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ON THE SPECTRUM OF THE BERGMAN-HILBERT MATRIX II

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ABSTRACT. We study a class of matrices (introduced by T. Kato) with principal homogeneous part, and use Mellin transform of the homogeneous kernel to determine spectral density of the positive infinite matrices.

1. **Introduction.** In the course of lifting Hankel operators on the Hardy space of the circle to Hankel operators on the Bergman space of the disk via the Schur multiplier

$$M = \left[\frac{\sqrt{(i+1)(j+1)}}{i+j+1}\right]_{i,j \geqq 0}$$

we studied the Bergman-Hilbert matrix A and its homogeneous companion B. We recall that

$$A = \left[\frac{\sqrt{(i+1)(j+1)}}{(i+j+1)^2}\right]_{i,j \geqq 0} \quad \text{and} \quad B = \left[\frac{\sqrt{(i+1)(j+1)}}{(i+j+2)^2}\right]_{i,j \geqq 0}.$$

In [2] it was shown that $A - B$ is compact and $1 = \|B\|_e = \|A\|_e < \|A\|$, and in particular A has eigenvalues, thus distinguishing its spectral properties from those of the Hilbert matrix [1]. In fact, the relationship between A and B turns out to be a particular case of the general form of matrices with principal homogeneous part studied by T. Kato [4]. For this and other reasons which we hope will be clear in this note, B turns out to be an interesting matrix in its own right. What makes B more amenable than A is that its entries are values of a homogeneous kernel evaluated at lattice points in the plane and the same homogeneous kernel induces a rather well-behaved integral operator.

2. Consider the integral operator K defined on $\mathcal{L}^2(0,\infty)$ which is induced by the kernel

$$k(x,y) = \frac{\sqrt{xy}}{(x+y)^2}.$$

Note that $k(i+1,\, j+1) = b_{ij}, i, j \geqq 0$. We first write down the spectrum of K using the standard technique of Mellin transforms to express K as a multiplication operator

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on $\mathcal{L}^2(\mathbf{R})$. We are grateful to P. G. Rooney for bringing this to our notice. if $\mathcal{M}$ denotes Mellin transform and $f \in \mathcal{L}^2(0, \infty)$, then we have

$$\mathcal{M}_{Kf}(s) = m(s)\mathcal{M}_f(s) \quad \text{where} \quad \mathcal{M}_f(s) = \int_0^\infty x^{s-1} f(x)dx,$$

$$\begin{aligned}\mathcal{M}_{Kf}(s) &= \int_0^\infty x^{s-1}(Kf)(x)dx \\ &= \int_0^\infty \int_0^\infty x^{s-1} \frac{\sqrt{xy}}{(x+y)^2} f(y)dydx \\ &= \int_0^\infty \frac{x^{s-\frac{1}{2}}}{(1+x)^2}\, dx \int_0^\infty y^{s-1} f(y)dy.\end{aligned}$$

Hence

$$m(s) = \int_0^\infty \frac{x^{s-\frac{1}{2}}}{(x+1)^2}\, dx = \left(\frac{1}{2} - s\right) \pi \csc \pi \left(s - \frac{1}{2}\right).$$

Hence $\sigma(K)$ = closure of range $\{m(\frac{1}{2} + it), t \in \mathbf{R}\} = \overline{\text{Range}}\{t \operatorname{csch} t,\ t \in (0, \infty)\} = [0, 1]$. However, B is not unitarily equivalent to K, it is unitarily equivalent to an integral operator whose kernel is not easily expressible in closed form [see 5]. Hence we must rely on getting whatever information we can on the spectral density of B through eigenvalues of finite sections of it.

The finite section of B is $B_{n,m} = [b_{ij}], m < i, j \leqq nm$; we compare it with $(K_{n,m}f)(x) = \int_m^{nm} k(x, y)f(y)dy$. Homogeneity of k implies that $K_{n,m}$ is in fact independent of m. For any $(a,\ b) \subseteq [0, 1]$, $M_n(a,\ b)$ denotes the number of eigenvalues of $K_{n,m}$ in $(a,\ b)$, and $X_{n,m}(a,\ b)$ the number of eigenvalues of $B_{n,m}$ in $(a,\ b)$. All we need for Proposition 2 below is that $X_{n,m}(a, b)$ can be arbitrarily large. We will show essentially that $X_{n,n}(a, b)$ behaves asymptotically like $(\log n)(F^{-1}(a) - F^{-1}(b))$, where $F(x) = x \operatorname{csch} x$; the precise result is a little weaker.

We rely on [6, Section 2.6]. We need a little more work as $k(x, y) = \sqrt{xy}/(x + y)^2$ is not a decreasing function in either variable.

LEMMA 1. *If* $(x, y) \in (i - 1,\ i] \times (j - 1,\ j]$ *with* $m < i, j \leqq nm$, *then*

$$\left|\frac{\partial k}{\partial x}\right| \leqq \frac{c}{m^2}, \quad \left|\frac{\partial k}{\partial y}\right| \leqq \frac{c}{m^2},$$

for a constant c.

PROOF.

$$\frac{\partial k}{\partial x} = \sqrt{\frac{y}{x}}\, \frac{(-3x + y)}{2(x + y)^3}.$$

On each segment $x + y = s,\ x \in [m,\ s - m]$, we will bound

$$\left(\frac{\partial k}{\partial x}\right)^2 = \frac{1}{4}\, \frac{(s - x)(s - 4x)^2}{xs^6} \leqq \frac{3}{4}\, \frac{|g(x)|}{s^4},$$

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where

$$g(x) = \frac{(s-x)(s-4x)}{x}.$$

Now note that on the segment $g(x)$ has a minimum at $x = s/2$ where $g(s/2) < 0$, also $g(s-m) < 0$ since $s \geqq 2m$; this means that max $|g(x)|$ is attained either at $x = m$ or at $x = s/2$. But $g/(m) < s^2/m$ (again because $s \geqq 2m$), giving

$$\left(\frac{\partial k}{\partial x}\right)^2 \leqq \frac{3}{4}\,\frac{1}{ms^3} \leqq \text{const.}/m^4;$$

and $|g(s/2)| = s$, giving

$$\left(\frac{\partial k}{\partial x}\right)^2 \leqq (2m)^{-4}.$$

Hence

$$\left|\frac{\partial k}{\partial x}\right| \leqq \frac{c}{m^2} \text{ for } x \geqq m, y \geqq m,$$

as desired. □

We write

$$\frac{F^{-1}(a) - F^{-1}(b)}{\pi^2} = \Phi(a,b)$$

with F as above.

PROPOSITION 1. *Given $(a,b) \subseteq (0,1)$ and $\epsilon > 0$, there exists n_0 such that for all $n \geqq n_0$ and $m = n^2$ we have*

$$\left|\frac{X_{n,m}(a,b)}{\log m} - \Phi(a,b)\right| < \epsilon.$$

PROOF. In order to connect $K_{n,m}$ to $B_{n,m}$ we define an isometry $\mathcal{U}_{n,m} : \mathbf{C}^{(n-1)m} \longrightarrow \mathcal{L}^2(m,\ nm)$ by

$$\mathcal{U}_{n,m}[x_i]_{i=1}^{(n-1)m} = \sum_i x_i \chi_{[m+i-1,\ m+i)}.$$

Note that $\mathcal{U}_{n,m} B_{n,m} \mathcal{U}_{n,m}^{-1}$ is an integral operator whose kernel is constant $= k(m+i-1,\ m+j-1)$ on each square $\{(x,y) : m+i-1 \leqq x < m+1,\ m+j-1 \leqq y < m+j\}$, and hence $T_{n,m} = K_{n,m} - \mathcal{U}_{n,m} B_{n,m} \mathcal{U}_{n,m}^{-1}$ is an integral operator whose kernel (being zero at one corner of each such square) is bounded by c/m^2 (see Lemma 1). Next we estimate the norm of $T_{n,m}$. As an operator on $\mathcal{L}^2(m,\ nm)$ with bounded kernel, it satisfies $\|T_{n,m}\| \leqq c(nm-m)/m^2 < cn/m$. It is enough to consider the special values for which $m = n^2$; then $\|T_{n,m}\| < c/n$.

Now given $\epsilon > 0$ first choose δ so that by changing $s,\ t$ by less than $\delta, \Psi(s,t)$ changes by less than $\epsilon/3$. Next we choose n_0 so that for $n \geqq n_0$, $\|K_{n,m} - \mathcal{U}_{n,m} B_{n,m} \mathcal{U}_{n,m}^{-1}\| < \delta$. Now Weyl's theorem says that two compact self-adjoint

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operators differing by less than δ in the sense of operator-norm must have corresponding eigenvalues differing by no more than δ. In particular, $M_m(a+\delta,\ b-\delta) \leqq X_{n,m}(a,b) \leqq M_m(a-\delta,\ b+\delta)$. By [6, 2.6(b)], using our calculation of $m(s)$ above, we may choose m large enough so that

$$\left|\frac{M_m(a+\delta, b-\delta)}{\log\ m} - \Phi(a+\delta,\ b-\delta)\right| < \epsilon/3$$

and

$$\left|\frac{M_m(a-\delta, b+\delta)}{\log\ m} - \Phi(a-\delta,\ b+\delta)\right| < \epsilon/3.$$

These inequalities give the conclusion of the Proposition directly: $X_{n,m}(a,b) \geqq M_m(a+\delta,\ b-\delta) > \log\ m(\Phi(a+\delta,\ b-\delta) - \epsilon/3) > \log m(\Phi(a,b) - \epsilon)$ and $X_{n,m}(a,b) \leqq M_n(a-\delta,\ b+\delta) < \log\ m(\Phi(a-\delta,\ b+\delta) + \epsilon/3) \leqq \log\ m(\Phi(a,b)+\epsilon)$. □

COROLLARY. *The inequalities in the previous proposition also hold for eigenvalues of finite sections of A.*

PROOF. We know that $A - B$ is Hilbert-Schmidt [2] and hence $\|A_{n,m} - B_{n,m}\|^2 < \sum_{i,j>m}^{nm} |a_{ij} - b_{ij}|^2$ with $\sum_{i,j\geqq 0} |a_{ij} - b_{ij}|^2 < \infty$. Hence, a simple application of Weyl's theorem gives the desired conclusion. □

REMARK. The same argument goes through for the class of matrices [4, Section 2] of the form $A = B - C$ where B is the principal homogeneous part of A and C is Hilbert-Schmidt. If $b_{ij} = K(i+\theta,\ j+\theta), \theta > 0$, and $K(t,\ 1)$ has Mellin transform on the critical line (Re $s = \frac{1}{2}$) which is one-to-one, then the finite sections of A and B have the spectral density described above.

PROPOSITION 2. $\sigma(B) = \sigma_e(B) = \sigma_e(A) = [0,\ 1]$.

PROOF. Suppose $\lambda \notin \sigma_e(B)$. As B is self-adjoint, λ is at most an isolated eigenvalue of finite multiplicity, and hence there exists $\delta > 0$ and a subspace $\mathcal{N}_1$ of finite codimension l (l being zero in case $\lambda \neq \sigma(B)$) such that $|((B-\lambda)x, x))| \geqq \delta\|x\|^2$ whenever $x \in \mathcal{N}_1$. But also if $0 < \epsilon$, by Proposition 1, there exists a projection P such that PBP has at least $l+1$ eigenvalues in $(\lambda - \epsilon,\ \lambda + \epsilon)$. Now we may choose pairwise orthogonal unit vectors x_i, $1 \leqq i \leqq l+1$ such that $Px_i = x_i$ and $PBPx_i = \lambda_i x_i$. If $\mathcal{N}_2 = \{\sum_{i=1}^{l+1} k_i x_i,\ k_i \in \mathbf{C}\}$ then dim $\mathcal{N}_2 = l+1$, while co-dim $\mathcal{N}_1 = l$. Hence projection on the orthocomplement of $\mathcal{N}_1$ when restricted to $\mathcal{N}_2$ has a nontrivial kernel. Now if x is a non-zero vector in $\mathcal{N}_2 \cap \mathcal{N}_1$ we have $x = \sum_{i=1}^{l+1} k_i x_i$ with $\|x_i\| = 1$, $\|x\|^2 = \sum_{i=1}^{l+1} |k_i|^2$ and $((B-\lambda)x, x) = \sum_{i=1}^{l+1} |k_i|^2(\lambda_i - \lambda)$. Thus

$$|((B-\lambda)x,\ x)| \leqq \sum_{i=1}^{l+1} |k_i|^2 |\lambda_i - \lambda| < \epsilon\|x\|^2.$$

Taking $\epsilon = \delta$ gives a contradiction since $x \in \mathcal{N}_1$. The compactness of $A - B$ [2] shows that $\sigma_e(A) \supseteq [0,\ 1]$. B is a positive self-adjoint operator of norm 1 [2] and hence $\sigma(B) \subseteq [0,\ 1]$. Since $\sigma_e(B) \subseteq \sigma(B)$, this completes the proof. □

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REMARK. As shown in Proposition 2, B cannot have isolated eigenvalues of finite multiplicity in $[0, 1]$. We conjecture that neither A nor B has an eigenvalue in $[0, 1]$.

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