, p. 353]. Thus, $\sigma(h') = \sigma(A_{k-2})$. Thirdly, if A_{k-2} is a polynomial, applying Lemma 3.1 to (3.10) gives $m(r,h') = O\{\log r + S(r,h')\}$ or $T(r,h') = O\{\log r\}$ n.e. as $r \to \infty$. Hence, h' is also a polynomial. Conversely, it is clear that if h' is a polynomial, then A_{k-2} is also a polynomial from (3.10). Fourthly, under the situation that both of h' and A_{k-2} are polynomials, calculating (3.10) directly gives: h' is constant if and only if A_{k-2} is constant, and $h' \equiv 0$ if and only if $A_{k-2} \equiv 0$.

From $\eta'/\eta = h'$, we get

$$\eta = \mathrm{e}^{h+c_1}.$$

This is the third formula in (3.5). And, from $f'/f = \eta - \frac{1}{2}(k-1)h'$, we get (3.3). Noting that $B_0 \equiv \cdots \equiv B_{k-3} \equiv 0$ and $a = \frac{1}{2}(k-1)h'$, we get (3.4) from (3.10) and (3.9). The proof is completed. □

Lemma 3.4. *Let $k \geqslant 2$ and $A, A_0, \ldots, A_{k-2}$ be entire functions with A non-constant. And suppose that there exits a constant $d < \sigma(A)$ such that $T(r,A_j) = S(r,A) + o(r^d)$, $j = 0, \ldots, k-2$, $\overline{N}(r,1/A) = S(r,A) + o(r^d)$ as $r \to \infty$. If the equation*

$$w^{(k)} + A_{k-2}w^{(k-2)} + \cdots + A_1 w' + (A_0 + A)w = 0 \tag{3.11}$$

possesses a solution $f \not\equiv 0$, providing that there exits a constant $d_1 < \sigma(A)$ such that $\overline{N}(r,1/f) = S(r,A) + o(r^{d_1})$ as $r \to \infty$, then we must have

$$f = \exp\left\{\omega\int_0^z \mathrm{e}^{h(t)}\,\mathrm{d}t - \tfrac{1}{2}(k-1)h(z) + c_2\right\}, \tag{3.12}$$

with $\omega^k = 1$ (i.e. ω is a kth root of unity), c_2 constant, and the coefficients $A_0, \ldots, A_{k-2}$ in (3.11) must satisfy (3.4), A must be zero-free and

$$A = -\mathrm{e}^{kh}, \qquad kh' = A'/A. \tag{3.13}$$

Moreover, equation (3.11) is determined uniquely by (3.13) and (3.4) (i.e. by A or h'), and it possesses k linearly independent zero-free solutions given by (3.12).

Proof. Clearly, we can assume that $d_1 > d$ without loss of generality. Substituting f for w in (3.11), we get

$$L_k(f) = -fA, \tag{3.14}$$

where $L_k(f)$ is defined in (3.2). Hence,

$$\begin{aligned}\overline{N}(r, 1/L_k(f)) &= \overline{N}(r, 1/(fA)) \\ &\leqslant \overline{N}(r, 1/f) + \overline{N}(r, 1/A) \\ &= S(r, A) + o(r^{d_1}), \quad \text{as } r \to \infty.\end{aligned} \tag{3.15}$$

In addition, from [**11**, Lemma 3.5], setting $\xi = f'/f$, we have

$$L_k(f)/f = \xi^k + P_{k-1}(\xi), \tag{3.16}$$

where $P_{k-1}(\xi)$ is a differential polynomial in ξ with degree $k-1$, each coefficient of which is a constant multiple of one of $A_0, \ldots, A_{k-2}$. Combining (3.14) and (3.16) gives

$$\xi^k + P_{k-1}(\xi) = -A. \tag{3.17}$$

It is easy to get, from (3.17),

$$\begin{aligned}T(r, A) &= O\left\{T(r, \xi) + \sum_{j=0}^{k-2} T(r, A_j)\right\} \\ &= O\{T(r, \xi) + S(r, A) + r^{d_1}\} \text{ n.e.}, \quad \text{as } r \to \infty,\end{aligned}$$

or

$$T(r, A) = O\{T(r, \xi) + r^{d_1}\} \text{ n.e.}, \quad \text{as } r \to \infty. \tag{3.18}$$

On the other hand, applying Lemma 3.1 to (3.17) gives

$$\begin{aligned}m(r, \xi) &= O\left\{\sum_{j=0}^{k-2} m(r, A_j) + m(r, A) + S(r, \xi)\right\} \\ &= O\{m(r, A) + r^{d_1} + S(r, \xi)\} \text{ n.e.}, \quad \text{as } r \to \infty.\end{aligned}$$

Since (note that f is entire)

$$N(r, \xi) = N(r, f'/f) = \overline{N}(r, 1/f) = S(r, A) + o(r^{d_1}) \text{ n.e.}, \quad \text{as } r \to \infty,$$

we get

$$T(r, \xi) = O\{m(r, A) + r^{d_1} + S(r, \xi)\} \text{ n.e.}, \quad \text{as } r \to \infty,$$

or

$$T(r,\xi)=O\{T(r,A)+r^{d_1}\} \text{ n.e.}, \quad \text{as } r\to\infty. \tag{3.19}$$

It is easy to see that $\sigma(\xi)=\sigma(A)$ from (3.18), (3.19) and fact (A) in [**3**, § 2, p. 353]. Hence, we get from (3.18), the assumptions of this lemma and (3.15), and, noting that f is entire,

$$T(r,A_j)=S(r,f'/f)+o(r^{d_1}), \quad j=0,\dots,k-2,$$
$$\overline{N}(r,f)+\overline{N}(r,1/f)+\overline{N}(r,1/L_k(f))=S(r,f'/f)+o(r^{d_1}) \text{ n.e.}, \quad \text{as } r\to\infty.$$

Therefore, the conditions of Lemma 3.3 are satisfied, and, noting that f'/f is not constant from (3.18), (3.3) and (3.4) hold. From (3.5) we get

$$L_k(f)/f=\mathrm{e}^{k(h+c_1)}. \tag{3.20}$$

If we set $\omega=\mathrm{e}^{c_1}$ is a kth root of unity, i.e. $\omega^k=1$, then we get (3.12) from (3.3), and (3.20) becomes

$$L_k(f)/f=\mathrm{e}^{kh}. \tag{3.21}$$

Thus, (3.12) provided k linearly independent zero-free solutions for the equation (3.21). Combining (3.14) and (3.21) gives

$$\mathrm{e}^{kh}=-A.$$

Hence (3.13) holds. The proof of Lemma 3.4 is completed. □

4. Proofs of main results

Proof of Theorem 2.1. (i) If A has at least one zero, then, from Lemma 3.4, for any solution $f\not\equiv 0$ of the equation (2.1), there must not exist any constant $d_1<\sigma(A)$ such that $\overline{N}(r,1/f)=S(r,A)+o(r^{d_1})$ as $r\to\infty$. Therefore, we must have $\overline{N}(r,1/f)\neq S(r,A)$, and there must not exist any constant $d_1<\sigma(A)$ such that $\overline{N}(r,1/f)=o(r^{d_1})$ as $r\to\infty$. From these, we obtain, respectively,

$$\overline{\lim_{\substack{r\to\infty\\ r\notin E}}}\frac{\overline{N}(r,1/f)}{T(r,A)}>0 \quad \text{and} \quad \bar{\lambda}(f)\geqslant\sigma(A),$$

where E is a set of r with finite linear measure.

(ii) If A has no zeros, then, from Lemma 3.4, for any solution $f\not\equiv 0$ of every equation (2.1), except the equation (2.2), there must not also exist any constant $d_1<\sigma(A)$ such that $\overline{N}(r,1/f)=S(r,A)+o(r^{d_1})$ as $r\to\infty$. The remainder of reasoning is the same as in (i). For any solution $f\not\equiv 0$ of the exceptional equation (2.2) that is not a constant multiple of some f_j, the proof is the same. □

Proof of Corollary 2.2. From (3.13), for the exceptional equation (2.2), we must have that $A \neq 0$, $\sigma(h') = \sigma(A'/A)$, $h' \equiv 0$ if and only if $A' \equiv 0$, h' is a polynomial if and only if A'/A is a polynomial. Thus, from Lemma 3.3, the equations that satisfy the conditions of Corollary 2.2 cannot be the exceptional equation (2.2). And, therefore, Corollary 2.2 holds from Theorem 2.1. □

Proof of Corollary 2.3. Since $A_{k-2} \equiv 0$, Corollary 2.3 holds from Corollary 2.2. □

Remark 4.1. From (3.7) (combining (3.4)), we can more easily give the exceptional equation (2.2) determined uniquely by A (or h') with any order k. See the following examples.

(1) For $k = 2$, (2.2) is just the equation

$$w'' - \{\tfrac{1}{4}(h'^2 - 2h'') + \mathrm{e}^{2h}\}w = 0,$$

which possesses two linearly independent zero-free solutions (2.3), i.e.

$$f_{1,2} = \exp\left\{\pm \int_0^z \mathrm{e}^{h(t)}\,\mathrm{d}t - \tfrac{1}{2}h(z) + c\right\}.$$

Setting $h = \varphi - \log 2$, $c = -\frac{1}{2}\log 2$, we get the example in [**3**, p. 356].

(2) For $k = 3$, (2.2) is just the equation

$$w''' - (h'^2 - 2h'')w' + (h''' - h'h'' - \mathrm{e}^{3h})w = 0,$$

which possesses three linearly independent zero-free solutions (2.3), i.e.

$$f_j = \exp\left\{\omega_j \int_0^z \mathrm{e}^{h(t)}\,\mathrm{d}t - h(z) + c\right\},$$

where $\omega_j^3 = 1$, $j = 1, 2, 3$. Setting $h = \frac{1}{3}P + \pi\mathrm{i}$, $\omega_j = -K_j$, $c = 0$, we get the example in [**1**, § 9].

(3) For $k = 4$, (2.2) is just the equation

$$w^{(4)} - \tfrac{5}{2}(h'^2 - 2h'')w'' + 5(h''' - h'h'')w' + (\tfrac{9}{16}h'^4 - \tfrac{9}{4}h'^2h'' - \tfrac{3}{2}h'h''' + \tfrac{3}{4}h''^2 + \tfrac{3}{2}h^{(4)} - \mathrm{e}^{4h})w = 0,$$

which possesses four linearly independent zero-free solutions (2.3), i.e.

$$f_j = \exp\left\{\omega_j \int_0^z \mathrm{e}^{h(t)}\,\mathrm{d}t - \tfrac{3}{2}h(z) + c\right\},$$

where $\omega_j^4 = 1$, $j = 1, \dots, 4$.

5. A problem

We can see that, from Theorem 2.1, the problems of complex oscillation for equation (1.2), under the assumptions in this theorem and that of $A(z)$ being of infinite order of growth, are completely resolved. Now, the remaining problem is whether there exists an equation of the form of (1.2) satisfying the conditions in Theorem 2.1 with $A(z)$ finite order of growth which possesses a solution $f \not\equiv 0$ with $\sigma(A) \leqslant \lambda(f) < \infty$. Concerning this problem, Bank, Langley and others have performed several important works to find the conditions such that any solution $f \not\equiv 0$ of the equation

$$w^{(k)} + Q_{k-2}w^{(k-2)} + \cdots + Q_1 w' + (Q_0 + R\mathrm{e}^P)w = 0$$

satisfies $\lambda(f) = \infty$, where $k \geqslant 2$, $Q_0, \dots, Q_{k-2}$, R and P are polynomials with $R \not\equiv 0$ and P non-constant (see, for example, [**1**, **7**, **8**, **12**]).

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