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Periodic Steady-state Solutions of a Liquid Film Model via a Classical Method

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Abstract. In this paper, periodic steady-state of a liquid film flowing over a periodic uneven wall is investigated via a classical method. Specifically, we analyze a long-wave model that is valid at the near-critical Reynolds number. For the periodic wall surface, we construct an iteration scheme in terms of an integral form of the original steady-state problem. The uniform convergence of the scheme is proved so that we can derive the existence and the uniqueness as well as the asymptotic formula of the periodic solutions.

1 Introduction

Investigations of liquid film flow over an inclined wall have been of great interest to many scientific researchers, as it arises in applications for many topics; see [1,7,8,11,12]. Many previous works dealt with the problem of a viscous liquid falling down an inclined wall with a flat surface, in which theoretical and numerical methods were applied to study the existence of the steady-state solution and its stability characteristics (*e.g.*, [2–4]). A change of flatness in the wall surface is more reasonable in practice, and it affects the liquid surface behavior. Recently, Tesuilko and Blyth [9] studied the effect of inertia on a film flowing on an uneven wall in the presence of an electric field. Tseluiko *et al* [10] worked on the model derived in [9], assuming that the flow variation as well as the variation in the wall shape in the flow direction are subtle. Ignoring the electric effects, they solved the steady-state problem numerically.

The purpose of this paper is to study this problem via a classical method. As in [5], by *classical methods in differential equations* we mean finite dimensional methods derived from what is called *classical analysis*. Whereas modern applied analysis is commonly used to cast differential equation problems (including boundary value problems) into infinite dimensional settings, so that degree theory or infinite-dimensional fixed point theorems can be applied to prove the existence of solutions, "classical analysis", in handling the same problems, often provides more information than the abstract approaches. In particular, the "classical analysis" methods used are more likely to be constructive in some sense and so can form the basis of numerical methods. They are sometimes more global, for instance, giving estimates of the size of a small parameter.

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In this paper, we consider a liquid film flowing over a periodic uneven wall inclined at an angle θ to the horizontal line. We introduce (x, y)-coordinates so that the flow is along the x^+-axis direction. Let $y = s(x)$ be the function that describes the wall surface topography; see Figure 1.

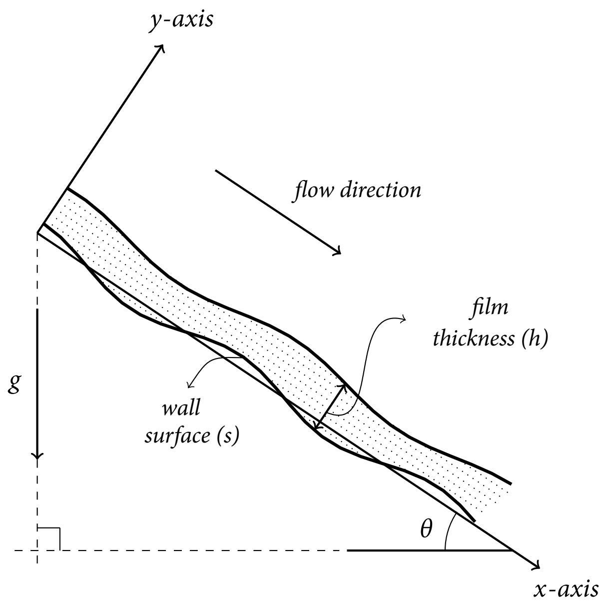

Figure 1: Liquid film flowing over an inclined uneven wall.

The flow is governed by the partial differential equation

$$h_t + q_x = 0; \tag{1.1}$$

see [10], where $h(x,t)$ is the dimensionless film thickness at time t and location x, and $q(x,t)$ is the flux rate given by

$$q = \frac{2}{3}h^3 + \frac{8R}{15}h^6 h_x - \frac{2\cot(\theta)}{3}h^3(h+s)_x + \frac{1}{3C}h^3(h+s)_{xxx}. \tag{1.2}$$

Here, R and C are the Reynolds and capillary numbers, respectively, which are given in terms of the liquid density, the liquid viscosity, and the surface tension.

Throughout this paper, we assume that the wall surface shape $s(x)$ satisfies

$$|s'(x)| \le a_1\epsilon \quad \text{and} \quad |s'''(x)| \le a_2\epsilon \tag{1.3}$$

for small positive number ϵ, and constants a_1, a_2. In [10], the authors considered two kinds of wall surface topography, a sinusoidal wall with $s(x) = A\cos(\frac{\pi x}{l})$ and a rectangular wall with $s(x) = A\tanh(\cos(\frac{\pi x}{l})/d)$. Here A is the amplitude, l is the period, and d is a constant such that the smaller the value of d the steeper the wall is. They assumed that A/l is small, which implies condition (1.3).

Using an analytical method, we will prove the existence of periodic steady-states to the partial differential equations. We give the details in three cases in terms of integral equations. The result not only provides the existence and the uniqueness of a periodic solution, but also gives a generalized asymptotic formula.

The rest of the paper is as follows. We prove the existence and the uniqueness of steady-state solution using the uniform convergence of the iteration scheme in Section 2. We shall split the proof into three cases, depending on the roots of the characteristic equation of the homogeneous differential equation associated with the steady-state problem. Conclusions are presented in Section 3.

2 Existence of Steady-state Solution

In this section, we seek a periodic steady-state solution, $h(x,t) = h_0(x)$, to (1.1)–(1.2). By (1.2), this is equivalent to find $h_0(x)$ that solves the ordinary differential equation $q'(x) = 0$ or $q(x) = q_0$ for a constant q_0 that is related to the flow flux of the model. For convenience and without loss of generality, we choose $q_0 = 2/3$, and the steady-state $h_0(x)$ from equation (1.2) satisfies

$$\frac{2}{3}h_0^3 + \frac{8R}{15}h_0^6 h_0' - \frac{2\cot(\theta)}{3}h_0^3(h_0+s)' + \frac{1}{3C}h_0^3(h_0+s)''' = \frac{2}{3}, \tag{2.1}$$

where prime denotes the derivative d/dx. When condition (1.3) holds, $h_0(x) = 1$ is an approximation solution to (2.1) (for any q_0, the approximation is $h_0(x) = \sqrt[3]{3q_0/2}$). This suggests that $h_0(x) = 1 + w(x)$ is the exact steady-state solution to (2.1) for some periodic small-amplitude function $w(x) \neq -1$. Substitute it into equation (2.1) and multiply the equation by $\frac{3C}{h_0^3}$ to get

$$\frac{8RC}{5}(3w + 3w^2 + w^3)w' + \Big(\frac{8RC}{5} - 2C\cot(\theta)\Big)w' - 2C\cot(\theta)s' + w''' + s''' = 2C\Big[\frac{1}{(1+w)^3} - 1\Big],$$

which is equivalent to

$$w''' + \Big(\frac{8RC}{5} - 2C\cot(\theta)\Big)w' + 6Cw = F(s', s''', w, w'), \tag{2.2}$$

where

$$F(s', s''', w, w')(x) = 2C\cot(\theta)s'(x) - s'''(x) + \frac{2Cw^2(x)}{(1+w(x))^3}\big[6 + 8w(x) + 3w^2(x)\big] - \frac{8RC}{5}(3w(x) + 3w^2(x) + w^3(x))w'(x).$$

Define

$$a := \frac{8RC}{5} - 2C\cot(\theta) \quad \text{and} \quad b := 6C. \tag{2.3}$$

The homogeneous part of the non-homogeneous equation (2.2) becomes

$$w''' + aw' + bw = 0. \tag{2.4}$$

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To find the fundamental set of solutions for the third-order homogeneous equation (2.4), which has the characteristic equation

$$r^3 + ar + b = 0, \tag{2.5}$$

we need the following lemma.

Lemma 2.1 (Cardano's Formula, see [6, formulas (50)–(51), chapter 4]) *The cubic algebraic equation* (2.5) *has the roots*

$$r_1 = \phi + \psi, \quad r_2 = -\frac{1}{2}(\phi + \psi) + \frac{\sqrt{3}}{2}(\phi - \psi)i, \quad \text{and} \quad r_3 = -\frac{1}{2}(\phi + \psi) - \frac{\sqrt{3}}{2}(\phi - \psi)i,$$

where

$$\phi = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \quad \text{and} \quad \psi = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}.$$

Moreover, let $\Delta = \frac{b^2}{4} + \frac{a^3}{27}$. *Then we have the following three cases:*

- *If* $\Delta = 0$, *then* (2.5) *has three real roots, at least two of which are equal. Here if* a *and* b *are not equal to* 0, *then the number of equal roots is exactly two.*
- *If* $\Delta < 0$, *then* (2.5) *has three real distinct roots.*
- *If* $\Delta > 0$, *then* (2.5) *has a real root and two conjugate complex roots.*

The three different possibilities in Lemma 2.1 divide our work into three subsections. In Subsection 2.1, we will show the existence of the steady-state solution $h_0(x)$ to (2.1) by proving the existence of a periodic solution $w(x)$ to (2.2) when a and b, defined in (2.3), satisfy $\Delta = 0$. After that, we will use the same idea in Subsections 2.2 and 2.3 to prove the existence when $\Delta < 0$ or $\Delta > 0$ is satisfied.

2.1 Existence of Steady-state when $\Delta = 0$

In the case $\frac{b^2}{4} + \frac{a^3}{27} = 0$, a must be negative, that is, $R < \frac{5}{4}\cot(\theta) := R_c$, where R_c is the critical Reynolds number (see [10]). In particular, $R = R_c - 15/4\sqrt[3]{4C}$. By applying Lemma 2.1, the characteristic equation (2.5) associated with the homogeneous equation (2.4) has a simple root $r = -2\alpha$, and a root of multiplicity 2, $r = \alpha$, where $\alpha = \sqrt[3]{3C}$. Then the fundamental set of solutions to the homogeneous equation (2.4) is

$$\{w_1, w_2, w_3\} = \{e^{-2\alpha x}, e^{\alpha x}, xe^{\alpha x}\},$$

with a constant Wronskian $W(w_1, w_2, w_3) = 9\alpha^2$. Using the variation-of-parameters method, the integral form of the non-homogeneous equation (2.2) becomes

$$w(x) = e^{-2\alpha x}\int_{-\infty}^{x} \frac{e^{2\alpha t}}{9\alpha^2} F(t)dt + e^{\alpha x}\int_{\infty}^{x} \frac{-(3\alpha t + 1)e^{-\alpha t}}{9\alpha^2} F(t)dt + xe^{\alpha x}\int_{\infty}^{x} \frac{3\alpha e^{-\alpha t}}{9\alpha^2} F(t)dt,$$

which can be further written as

$$w(x) = \frac{1}{9\alpha^2}\int_{-\infty}^{x} e^{2\alpha(t-x)}F(t)dt + \frac{1}{3\alpha}\int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}F(t)dt + \frac{1}{9\alpha^2}\int_{x}^{\infty} e^{-\alpha(t-x)}F(t)dt. \tag{2.6}$$

In order to construct a better iteration scheme for $w(x)$ in a simple functional space so that the estimate of the norm of the integral operator becomes affordable, we want to remove the derivative term w' in the right-hand side of (2.6) and rewrite it as a functional of $w(x)$ only. To do this, we substitute the formula $F(t)$ and integrate the w'-term by parts. The first term in the right-hand side of (2.6) becomes

$$\begin{aligned}
&\int_{-\infty}^{x} e^{2\alpha(t-x)}F(t)dt \\
&= \int_{-\infty}^{x} e^{2\alpha(t-x)}\Big\{2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}(6+8w+3w^2)\Big\}dt \\
&\quad - \frac{8RC}{5}\int_{-\infty}^{x} e^{2\alpha(t-x)}(3w+3w^2+w^3)w'dt \\
&= \int_{-\infty}^{x} e^{2\alpha(t-x)}\Big\{2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}(6+8w+3w^2)\Big\}dt \\
&\quad - \frac{2RC}{5}(w^4+4w^3+6w^2) + \frac{4RC\alpha}{5}\int_{-\infty}^{x} e^{2\alpha(t-x)}(w^4+4w^3+6w^2)dt.
\end{aligned}$$

Similarly, for the second and last terms, we have

$$\begin{aligned}
&\int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}F(t)dt \\
&= \int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}\Big\{2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}(6+8w+3w^2)\Big\}dt \\
&\quad + \frac{2RC}{5}\int_{x}^{\infty}(1-\alpha(t-x))e^{-\alpha(t-x)}(w^4+4w^3+6w^2)dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_{x}^{\infty} e^{-\alpha(t-x)}F(t)dt \\
&= \int_{x}^{\infty} e^{-\alpha(t-x)}\Big\{2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}(6+8w+3w^2)\Big\}dt \\
&\quad + \frac{2RC}{5}(w^4+4w^3+6w^2) - \frac{2RC\alpha}{5}\int_{x}^{\infty} e^{-\alpha(t-x)}(w^4+4w^3+6w^2)dt.
\end{aligned}$$

Now, we define functions G, H, and Q by

$$\begin{aligned}
G(s) &:= 2C\cot(\theta)s' - s''', \\
H(w) &:= 2C\frac{w^2}{(1+w)^3}(6+8w+3w^2), \\
Q(w) &:= \frac{2RC}{5}(w^4+4w^3+6w^2).
\end{aligned} \tag{2.7}$$

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Then we re-write the integral equation (2.6) in the form

$$w(x) = T_0(G)(x) + T_1(H)(x) + T_2(Q)(x) := T(w(x)), \tag{2.8}$$

where

(2.9)

$$\begin{aligned} T_0(G)(x) &= \frac{1}{9\alpha^2}\int_{-\infty}^{x} e^{2\alpha(t-x)}G(s(t))dt + \frac{1}{3\alpha}\int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}G(s(t))dt \\ &\quad + \frac{1}{9\alpha^2}\int_{x}^{\infty} e^{-\alpha(t-x)}G(s(t))dt, \\ T_1(H)(x) &= \frac{1}{9\alpha^2}\int_{-\infty}^{x} e^{2\alpha(t-x)}H(w(t))dt + \frac{1}{3\alpha}\int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}H(w(t))dt \\ &\quad + \frac{1}{9\alpha^2}\int_{x}^{\infty} e^{-\alpha(t-x)}H(w(t))dt, \\ T_2(Q)(x) &= \frac{2}{9\alpha}\int_{-\infty}^{x} e^{2\alpha(t-x)}Q(w(t))dt - \frac{1}{3}\int_{x}^{\infty}(t-x)e^{-\alpha(t-x)}Q(w(t))dt \\ &\quad + \frac{2}{9\alpha}\int_{x}^{\infty} e^{-\alpha(t-x)}Q(w(t))dt. \end{aligned}$$

To find a periodic function $w(x)$ that satisfies equation (2.8), we define an iteration scheme with the initial periodic function $w_0(x)$ as

$$\begin{aligned} w_0(x) &= T_0(G)(x), \\ w_{n+1}(x) &= T(w_n)(x), \text{ for } n \geq 0. \end{aligned} \tag{2.10}$$

Obviously, the operator T maps a periodic function into a periodic function with the same prime period. We will show that the series $\sum_{n=1}^{\infty}(w_n - w_{n-1})$ converges uniformly for x in $(-\infty, \infty)$. Then the required periodic solution $w(x)$ can be obtained by the limit

$$w(x) = \lim_{n\to\infty} w_n(x) = w_0(x) + \sum_{i=1}^{\infty}\big(w_i(x) - w_{i-1}(x)\big).$$

First of all, we want to estimate the initial function $w_0(x)$. Note that

$$\begin{aligned} |w_0(x)| \leq \|G(s(x))\| \Big\{ \frac{1}{9\alpha^2}\Big|\int_{-\infty}^{x} e^{2\alpha(t-x)}dt\Big| + \frac{1}{3\alpha}\Big|\int_{x}^{\infty}(t-x)e^{-\alpha(t-t)}dt\Big| \\ + \frac{1}{9\alpha^2}\Big|\int_{x}^{\infty} e^{-\alpha(t-x)}dt\Big|\Big\} \\ = \frac{1}{2\alpha^3}\|G(s(x))\|, \end{aligned}$$

where $\|\cdot\|$ is the maximum norm. This means that we can determine the bound of the periodic function $w_0(x)$ by the bound of $s(x)$, that is, for $s(x)$ satisfying inequalities in (1.3) and using the definition of $G(s)$, we have

$$|w_0(x)| \leq \|w_0(x)\| \leq B\epsilon < \frac{1}{2}, \tag{2.11}$$

where $B = \frac{1}{2\alpha^3}(2C\cot(\theta)a_1 + a_2)$, and ϵ is sufficiently small (less than ϵ_0 below).

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Now we are ready to show the uniform convergence of the series $\sum_{n=1}^{\infty}(w_n - w_{n-1})$. To this end, we define the constants

$$M_1 := \sup_{|w|\le\frac{1}{2}} |H''(w)|, \qquad M_2 := \sup_{|w|\le\frac{1}{2}} |Q''(w)|,$$
$$M := \frac{1}{2\alpha^3}M_1 + \frac{2}{3\alpha^2}M_2, \qquad \beta := 2MB. \tag{2.12}$$

We will show that there exists a constant ϵ_0 such that for $0 < \epsilon < \epsilon_0$, we have

$$|w_n - w_0| \le \beta\epsilon\|w_0\|, \qquad n = 1, 2, 3, \dots, \tag{2.13}$$

and

$$|w_n - w_{n-1}| \le (2\beta\epsilon)^n\|w_0\|, \qquad n = 1, 2, 3, \dots. \tag{2.14}$$

Indeed, for $n = 1$, we use the iteration definition (2.10) and (2.8) to have

$$|w_1 - w_0| = |T(w_0) - w_0| \le |T_1(H(w_0))| + |T_2(Q(w_0))|. \tag{2.15}$$

Using Taylor expansion, $Q(w) = Q''(v)w^2$ for $v \in (0, w)$ and $|w| < \frac{1}{2}$. This implies

$$\|Q(w_0)\| \le M_2\|w_0\|^2. \tag{2.16}$$

Similarly,

$$\|H(w_0)\| \le M_1\|w_0\|^2. \tag{2.17}$$

By using (2.9), (2.16), and (2.17) in (2.15) yields

$$|w_1 - w_0| \le M\|w_0\|^2. \tag{2.18}$$

Hence, from inequality (2.11), we have

$$|w_1 - w_0| \le MB\epsilon\|w_0\| \le \beta\epsilon\|w_0\|,$$

which proves that inequalities (2.13) and (2.14) hold for $n = 1$. To complete our argument, we assume, by induction, that inequalities (2.13) and (2.14) are true for $n = k$. This gives $|w_k| \le (1 + \beta\epsilon)B\epsilon \le \frac{1}{2}$ as long as $\epsilon < \epsilon_0$ for a given small ϵ_0. We need to show that both of (2.13) and (2.14) hold true for $n = k + 1$. Actually, we have

$$\begin{aligned}
|w_{k+1} - w_0| &= |T(w_k) - w_0| \\
&\le |T_1(H(w_k))| + |T_2(Q(w_k))| \\
&\le M\|w_k\|^2 && \text{similar to (2.18)} \\
&\le M(1 + \beta\epsilon)^2\|w_0\|^2 && \text{from our assumption} \\
&\le BM(1 + \beta\epsilon)^2\epsilon\|w_0\| && \text{using (2.11)} \\
&\le \beta\epsilon\|w_0\|.
\end{aligned}$$

This implies that the inequality (2.13) is satisfied for all n. Here, we have assumed that ϵ is sufficiently small so that $(1 + \beta\epsilon)^2 \le 2$ for $\epsilon < \epsilon_0$. For inequality (2.14), we have

$$\begin{aligned}
|w_{k+1} - w_k| &= |T(w_k) - T(w_{k-1})| \\
&\le |T_1(H(w_k) - H(w_{k-1}))| + |T_2(Q(w_k) - Q(w_{k-1}))|.
\end{aligned} \tag{2.19}$$

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By the Mean Value Theorem, for $0 \le \theta \le 1$, we get

$$\begin{aligned}\|Q(w_k) - Q(w_{k-1})\| &\le \|Q'(\theta w_k + (1-\theta)w_{k-1})\| \cdot \|w_k - w_{k-1}\| \\ &= \|Q''(v)\| \cdot \|\theta w_k + (1-\theta)w_{k-1}\| \cdot \|w_k - w_{k-1}\| \quad \text{for some } v \\ &\le M_2(1+\beta\epsilon)\|w_0\| \cdot \|w_k - w_{k-1}\|,\end{aligned}$$

and similarly,

$$\|H(w_k) - H(w_{k-1})\| \le M_1(1+\beta\epsilon)\|w_0\| \cdot \|w_k - w_{k-1}\|.$$

Hence, inequality (2.19) implies

$$\begin{aligned}|w_{k+1} - w_k| &\le M(1+\beta\epsilon)\|w_0\|\|w_k - w_{k-1}\| \le M(1+\beta\epsilon)(2\beta\epsilon)^k\|w_0\|^2 \\ &\le MB\epsilon(1+\beta\epsilon)(2\beta\epsilon)^k\|w_0\| \le (2\beta\epsilon)^{k+1}\|w_0\|,\end{aligned}$$

which proves that inequality (2.14) is true for all n. By the well-known Weierstrass M-test, series

$$w_0(x) + \sum_{n=1}^{\infty}\big(w_n(x) - w_{n-1}(x)\big)$$

is uniformly convergent for $x \in (-\infty, \infty)$. Consequently, we have the following theorem.

Theorem 2.1 *Assume that a and b, defined in* (2.3), *satisfy* $\frac{b^2}{4} + \frac{a^3}{27} = 0$. *There exists a small* ϵ_0 *such that for* $\epsilon < \epsilon_0$, (2.1) *has a solution* $h_0(x) = 1 + w(x)$, *where* $w(x)$ *is a solution of the differential equation* (2.2) *with the asymptotic expansion*

$$w(x) = w_0(x) + \sum_{n=1}^{\infty}\big(w_n(x) - w_{n-1}(x)\big),$$

and $w_n(x)$, $n = 0, 1, 2, \dots$, *are defined in* (2.10).

Remark 2.1 Based on (2.13) and (2.14), Theorem 2.1 also provides a generalized asymptotic expansion to the periodic steady-state solution.

2.2 Existence of Steady-state when $\Delta < 0$

In this subsection, we study the existence of periodic steady-state in the case $\frac{b^2}{4} + \frac{a^3}{27} < 0$. The fundamental set of solutions to the homogeneous equation (2.4), in this case, is $\{w_1, w_2, w_3\} = \{e^{r_1 x}, e^{r_2 x}, e^{r_3 x}\}$, where r_1, r_2, and r_3 are the real distinct roots of the characteristic equation (2.5) defined in Lemma 2.1, with a constant Wronskian

$$\widehat{W} := W(w_1, w_2, w_3) = r_2 r_3(r_3 - r_2) - r_1 r_3(r_3 - r_1) + r_1 r_2(r_2 - r_1).$$

Note that, when $\Delta < 0$, we have $r_1 < 0$ and $r_2, r_3 > 0$. Then using the variation-of-parameters method, we have the following integral form of the non-homogeneous differential equation (2.2):

$$w(x) = C_1 \int_{-\infty}^{x} e^{-r_1(t-x)}F(t)dt + C_2 \int_{x}^{\infty} e^{-r_2(t-x)}F(t)dt + C_3 \int_{x}^{\infty} e^{-r_3(t-x)}F(t)dt, \tag{2.20}$$

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where

$$C_1 = \frac{r_3 - r_2}{\widehat{W}}, \quad C_2 = \frac{r_3 - r_1}{\widehat{W}}, \quad \text{and} \quad C_3 = \frac{-(r_2 - r_1)}{\widehat{W}}.$$

Substitute $F(t)$ and integrate the w'-term by parts to get

$$\begin{aligned}
&\int_{-\infty}^{x} e^{-r_1(t-x)} F(t)\,dt \\
&\quad = \int_{-\infty}^{x} e^{-r_1(t-x)} \Big\{ 2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}(6+8w+3w^2) \Big\}\,dt \\
&\qquad - \frac{2RC}{5}(w^4+4w^3+6w^2) - \frac{2RCr_1}{5}\int_{-\infty}^{x} e^{-r_1(t-x)}(w^4+4w^3+6w^2)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_{x}^{\infty} e^{-r_i(t-x)} F(t)\,dt \\
&\quad = \int_{x}^{\infty} e^{-r_i(t-x)} \Big\{ 2C\cot(\theta)s' - s''' + \frac{2Cw^2}{(1+w)^3}(6+8w+3w^2) \Big\}\,dt \\
&\qquad + \frac{2RC}{5}(w^4+4w^3+6w^2) - \frac{2RCr_i}{5}\int_{x}^{\infty} e^{-r_i(t-x)}(w^4+4w^3+6w^2)\,dt,
\end{aligned}$$

for $i = 2, 3$. In terms of $G(s)$, $H(w)$, and $Q(w)$ defined in (2.7), the integral equation (2.20) can be written in the form

$$w(x) = \widehat{T}_0(G)(x) + \widehat{T}_1(H)(x) + \widehat{T}_2(Q)(x) := \widehat{T}(w(x)),$$

where

$$\begin{aligned}
\widehat{T}_0(G)(x) &= C_1 \int_{-\infty}^{x} e^{-r_1(t-x)} G(s(t))\,dt + \sum_{i=2}^{3} C_i \int_{x}^{\infty} e^{-r_i(t-x)} G(s(t))\,dt, \\
\widehat{T}_1(H)(x) &= C_1 \int_{-\infty}^{x} e^{-r_1(t-x)} H(w(t))\,dt + \sum_{i=2}^{3} C_i \int_{x}^{\infty} e^{-r_i(t-x)} H(w(t))\,dt,
\end{aligned}$$

and

$$\widehat{T}_2(Q)(x) = -C_1 r_1 \int_{-\infty}^{x} e^{-r_1(t-x)} Q(w(t))\,dt - \sum_{i=2}^{3} C_i r_i \int_{x}^{\infty} e^{-r_i(t-x)} Q(w(t))\,dt.$$

Similar to the previous subsection, we define an iteration scheme

$$\begin{aligned}
\widehat{w}_0(x) &= \widehat{T}_0(G)(x), \\
\widehat{w}_{n+1}(x) &= \widehat{T}(\widehat{w}_n)(x), \quad \text{for } n \geq 0,
\end{aligned} \tag{2.21}$$

and later use the following constants:

$$\begin{aligned}
&\widehat{B} := (2C\cot(\theta)a_1 + a_2)\sum_{i=1}^{3}\Big|\frac{C_i}{r_i}\Big|, \\
&\widehat{M} := M_1 \sum_{i=1}^{3}\Big|\frac{C_i}{r_i}\Big| + M_2 \sum_{i=1}^{3}|C_i|, \\
&\widehat{\beta} := 2\widehat{M}\widehat{B},
\end{aligned}$$

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where M_1 and M_2 are the same as those in (2.12). The operator $\widehat{T}$ maps periodic functions into periodic functions. Then we can apply the same technique used in the previous subsection to show that there exists an $\epsilon_0 > 0$ such that for sufficiently small $\epsilon < \epsilon_0$, the inequalities

$$|\widehat{w}_0| \le \|\widehat{w}_0\| \le \widehat{B}\epsilon,$$
$$|\widehat{w}_n - \widehat{w}_0| \le \widehat{\beta}\epsilon\|\widehat{w}_0\|, \quad n = 1, 2, 3, \dots,$$
$$|\widehat{w}_n - \widehat{w}_{n-1}| \le (2\widehat{\beta}\epsilon)^n\|\widehat{w}_0\|, \quad n = 1, 2, 3, \dots,$$

hold. Hence, the Weierstrass M-test implies that series

$$\widehat{w}_0(x) + \sum_{n=1}^{\infty}\big(\widehat{w}_n(x) - \widehat{w}_{n-1}(x)\big)$$

is uniformly convergent for $x \in (-\infty, \infty)$. Then the following result is valid.

Theorem 2.2 *Assume that a and b, defined in* (2.3)*, satisfy* $\frac{b^2}{4} + \frac{a^3}{27} < 0$. *There exists a constant* $\epsilon_0 > 0$ *such that for* $\epsilon < \epsilon_0$, (2.1) *has a periodic solution* $h_0(x) = 1 + w(x)$, *where* $w(x)$ *is a solution of the differential equation* (2.2) *with the asymptotic expansion*

$$w(x) = \widehat{w}_0(x) + \sum_{n=1}^{\infty}\big(\widehat{w}_n(x) - \widehat{w}_{n-1}(x)\big),$$

and $\widehat{w}_n(x)$, $n = 0, 1, 2, \dots$, *are defined in* (2.21).

2.3 Existence of Steady-state when $\Delta > 0$

When $\Delta > 0$, Lemma 2.1 implies that the characteristic equation (2.5), associated with the homogeneous equation (2.4), has a real root r and two complex conjugate roots $u \pm iv$, where r, u, and v can be defined in terms of ϕ and ψ in Lemma 2.1. The fundamental set of solutions is $\{w_1, w_2, w_3\} = \{e^{rx}, e^{ux}\cos(vx), e^{ux}\sin(vx)\}$, with a constant Wronskian

$$\overline{W} := W(w_1, w_2, w_3) = v(2r^2 + u^2 + v^2).$$

Note that, since $b > 0$, we have $r < 0$ and $u > 0$, with $r + 2u = 0$. Hence, the integral form of the differential equation (2.2), in this case, is

$$w(x) = e^{rx}\int_{-\infty}^{x}\frac{W_1(t)}{\overline{W}}F(t)dt + e^{ux}\cos(vx)\int_{\infty}^{x}\frac{W_2(t)}{\overline{W}}F(t)dt + e^{ux}\sin(vx)\int_{\infty}^{x}\frac{W_3(t)}{\overline{W}}F(t)dt,$$

where

$$W_1(t) = ve^{-rt}, \quad W_2(t) = -\big[(u-r)\sin(vt) + v\cos(vt)\big]e^{-ut},$$
$$W_3(t) = \big[(u-r)\cos(vt) - v\sin(vt)\big]e^{-ut}.$$

This integral form can be written as

$$w(x) = \frac{v}{\overline{W}}\int_{-\infty}^{x} e^{-r(t-x)}F(t)dt + \int_{x}^{\infty} g(x,t)e^{-u(t-x)}F(t)dt, \tag{2.22}$$

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where $g(x,t)$ is given by

$$g(x,t)=\frac{1}{\overline{\overline{W}}}\Big[\,(u-r)\sin(v(t-x))+v\cos(v(t-x))\Big].$$

We write the integrals in (2.22) as

$$\begin{aligned}
&\frac{v}{\overline{\overline{W}}}\int_{-\infty}^{x}e^{-r(t-x)}F(t)dt\\
&\quad=\frac{v}{\overline{\overline{W}}}\int_{-\infty}^{x}e^{-r(t-x)}\Big\{2C\cot(\theta)s'-s'''+\frac{2Cw^2}{(1+w)^3}(6+8w+3w^2)\Big\}dt\\
&\qquad-\frac{2RCv}{5\overline{\overline{W}}}(w^4+4w^3+6w^2)-\frac{2RCrv}{5\overline{\overline{W}}}\int_{-\infty}^{x}e^{-r(t-x)}(w^4+4w^3+6w^2)dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_{x}^{\infty}g(x,t)e^{-u(t-x)}F(t)dt\\
&\quad=\int_{x}^{\infty}e^{-u(t-x)}g(x,t)\Big\{2C\cot(\theta)s'-s'''+\frac{2Cw^2}{(1+w)^3}(6+8w+3w^2)\Big\}dt\\
&\qquad+\frac{2RCv}{5\overline{\overline{W}}}(w^4+4w^3+6w^2)\\
&\qquad+\frac{2RC}{5}\int_{x}^{\infty}[g_t(x,t)-ug(x,t)]e^{-u(t-x)}(w^4+4w^3+6w^2)dt.
\end{aligned}$$

From this, the formula of $w(x)$ in (2.22) can be expressed as

$$w(x)=\overline{T}_0(G)(x)+\overline{T}_1(H)(x)+\overline{T}_2(Q)(x):=\overline{T}\big(w(x)\big),$$

where $G(s), H(w)$, and $Q(w)$ are defined in (2.7), and

$$\begin{aligned}
\overline{T}_0(G)(x)&=\frac{v}{\overline{\overline{W}}}\int_{-\infty}^{x}e^{-r(t-x)}G(s(t))dt+\int_{x}^{\infty}g(x,t)e^{-u(t-x)}G(s(t))dt,\\
\overline{T}_1(H)(x)&=\frac{v}{\overline{\overline{W}}}\int_{-\infty}^{x}e^{-r(t-x)}H(w(t))dt+\int_{x}^{\infty}g(x,t)e^{-u(t-x)}H(w(t))dt,\\
\overline{T}_2(Q)(x)&=-\frac{vr}{\overline{\overline{W}}}\int_{-\infty}^{x}e^{-r(t-x)}Q(w(t))dt\\
&\qquad+\int_{x}^{\infty}[g_t(x,t)-ug(x,t)]e^{-u(t-x)}Q(w(t))dt.
\end{aligned}$$

Similar to the previous cases, we define an iteration scheme, for this case, as

$$\begin{aligned}
\overline{w}_0(x)&=\overline{T}_0(G)(x),\\
\overline{w}_{n+1}(x)&=\overline{T}(\overline{w}_n)(x),\ \text{for } n\geq 0.
\end{aligned}\tag{2.23}$$

Then we can show that, there exists an $\epsilon_0>0$ such that for $\epsilon<\epsilon_0$, the inequalities

$$|\overline{w}_0|\leq\|\overline{w}_0\|\leq\overline{B}\epsilon,\quad |\overline{w}_n-\overline{w}_{n-1}|\leq\overline{\beta}\epsilon\|\overline{w}_0\|,$$

and

$$|\overline{w}_n-\overline{w}_{n-1}|\leq(2\overline{\beta}\epsilon)^n\|\overline{w}_0\|,\quad n=1,2,3,\ldots,$$

hold, where

$$\overline{B} = \left(2C\cot(\theta)a_1 + a_2\right)\left\{\left|\frac{v}{r\overline{W}}\right| + \left|\frac{v}{u\overline{W}}\right|\right\},$$
$$\overline{M} := M_1\left\{\left|\frac{v}{r\overline{W}}\right| + \left|\frac{v}{u\overline{W}}\right|\right\} + M_2\left\{2\left|\frac{v}{\overline{W}}\right| + \left|\frac{v(u-r)}{u\overline{W}}\right|\right\},$$

and

$$\overline{\beta} := 2\overline{M}\overline{B},$$

with the same constants M_1 and M_2 defined in (2.12). Note that $g(x,x)$ and $g_t(x,x)$ are bounded and satisfy

$$\|g(x,x)\| \le \left|\frac{v}{\overline{W}}\right|, \quad \|g_t(x,x)\| \le \left|\frac{v(u-r)}{\overline{W}}\right|.$$

Then, the uniform convergence of

$$\overline{w}_0(x) + \sum_{n=1}^{\infty}\left(\overline{w}_n(x) - \overline{w}_{n-1}(x)\right)$$

is confirmed for $x \in (-\infty, \infty)$. Hence, we obtain the following theorem.

Theorem 2.3 *Assume that a and b, defined in* (2.3)*, satisfy* $\frac{b^2}{4} + \frac{a^3}{27} > 0$. *There exists an* $\epsilon_0 > 0$ *such that for* $\epsilon < \epsilon_0$, (2.1) *has a periodic solution* $h_0(x) = 1 + w(x)$, *where* $w(x)$ *is a solution of the differential equation* (2.2) *with the asymptotic expansion*

$$w(x) = \overline{w}_0(x) + \sum_{n=1}^{\infty}\left(\overline{w}_n(x) - \overline{w}_{n-1}(x)\right),$$

and $\overline{w}_n(x)$, $n = 0, 1, 2, \dots$, *are defined in* (2.23).

3 Conclusions

We analytically study the flow of a liquid film over an inclined periodic uneven wall governed by a long-wave model. The existence of a periodic steady-state solution is proved using asymptotic expansion.

We start by constructing an iteration scheme in terms of integral forms from this steady-state problem to find periodic solutions in the form $h_0(x) = 1 + w(x)$, where $w(x)$ is the solution to (2.2). Three distinct cases have been handled in terms of the values of R, C, and θ. For each case, we prove the existence and find an asymptotic formula for $w(x)$.

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