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ON RINGS WHOSE MODULES OF FINITE LENGTH ARE ALL CYCLIC

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We give some characterisations of rings R whose modules with composition series are all cyclic. In particular, we prove that all left R-modules of finite length are cyclic if and only if R has no nonzero Artinian factor rings.

1. INTRODUCTION

In D-module theory, it is useful that every holonomic module over a Weyl algebra is cyclic and every holonomic module over the sheaf D_X of holomorphic differential operators on a complex analytic manifold X is locally cyclic (see [**3**, Corollary 2.6, Chapter 10] and [**2**, Proposition 3.1.5]). This follows from the fact that, if R is a simple left Noetherian, non-Artinian ring and M is a finitely generated Artinian left R-module, then M is a cyclic module. (see [**3**, Theorem 2.5, p. 90] and [**2**, Proposition 1.1.35].) In this paper, we consider when the class of cyclic modules contains all modules of finite length. Let R be a ring. We shall prove that every left R-module of finite length is cyclic if and only if R has no simple left and right Artinian factor rings. Consequently we know that every left R-module of finite length is cyclic if and only if every right R-module of finite length is cyclic. We call a ring R a finite length is cyclic-ring if R satisfies these equivalent conditions. We shall give some charactrisations of a finite ring is cyclic-ring and using those, we shall prove that if R is a finite ring is cyclic-ring, then every finite normalising extension of R is also a finite ring is cyclic-ring.

2. MODULES OF FINITE LENGTH

We begin with the following general considerations.

Let M be a left R-module. For any subset X of M, $\mathrm{Ann}_R(X)$ denotes the annihilator of X in R. For each $m \in M$ and each submodule N of M, we set $(N : m) = \{a \in R \mid am \in N\}$. For any positive integer n, $M^{(n)}$ denotes the direct sum of n copies of M.

LEMMA 2.1. *Let R be a ring and let M be a left R-module with composition series $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$. If, for each $m \in M$, and each $i = 1, \ldots, n-1$, $(M_{i+1} : m) \not\subset \mathrm{Ann}_R(M_i/M_{i+1})$, then M is cyclic.*

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PROOF: Take $x \in M_0 \setminus M_1$ and $y \in M_1 \setminus M_2$. Then $Rx + Ry + M_2 = M$. Since $(M_2 : x) \not\subset \operatorname{Ann}_R(M_1/M_2)$ and since $Ry + M_2 = M_1$, there exists $a \in R$ such that $(M_2 : x)ay + M_2 = M_1$. Hence $R(x + ay) + M_2 = M$. Continuing this process, we obtain $z \in M$ such that $Rz = Rz + M_n = M$. ∎

PROPOSITION 2.2. *Let R be a ring and let M be a left R-module of finite length. Suppose that, for each composition factor N of M, the primitive factor ring $R/\operatorname{Ann}_R(N)$ is not left Artinian. Then M is cyclic.*

PROOF: Consider a composition series $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$. We shall prove that the condition in Lemma 2.1 is satisfied. Suppose, to the contrary, that $(M_{i+1} : m) \subset \operatorname{Ann}_R(M_i/M_{i+1})$ for some $m \in M$ and i. Let $\overline{m}$ denote the homomorphic image of m in M/M_{i+1}. Then $R\overline{m} \cong R/(M_{i+1} : m)$. Then $R/\operatorname{Ann}_R(M_i/M_{i+1})$ is a homomorphic image of $R\overline{m}$, and hence it is left Artinian, a contradiction. ∎

COROLLARY 2.3. *Let R be a ring and let M be a simple left R-module. If the primitive factor ring $R/\operatorname{Ann}_R(M)$ is not left Artinian, then $M^{(n)}$ is cyclic for any positive integer n.*

We give some characterisations of rings all of whose modules of finite length are cyclic, which generalise [**3**, Theorem 2.5].

THEOREM 2.4. *Let R be a ring. Then the following statements are equivalent:*

(1) *Any left R-module of finite length is cyclic.*

(2) *There is a positive integer n such that any left R-module of finite length is generated by n elements.*

(3) *Every finitely cogenerated left R-module has an essential cyclic submodule.*

(4) *For any simple left R-module M and any positive integer n, $M^{(n)}$ is cyclic.*

(5) *R has no left Artinian factor rings.*

(6) *R has no simple left Artinian factor rings.*

(1′)–(6′) *The left-right symmetric versions of* (1)–(6).

PROOF: The equivalence of (5) and (6) and the implications (1) ⇒ (4) and (1) ⇒ (2) are clear.

(1) ⇒ (3). Let X be a finitely cogenerated left R-module. Then X has a finitely generated essential socle S and S is cyclic by hypothesis.

(2) ⇒ (4). Assume that any left R-module of finite length is generated by n elements. Let M be a simple left R-module and consider the direct sum $M^{(mn)}$ of mn copies of M. By hypothesis, there exists n elements $x_1, \dots, x_n \in M^{(mn)}$ which generate $M^{(mn)}$. For any left R-module X, let $L(X)$ denote the composition length of X. Then we have $mn = L(M^{(mn)}) \leqslant L(Rx_1) + \cdots + L(Rx_n)$. Hence, $L(Rx_i) \geqslant m$ for some $i \in \{1, \dots, n\}$. Since Rx_i is a submodule of the completely reducible module $M^{(mn)}$, this implies that $Rx_i \cong M^{(m+k)}$ for some $k \geqslant 0$. Then $M^{(m)} (\cong M^{(m+k)}/M^{(k)})$ is also cyclic.

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(3) $\Rightarrow$ (4). Let M be a simple left R-module. Then, by hypothesis, $M^{(n)}$ has an essential cyclic submodule for any positive integer n. Since $M^{(n)}$ is completely reducible, this implies that $M^{(n)}$ itself is cyclic.

(6) $\Leftrightarrow$ (6'). It is well-known that a ring is a left Artinian simple ring if and only if it is a right Artinian simple ring.

(4) $\Rightarrow$ (6). Suppose that R/I is a left Artinian ring for some ideal I. We may assume that R/I is a simple ring. Then we can easily see that $R/I \oplus R/I$ is not a cyclic left R-module.

(5) $\Rightarrow$ (1). This follows from Proposition 2.2. This completes the proof. ∎

A ring R is called a *finite ring is cyclic-ring* if R satisfies the equivalent conditions of Theorem 2.4.

Corollary 2.5. *Let S be a finite normalising extension of a ring R. If R is a finite ring is cyclic-ring, then S is a finite ring is cyclic-ring.*

Proof: This follows from Theorem 2.4 and [**4**, Corollary 10.1.11]. ∎

We note that the corollary above also follows more directly from [**4**, Proposition 10.1.9(iii)].

Corollary 2.6. *Let S be an extension of a ring R such that ${}_RS$ is finitely generated and $S = R \oplus I$ where I is a two-sided ideal of S. If S is a finite ring is cyclic-ring, then R is a finite ring is cycli-ring.*

Proof: Let K be a proper ideal of R. Then SKS is an ideal of S such that $SKS \cap R = K$. Hence S/SKS is finitely generated as a left R/K-module. Since S/SKS is not left Artinian, R/K is also not left Artinian. Therefore this follows from Theorem 2.4. ∎

Corollary 2.7. *The class of finite ring is cyclic-rings is Morita stable.*

Proof: This follows from the fact that the class of right Artinian rings are Morita stable (See [**1**, Corollary 21.9]). ∎

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