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ON PARABOLIC SUBMONOIDS OF A CLASS OF SINGULAR ARTIN MONOIDS

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Abstract

This paper concerns parabolic submonoids of a class of monoids known as *singular Artin monoids*. The latter class includes the singular braid monoid — a geometric extension of the braid group, which was created for the sole purpose of studying Vassiliev invariants in knot theory. However, those monoids may also be construed (and indeed, are defined) as a formal extension of Artin groups which, in turn, naturally generalise braid groups. It is the case, by van der Lek and Paris, that standard parabolic subgroups of Artin groups are canonically isomorphic to Artin groups. This naturally invites us to consider whether the same holds for parabolic submonoids of singular Artin monoids. We show that it is in fact true when the corresponding Coxeter matrix is of 'type FC'; hence generalising Corran's result in the 'finite type' case.

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1. Preliminaries

We begin with some formal definitions. Let I be a finite indexing set, and let $M = (m_{ij})_{i,j\in I}$ denote the matrix, indexed by the elements of I, that satisfies:

(i) $m_{ii} = 1$ if $i \in I$;
(ii) $m_{ij} = m_{ji} \in \{2, 3, 4, \dots, \infty\}$ whenever $i, j \in I$ and $i \neq j$.

Such a matrix is known as a *Coxeter matrix*. Every Coxeter matrix M may be associated with a graph Γ^M defined as follows:

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(i) I is the set of vertices of Γ^M;

(ii) any two nodes $i, j \in I$ are joined by an edge if $m_{ij} \geq 3$;

(iii) the edge joining two vertices i and j is labelled by m_{ij} if $m_{ij} \geq 4$; edge labels are suppressed whenever $m_{ij} = 3$.

Such a graph is referred to as a *Coxeter graph of type M*. Now let $S = \{\sigma_i \mid i \in I\}$ be a set in one-to-one correspondence with I. If X is a set, then X^* denotes the free monoid generated by X. If q is a natural number and $i, j \in I$ then $\langle\sigma_i\sigma_j\rangle^q$ indicates the alternating product $\sigma_i\sigma_j\sigma_i \cdots$ of length q (that is, with q factors). The *Artin group of type M*, G_M, is the group generated by S subject to the relations

$$\langle\sigma_i\sigma_j\rangle^{m_{ij}} = \langle\sigma_j\sigma_i\rangle^{m_{ij}} \quad \text{for } i, j \in I, m_{ij} \neq \infty;$$

these relations are denoted by $\mathscr{R}_1$ and called the *braid relations*. In arguments below we regard a relation formally as an ordered pair of words. For example, the relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ becomes the ordered pair $(\sigma_i\sigma_j, \sigma_j\sigma_i)$. If X is a set of ordered pairs of words then $X^\Sigma = \{(U, V) \mid (U, V) \text{ or } (V, U) \in X\}$. The *Coxeter group of type M*, W_M, is the group generated by S subject to the preceding braid relations $\mathscr{R}_1$, together with the relations $\sigma_i^2 = 1$ for every i in I. Hence, Coxeter groups arise as quotient groups of Artin groups. If W_M is finite then M (or Γ^M) is said to be of *finite type* or *spherical type*. A Coxeter group is finite precisely when its graph is a finite disjoint union of the graphs shown in Figure 1 (see, for example, [14, 21]) .

The first, and arguably the most well-known, (non-abelian) example of an Artin group is the braid group established in 1925 by Artin [3]; thus the terminology *Artin group* suggested by Brieskorn and Saito [9]. Indeed, Artin groups are also known as *generalised braid groups*. Observe that $\mathcal{B}_{n+1}$, the braid group on $n+1$ strings, arises from the special case when $I = \{1, \ldots, n\}$, $m_{ij} = 3$ when $|i - j| = 1$, and $m_{ij} = 2$ when $|i - j| \geq 2$. Its associated Coxeter graph is referred to as type A_n (shown in Figure 1), and the corresponding Coxeter group is the symmetric group on $n + 1$ letters.

We now extend Artin groups as follows [11, 19]: put $T = \{\tau_i \mid i \in I\}$, and let $S^{-1} = \{\sigma_i^{-1} \mid i \in I\}$, the set of formal inverses of S. The *singular Artin monoid of type M*, denoted by SG_M, is the monoid generated by $S \cup S^{-1} \cup T$ and has as its defining relations the set $\mathscr{R}$, which is comprised of the free group relations $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$, the braid relations $\mathscr{R}_1$, and the relations $\mathscr{R}_2$ listed below:

$$\tau_i\sigma_i = \sigma_i\tau_i \quad \text{for all } i \text{ in } I;$$

$$\tau_i\langle\sigma_j\sigma_i\rangle^{m_{ij}-1} = \begin{cases} \langle\sigma_j\sigma_i\rangle^{m_{ij}-1}\tau_j & \text{if } m_{ij} < \infty \text{ and is odd, or} \\ \langle\sigma_j\sigma_i\rangle^{m_{ij}-1}\tau_i & \text{if } m_{ij} < \infty \text{ and is even;} \end{cases}$$

$$\tau_i\tau_j = \tau_j\tau_i \quad \text{if } m_{ij} = 2.$$

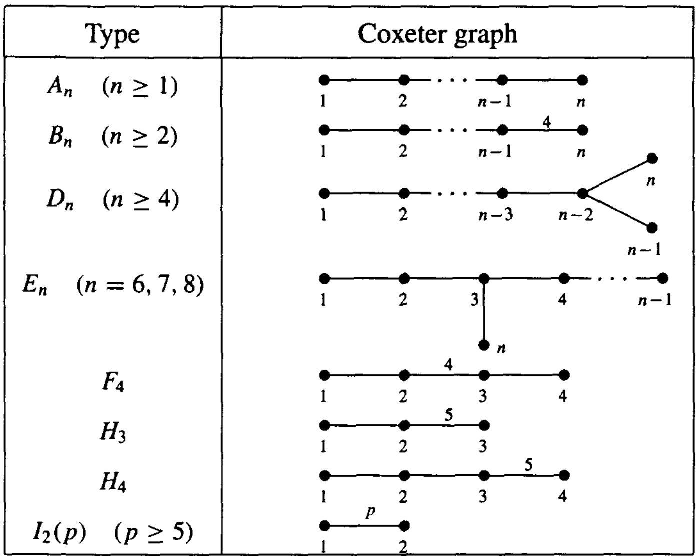

FIGURE 1. The irreducible Coxeter graphs of finite type. Unlabelled edges have value 3.

We define the *positive singular Artin monoid of type M* to be the monoid generated by $S \cup T$ and the set of defining relations comprised of both $\mathscr{R}_1$ and $\mathscr{R}_2$ listed above.

REMARK 1. The special case when the singular Artin monoid is of type A_n may be familiar to some readers as the *singular braid monoid on* $n+1$ *strings*, $\mathcal{SB}_{n+1}$, which was introduced by Baez [4] and Birman [8] in their study of knot invariants. We remark that, although singular Artin monoids are defined (abstractly) by the above generators and relations, $\mathcal{SB}_{n+1}$ was originally introduced geometrically in [4, 8] and was then shown (in [8, Lemma 3] and a subcase of [20, Theorem 2.1]) to admit the preceding presentation.

Where it does not cause confusion, elements of G_M and $\mathcal{S}G_M$ may be referred to by words which represent them. If A and B are elements of $(S \cup T \cup S^{-1})^*$, we write $A \approx B$ if A can be transformed into B by the use of the set of defining relations of $\mathcal{S}G_M$, and $A = B$ if the two words are equal letter by letter.

2. Parabolic subgroups and submonoids

Now let J be any subset of I. Recall that M is a Coxeter matrix over the finite indexing set I. Denote by M_J the submatrix of M containing the entries indexed by J; it is clear that M_J is also a Coxeter matrix. In accordance with Corran [12, Section 5], we use the notation: $S_J = \{\sigma_j \mid j \in J\}$, $S_J^{-1} = \{\sigma_j^{-1} \mid j \in J\}$, $T_J = \{\tau_j \mid j \in J\}$.

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We denote by $\mathscr{R}_{1_J}$ and $\mathscr{R}_J$ the defining relations of G_{M_J} and $\mathcal{S}G_{M_J}$ respectively. Then by the definition of these relations it is evident that $\mathscr{R}_{1_J} \subseteq \mathscr{R}_1$ and $\mathscr{R}_J \subseteq \mathscr{R}$. The subgroups of W_M and G_M generated by S_J are denoted by W_M^J and G_M^J and are called the *standard parabolic subgroups of* W_M and G_M respectively. Let P_J denote the submonoid of $\mathcal{S}G_M$ generated by $S_J \cup S_J^{-1} \cup T_J$; that is, the set of equivalence classes of words over $S_J \cup S_J^{-1} \cup T_J$ under $\approx$. Then P_J is referred to as the *parabolic submonoid defined by* J [12, Section 5]. Notice that G_M^J is a homomorphic image of the Artin group G_{M_J}. Lek [23] and Paris [24] showed that this homomorphism is an isomorphism. This result was first discovered for Artin groups of finite type in [9, 15] (it was also later proved in [10]); for 'extra-large' type Artin systems in [2]; and was gradually extended to include all types in [23, 24]. It is also well-known that the subgroup W_M^J is canonically isomorphic to the Coxeter group associated with the matrix M_J; its graph Γ^{M_J} is the full subgraph of Γ^M generated by S_J.

An analogous result holds for singular Artin monoids of finite type: namely, that $\mathcal{S}G_{M_J}$ naturally injects into $\mathcal{S}G_M$ whenever M is of finite type, so that the image of that embedding is precisely P_J [12, Proposition 33]. Hence the following holds.

THEOREM 2 (Corran [12, Theorem 34]). *Parabolic submonoids of singular Artin monoids of finite type are (isomorphic to) singular Artin monoids.*

The Coxeter matrix M is said to be *right-angled* if

$$m_{ij} \in \{2, \infty\} \quad \text{for } i, j \in I,\ i \neq j.$$

Right-angled Artin groups are also known as *graph groups* or *free partially commutative groups* [19]. Their applications extend to areas such as random walks, parallel computation and cohomology of groups (see, for example, [7]). The Coxeter matrix M is said to be of *type FC* if it satisfies the ensuing condition:

- For every $J \subseteq I$, either W_{M_J} is finite or $m_{st} = \infty$ for some $s, t \in J$, $s \neq t$.

For example, the Coxeter group associated with the graph shown in Figure 2 is of type *FC* [18]. The terminology *FC* refers to 'flag complex'; it is introduced in [10] where the reader can find a detailed exposition and classification of such types.

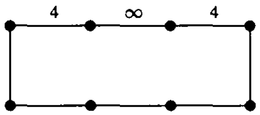

FIGURE 2.

REMARK 3. Observe that both right-angled and finite type Artin groups are of type *FC*. Furthermore, if M is of type *FC* and $J \subseteq I$, then M_J (the submatrix of M containing the entries indexed by J) is also a Coxeter matrix of type *FC*.

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In [12, Section 5], Corran postulates that although it is not clear how to generalise Theorem 2 to include singular Artin monoid of all types, she suspects that it does hold for arbitrary types. The object of this paper is to extend this theorem of Corran to singular Artin monoids of type FC. That is, we prove the following result.

THEOREM 4. *Parabolic submonoids of singular Artin monoids of type FC are (isomorphic to) singular Artin monoids.*

Except when explicitly stated, we assume throughout this paper that M is of any type. If V and W are words over $S_J \cup S_J^{-1} \cup T_J$ and represent the same element of $\mathcal{S}G_{M_J}$, write $V \approx_J W$. By [23, 24], we have the following.

THEOREM 5. *Let U, V be words over $S_J \cup S_J^{-1}$ such that $U \approx V$. Then $U \approx_J V$.*

In [16], it was shown that the singular braid monoid on $n+1$ strings (that is, the singular Artin monoid of type A_n) can be embedded in a group. The group constructed by the authors relies heavily on the geometry of singular braids in space; more specifically, it has a geometric interpretation as singular braids with two types of (cancelling) singularities. By employing purely algebraic methods, Paris [25] gave another proof of the fact that singular braid monoids inject into groups. In fact, all singular Artin monoids embed into groups. This was shown (chronologically and with completely different proofs) in [5, 22, 19]. An evident corollary of this is that left and right cancellation hold in $\mathcal{S}G_M$.

PROPOSITION 6. *Let C, W, V be words over $S \cup S^{-1} \cup T$ such that either $CW \approx CV$ or $WC \approx VC$. Then $W \approx V$.*

The next proposition is a subcase of what is known as the 'FRZ' property [19, Proposition 4.1]. The property was first discovered in [17, Theorem 7.1] for the singular braid monoid on $n+1$ strings (defined in Remark 1); it was later shown to hold for singular Artin monoids of finite type [12, Theorem 31] and of type FC [19, Proposition 4.1]. By [1, Appendix], the FRZ property holds for positive singular Artin monoids of any type.

PROPOSITION 7. *Assume M is of type FC. Let U be a word over $S \cup S^{-1} \cup T$, $s, t \in I$ and suppose $\sigma_s U \approx U \sigma_t$. Then $\tau_s U \approx U \tau_t$.*

3. Proof of the main theorem

Let U, V be words over $S \cup S^{-1} \cup T$. We say U and V *differ by an elementary transformation* if there are words X and Y and a relation $(\kappa_1, \kappa_2) \in (\mathcal{R}_1 \cup \mathcal{R}_2)^{\Sigma}$ such that $V = X\kappa_1 Y$ and $U = X\kappa_2 Y$. We say that a word V is *obtained from U by a trivial insertion* if there are words X, Y and a letter $a \in S \cup S^{-1}$ such that $U = XY$ and $V = Xaa^{-1}Y$. In this case we also say that *U is obtained from V by a trivial deletion.*

Define a monoid homomorphism $\mathcal{N}$ from $\mathcal{S}G_M$ to $(\mathbb{Z}, +)$ by

$$\mathcal{N} : \sigma_i^{\pm 1} \mapsto 0, \ \tau_i \mapsto 1 \quad \text{for } i \in I.$$

Thus $\mathcal{N}$ counts the number of taus in any given word. Now let W be a word over $S \cup S^{-1} \cup T$, and suppose $\mathcal{N}(W) = k \geq 1$. Then there are words W_i over $S \cup S^{-1}$ and generators $\tau_{a_i} \in T$ such that $W = W_0 \tau_{a_1} W_1 \tau_{a_2} \cdots W_{k-1} \tau_{a_k} W_k$. For $r = 1, \ldots, k$, let

$$\rho_r(W) = W_0 \tau_{a_1} W_1 \cdots \tau_{a_{r-1}} W_{r-1} \sigma_{a_r} W_r \tau_{a_{r+1}} W_{r+1} \cdots \tau_{a_k} W_k$$

and

$$\theta_r(W) = W_0 \tau_{a_1} W_1 \cdots \tau_{a_{r-1}} W_{r-1} W_r \tau_{a_{r+1}} W_{r+1} \cdots \tau_{a_k} W_k.$$

Hence both ρ_r and θ_r reduce the number of taus of W by 1. We observe that ρ_r has been previously defined in, for example, [6, 13].

LEMMA 8. *Let W, V be words over $S \cup S^{-1} \cup T$ such that $W \approx V$, and suppose $\mathcal{N}(W) = k$ is at least 1. Then for every $r \in \{1, \ldots, k\}$ there exists an $s \in \{1, \ldots, k\}$ such that $\rho_r(W) \approx \rho_s(V)$ and $\theta_r(W) \approx \theta_s(V)$.*

PROOF. Let r be any integer such that $1 \leq r \leq k$. Since $W \approx V$, there is a sequence $Z_1, \ldots, Z_t$ of words over $S \cup S^{-1} \cup T$ such that $W = Z_1 \approx Z_2 \approx \cdots \approx Z_t = V$ and Z_{i+1} is obtained from Z_i by an elementary transformation or by a trivial deletion or insertion. If $t = 1$ the result is trivial and hence starts an induction. Suppose then that t is least 2. If Z_2 is obtained from Z_1 by a trivial deletion or insertion, it is evident that $\rho_r(Z_1) \approx \rho_r(Z_2)$ and $\theta_r(Z_1) \approx \theta_r(Z_2)$. So assume that Z_1 and Z_2 differ by an elementary transformation. If the relation involves any σ, then inspection of $\mathscr{R}_1 \cup \mathscr{R}_2$ gives $\rho_r(Z_1) \approx \rho_r(Z_2)$ and $\theta_r(Z_1) \approx \theta_r(Z_2)$. Hence suppose the relation is of the form $(\tau_i \tau_j, \tau_j \tau_i)$ where $m_{ij} = 2$. Then we see that either

$$\rho_r(Z_1) \approx \rho_{r+1}(Z_2) \qquad \text{and} \qquad \theta_r(Z_1) = \theta_{r+1}(Z_2);$$

or

$$\rho_r(Z_1) \approx \rho_r(Z_2) \qquad \text{and} \qquad \theta_r(Z_1) \approx \theta_r(Z_2);$$

or

$$\rho_r(Z_1) \approx \rho_{r-1}(Z_2) \qquad \text{and} \qquad \theta_r(Z_1) = \theta_{r-1}(Z_2).$$

Thus there exists an integer $q \in \{1, \ldots, k\}$ such that

$$\rho_r(Z_1) \approx \rho_q(Z_2) \qquad \text{and} \qquad \theta_r(Z_1) \approx \theta_q(Z_2).$$

By the inductive hypothesis, we deduce that $\rho_q(Z_2) \approx \rho_s(Z_t)$ and $\theta_q(Z_2) \approx \theta_s(Z_t)$ for some $s \in \{1, \ldots, k\}$, whence $\rho_r(W) = \rho_r(Z_1) \approx \rho_q(Z_2) \approx \rho_s(Z_t) = \rho_s(V)$, and similarly $\theta_r(W) \approx \theta_s(V)$, as required. The result now follows by induction. □

THEOREM 9. *Suppose M is of type FC. Let $J \subseteq I$, and suppose $U \approx V$ where U and V are words over $S_J \cup S_J^{-1} \cup T_J$. Then $U \approx_J V$.*

PROOF. Let U, V be words over $S_J \cup S_J^{-1} \cup T_J$ such that $U \approx V$, and put $\mathcal{N}(U) = k$. If $k = 0$, the result follows by Theorem 5 and starts an induction. So suppose $k \geq 1$. Then there exists an $a \in J$ and words X_1, X_2 over $S_J \cup S_J^{-1}$ and $S_J \cup S_J^{-1} \cup T_J$ respectively such that $U = X_1 \tau_a X_2$. Thus

$$X_1 \tau_a X_2 = U \approx V, \tag{1}$$

so by Lemma 8, there exists an $r \in \{1, \ldots, k\}$ such that

$$X_1 \sigma_a X_2 = \rho_1(U) \approx \rho_r(V) \quad \text{and} \quad X_1 X_2 = \theta_1(U) \approx \theta_r(V). \tag{2}$$

Since $\mathcal{N}(U) = \mathcal{N}(V) = k \geq 1$, there are words Y_1, Y_2 over $S_J \cup S_J^{-1} \cup T_J$ and a generator $\tau_b \in T_J$ such that

$$V = Y_1 \tau_b Y_2, \quad \text{where } \mathcal{N}(Y_1) = r - 1 \text{ and } \mathcal{N}(Y_2) = k - r. \tag{3}$$

Then

$$\rho_r(V) = Y_1 \sigma_b Y_2 \quad \text{and} \quad \theta_r(V) = Y_1 Y_2,$$

so by (2),

$$X_1 \sigma_a X_2 \approx Y_1 \sigma_b Y_2 \quad \text{and} \quad X_1 X_2 \approx Y_1 Y_2. \tag{4}$$

By noting that X_1 is over $S_J \cup S_J^{-1}$, we deduce that X_1^{-1} is also over $S_J \cup S_J^{-1}$, so by (4), we obtain

$$\sigma_a X_2 \approx X_1^{-1} Y_1 \sigma_b Y_2 \quad \text{and} \quad X_2 \approx X_1^{-1} Y_1 Y_2; \tag{5}$$

moreover by (3), we see that

$$k - 1 = \mathcal{N}(Y_1 Y_2) = \mathcal{N}(X_1^{-1} Y_1 Y_2) \geq \mathcal{N}(Y_1) = \mathcal{N}(\sigma_a X_1^{-1} Y_1). \tag{6}$$

Observe that by (5), $\sigma_a X_1^{-1} Y_1 Y_2 \approx \sigma_a X_2 \approx X_1^{-1} Y_1 \sigma_b Y_2$, so by Proposition 6,

$$\sigma_a X_1^{-1} Y_1 \approx X_1^{-1} Y_1 \sigma_b. \tag{7}$$

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Since $a, b \in J$, and X_1^{-1}, X_2, Y_1, Y_2 are all words over $S_J \cup S_J^{-1} \cup T_J$, (6) and (7) together with the inductive hypothesis give

$$\sigma_a X_1^{-1} Y_1 \approx_J X_1^{-1} Y_1 \sigma_b; \tag{8}$$

furthermore, by (6) and the inductive hypothesis applied to the second equivalence of (5), we also infer that

$$X_2 \approx_J X_1^{-1} Y_1 Y_2. \tag{9}$$

By Remark 3, M_J is also of type FC; thus Proposition 7 may be applied to (8), and this yields

$$\tau_a X_1^{-1} Y_1 \approx_J X_1^{-1} Y_1 \tau_b. \tag{10}$$

Hence

$$\begin{aligned} U &= X_1 \tau_a X_2 && \text{by (1)} \\ &\approx_J X_1 \tau_a X_1^{-1} Y_1 Y_2 && \text{by (9)} \\ &\approx_J X_1 X_1^{-1} Y_1 \tau_b Y_2 && \text{by (10)} \\ &\approx_J Y_1 \tau_b Y_2 = V && \text{by (3).} \end{aligned}$$

The result now follows by induction. □

PROOF OF THEOREM 4. Suppose M is of type FC. Recall that P_J denotes the submonoid of $\mathcal{S}G_M$ generated by $S_J \cup S_J^{-1} \cup T_J$. By Theorem 9, $\mathcal{S}G_{M_J}$ naturally embeds in $\mathcal{S}G_M$ with image P_J. Hence the parabolic submonoid P_J is canonically isomorphic to the singular Artin monoid $\mathcal{S}G_{M_J}$. □

REMARK 10. The reader may notice that the only part of the proof of Theorem 9 that requires the FC condition is Proposition 7. Hence in order to strengthen Theorem 4 to show that it holds for singular Artin monoids of arbitrary type it suffices to prove Proposition 7 for any Coxeter matrix M; the proof would then proceed unmodified to that of Theorem 9.

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