Proceedings of the Edinburgh Mathematical Society (2001) **44**, 241–248 ©

A NON-EXTENDABLE BOUNDED LINEAR MAP BETWEEN C^*-ALGEBRAS

NARUTAKA OZAWA

Texas A&M University, College Station, TX77843, USA and
University of Tokyo, Komaba, 153-8914, Japan (ozawa@math.tamu.edu)

(Received 10 September 1999)

Abstract We present an example of a C^*-subalgebra A of $\mathbb{B}(H)$ and a bounded linear map from A to $\mathbb{B}(K)$ which does not admit any bounded linear extension. This generalizes the result of Robertson and gives the answer to a problem raised by Pisier. Using the same idea, we compute the exactness constants of some Q-spaces. This solves a problem raised by Oikhberg. We also construct a Q-space which is not locally reflexive.

Keywords: bounded linear extensions; non-exact Q-spaces

AMS 2000 *Mathematics subject classification:* Primary 46L05
Secondary 46L07

1. A non-extendable bounded linear map

Definition 1.1. Let X be a closed subspace of a Banach space Y. We say X is complemented in Y if there is a bounded linear projection from Y onto X. We say X is weakly complemented in Y if there is a bounded linear map T from Y to X^{**} such that $T|_X = \iota_X$, where ι_X is the canonical inclusion map of X into X^{**}.

Let A be a C^*-algebra. If there is a faithful $*$-representation $A \subset \mathbb{B}(H)$ such that A is (weakly) complemented in $\mathbb{B}(H)$, then it follows from the injectivity of $\mathbb{B}(H)$ that for any faithful $*$-representation $A \subset \mathbb{B}(K)$, A is (weakly) complemented in $\mathbb{B}(K)$. A C^*-algebra A has the weak expectation property (WEP) [**13**] if for every faithful $*$-representation $A \subset \mathbb{B}(H)$, there is a complete contraction $T\colon \mathbb{B}(H) \to A^{**}$ such that $T|_A = \iota_A$.

Lemma 1.2. *Let A be a C^*-algebra and let $A \subset \mathbb{B}(H)$ be the universal representation, i.e. $\overline{A}^{\text{ultraweak}} = A^{**}$. If A is weakly complemented in $\mathbb{B}(H)$ and is locally reflexive, then A^{**} is complemented in $\mathbb{B}(H)$.*

Proof. Let $T\colon \mathbb{B}(H) \to A^{**}$ be a bounded linear map such that $T|_A = \mathrm{id}_A$. Let I be a set of all pairs $i = (E, F)$ consisting of finite-dimensional subspaces E of A^{**} and F of $\mathbb{B}(H)_*$. I is then directedly ordered by inclusions. Fix $i = (E, F)$ in I. Since A is locally reflexive, there is a map $S_i\colon E \to A$ with $\|S_i\|_{\text{cb}} \leqslant 1 + (1/\dim(E))$ such that

https://doi.org/10.1017/S0013091599000978 Published online by Cambridge University Press

$\langle S_i(e), f\rangle = \langle e, f\rangle$ for all $e \in E$ and $f \in F$. Since $\mathbb{B}(H)$ is injective, we can extend S_i to $\bar{S}_i\colon \mathbb{B}(H) \to \mathbb{B}(H)$ with $\|\bar{S}_i\|_{\mathrm{cb}} \leqslant \|S_i\|_{\mathrm{cb}}$. Define $T_i\colon \mathbb{B}(H) \to A^{**}$ by $T_i = T \circ \bar{S}_i$. We then have $\limsup \|T_i\| \leqslant \|T\|$ and $\lim\langle T_i(e), f\rangle = \langle e, f\rangle$ for all $e \in A^{**}$ and $f \in \mathbb{B}(H)_*$. Let $\tilde{T}\colon \mathbb{B}(H) \to A^{**}$ be a cluster point of the net $\{T_i\}_i$ in the point-ultraweak topology. $\tilde{T}$ is then the desired bounded linear projection. □

Next we use results due to De Cannière and Haagerup and due to Kirchberg. Let $C^*_\lambda(\mathbb{F}_n)$ be the reduced group C^*-algebra of the free group $\mathbb{F}_n$ with n generators ($n \geqslant 2$). Then, by [**3**], $C^*_\lambda(\mathbb{F}_n)$ has the complete metric approximation property. Thus, $C^*_\lambda(\mathbb{F}_n)$ is exact and *a fortiori* is locally reflexive [**12**]. By [**11**], there are a C^*-algebra B with the WEP and a surjective $*$-homomorphism π from B onto $C^*_\lambda(\mathbb{F}_n)$. Since $C^*_\lambda(\mathbb{F}_n)$ has the metric approximation property, there is a contractive linear lifting $\varphi\colon C^*_\lambda(\mathbb{F}_n) \to B$, i.e. $\pi \circ \varphi = \mathrm{id}_{C^*_\lambda(\mathbb{F}_n)}$. (There is even a unital k-positive lifting for each $k \in \mathbb{N}$ (see [**20**]).) We need one more ingredient due to Haagerup and Pisier. Let $\mathrm{VN}(\mathbb{F}_n)$ be the group von Neumann algebra of the free group $\mathbb{F}_n$ with n generators ($n \geqslant 2$) and let $VN(\mathbb{F}_n) \subset \mathbb{B}(K)$ be any faithful $*$-representation. $\mathrm{VN}(\mathbb{F}_n)$ is then not complemented in $\mathbb{B}(K)$ (see Corollary 4.9 in [**8**]).

Lemma 1.3. *Let $C^*_\lambda(\mathbb{F}_n) \subset \mathbb{B}(H)$ be any faithful $*$-representation. Then $C^*_\lambda(\mathbb{F}_n)$ is not weakly complemented in $\mathbb{B}(H)$.*

Proof. If $C^*_\lambda(\mathbb{F}_n)$ is weakly complemented in $\mathbb{B}(H)$, then by Lemma 1.2 and the preceding remarks, $C^*_\lambda(\mathbb{F}_n)^{**}$ is complemented in $\mathbb{B}(K)$ for any faithful $*$-representation $C^*_\lambda(\mathbb{F}_n)^{**} \subset \mathbb{B}(K)$. Since the group von Neumann algebra $\mathrm{VN}(\mathbb{F}_n)$ is complemented in $C^*_\lambda(\mathbb{F}_n)^{**}$, *a fortiori* it is complemented in $\mathbb{B}(K)$. This contradicts Corollary 4.9 in [**8**]. □

Theorem 1.4. *Let $C^*_\lambda(\mathbb{F}_n) \subset \mathbb{B}(H)$ be a faithful $*$-representation. Let $B \subset \mathbb{B}(K)$ be a C^*-subalgebra with the WEP and π be a surjective $*$-homomorphism from B onto $C^*_\lambda(\mathbb{F}_n)$. If $\varphi\colon C^*_\lambda(\mathbb{F}_n) \to B$ is a bounded linear lifting of π, then there is no bounded linear extension $\bar{\varphi}\colon \mathbb{B}(H) \to \mathbb{B}(K)$ of φ.*

Proof. Suppose that there is a bounded linear extension $\bar{\varphi}\colon \mathbb{B}(H) \to \mathbb{B}(K)$ of φ. Since B has the WEP, there is a bounded linear map $\psi\colon \mathbb{B}(K) \to B^{**}$ such that $\psi|_B = \iota_B$. Let us define $T\colon \mathbb{B}(H) \to C^*_\lambda(\mathbb{F}_n)^{**}$ by $T = \pi^{**} \circ \psi \circ \bar{\varphi}$. T is then a bounded linear map and $T|_{C^*_\lambda(\mathbb{F}_n)} = \iota_{C^*_\lambda(\mathbb{F}_n)}$. This contradicts Lemma 1.3. □

Remark 1.5. Since $C^*_\lambda(\mathbb{F}_n)$ is weakly complemented in $\mathrm{VN}(\mathbb{F}_n)$ (see the proof of Lemma 7.6 in [**11**]), there is a bounded linear map from $\mathrm{VN}(\mathbb{F}_n)$ to $\mathbb{B}(K)$ without bounded linear extension to $\mathbb{B}(\ell_2(\mathbb{F}_n))$.

Problem 1.6. Let A be a C^*-subalgebra of $\mathbb{B}(H)$ and assume that any bounded linear map from A to $\mathbb{B}(H)$ extends to a bounded linear map on $\mathbb{B}(H)$. Is A weakly complemented in $\mathbb{B}(H)$?

2. Computing exactness constants of some Q-spaces

We will compute the exactness constant of some Q-spaces. A Q-space is a quotient operator space of a minimal operator space. By the duality between maximal and minimal operator spaces, a dual operator space of a subspace of a maximal operator space is a Q-space. See [**1**] and [**2**] for details.

Let E be a finite-dimensional operator space. For any C^*-algebra B and any closed two-sided ideal J in B, there is a canonical isomorphism $T_E\colon (E\otimes B)/(E\otimes J)\to E\otimes(B/J)$, where $\otimes$ means the minimal tensor product. Let C be a constant. We say E is C-exact if $\|T_E^{-1}\|\leqslant C$ for all choices of B and J. By the canonical isometric identification $E\otimes X = CB(E^*, X)$ for an operator space X (see [**2**, **4**]), E is C-exact if any complete contraction from the dual operator space E^* to any quotient C^*-algebra B/J has a lifting with cb-norm $\leqslant C$. For an infinite-dimensional operator space X, we say X is C-exact if every finite-dimensional operator subspace of X is C-exact and we say X is exact if it is C-exact for some constant C. The exactness constant $\mathrm{ex}(X)$ of X is defined by $\mathrm{ex}(X)=\inf\{C : X \text{ is } C\text{-exact}\}$. See [**17**] for details.

Define an operator space $E_n\subset \mathbb{M}_n\oplus\mathbb{M}_n\subset\mathbb{M}_{2n}$ by $E_n=\operatorname{span}\{e_{k1}\oplus e_{1k} : k=1,2,\dots,n\}$, where $\{e_{jk}\}$ is a standard matrix unit in $\mathbb{M}_n$. By Proposition 1.3 in [**8**], there are two maps $w\colon E_n\to C^*_\lambda(\mathbb{F}_n)$ and $v\colon C^*_\lambda(\mathbb{F}_n)\to E_n$ such that $v\circ w=\mathrm{id}_{E_n}$ and $\|v\|_{\mathrm{cb}}\leqslant 2$, $\|w\|_{\mathrm{cb}}\leqslant 1$. By Lemma 4.2 in [**8**], any projection P from $\mathbb{M}_{2n}$ onto E_n has cb-norm $\geqslant \frac{1}{2}(1+\sqrt{n})$. By Smith's lemma (Theorem 2.1 in [**21**]), we have $\|P\|_{\mathrm{cb}}=\|P\|_{2n}$ for any map $P\colon \mathbb{M}_{2n}\to E_n$. Hence, by a standard averaging argument (see [**8**, **20**]), we have $\|P\|\geqslant\frac{1}{2}(1+\sqrt{n})$ for any projection P from $\mathbb{M}_{2n}(\mathbb{M}_{2n})$ onto $\mathbb{M}_{2n}(E_n)$. On the other hand, by Remark 4.3 in [**8**], there is a projection Q from $\mathbb{M}_{2n}(\mathbb{M}_{2n})$ onto $\mathbb{M}_{2n}(E_n)$ with $\|Q\|=\frac{1}{2}(1+\sqrt{n})$. See [**8**] for details.

Theorem 2.1. *We equip $\mathbb{M}_{2n}(E_n)$ with a new operator space structure induced by the canonical embedding into $\max(\mathbb{M}_{2n}(\mathbb{M}_{2n}))$ and denote the resultant operator space by F_n, i.e. $F_n=\mathbb{M}_{2n}(E_n)$ as a Banach space and $F_n\subset\max(\mathbb{M}_{2n}(\mathbb{M}_{2n}))$ as an operator space. F_n^* is then a $4n^3$-dimensional Q-space such that*

$$\tfrac{1}{4}(1+\sqrt{n})\leqslant \mathrm{ex}(F_n^*)\leqslant \|\mathrm{id}\colon \min(F_n^*)\to F_n^*\|_{\mathrm{cb}}\leqslant\tfrac{1}{2}(1+\sqrt{n}).$$

Proof. We note that the formal identity $J\colon F_n\to\mathbb{M}_{2n}(E_n)$ is completely contractive. Let v and w be as in the preceding remarks and let $\tilde{w}\colon F_n\to\mathbb{M}_{2n}(C^*_\lambda(\mathbb{F}_n))$ be a complete contraction defined by $\tilde{w}=(\mathrm{id}_{\mathbb{M}_{2n}}\otimes w)\circ J$. By [**11**], there are a C^*-algebra B with the WEP and a surjective $*$-homomorphism π from B onto $\mathbb{M}_{2n}(C^*_\lambda(\mathbb{F}_n))$. Suppose that F_n^* is C-exact. By definition, there is a lifting $\varphi\colon F_n\to B$ of $\tilde{w}$ with $\|\varphi\|_{\mathrm{cb}}\leqslant C$. Since B has the WEP, φ extends to $\bar{\varphi}\colon \max(\mathbb{M}_{2n}(\mathbb{M}_{2n}))\to B^{**}$ with $\|\bar{\varphi}\|_{\mathrm{cb}}\leqslant C$. Let us define $P\colon\mathbb{M}_{2n}(\mathbb{M}_{2n})\to\mathbb{M}_{2n}(E_n)$ by $P=(\mathrm{id}_{\mathbb{M}_{2n}}\otimes v)^{**}\circ\pi^{**}\circ\bar{\varphi}$. P is then a projection with $\|P\|\leqslant 2C$. Thus, we have $C\geqslant\frac{1}{4}(1+\sqrt{n})$. This proves the first inequality. Since $\mathrm{ex}(\min(F_n^*))=1$, we have the second inequality.

Next, let Q be a projection as in the preceding remarks. We then have

$$\|\mathrm{id}\colon F_n\to\max(F_n)\|_{\mathrm{cb}}\leqslant\|Q\colon \max(\mathbb{M}_{2n}(\mathbb{M}_{2n}))\to\max(F_n)\|_{\mathrm{cb}}=\|Q\|=\tfrac{1}{2}(1+\sqrt{n}).$$

https://doi.org/10.1017/S0013091599000978 Published online by Cambridge University Press

Taking the dual of this identity map, we obtain the third inequality. □

Since an ℓ_∞-sum of Q-spaces is also a Q-space, we obtain the next corollary.

Corollary 2.2. *There is a Q-space which is not exact.*

Problem 2.3. What is the asymptotic behaviour of the constant

$$\sup\{\mathrm{ex}(E) : E \text{ an } n\text{-dimensional Q-space}\}$$

as n tends to infinity?

3. A Q-space which is not locally reflexive

Definition 3.1. Let $C \geqslant 1$ be a constant. We say an operator space X is C-locally reflexive if for any finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$ and any $\varepsilon > 0$, there is a map $\varphi\colon E \to X$ with $\|\varphi\|_{\mathrm{cb}} < C + \varepsilon$ such that $\langle \varphi(e), f\rangle = \langle e, f\rangle$ for all $e \in E$ and $f \in F$. We say an operator space X is locally reflexive if X is C-locally reflexive for some constant C.

We note that a C^*-algebra is 1-locally reflexive if it is locally reflexive and that a subspace of a C-locally reflexive operator space is also C-locally reflexive. See [**5**] for details. Now, let us construct a Q-space which is not locally reflexive. First, we need a lemma due to Oikhberg. Let us recall that the cb version of the Banach–Mazur distance between two completely isomorphic operator spaces X and Y is defined by

$$d_{\mathrm{cb}}(X, Y) = \inf\{\|\varphi\|_{\mathrm{cb}}\|\varphi^{-1}\|_{\mathrm{cb}} : \varphi \text{ a completely bounded isomorphism from } X \text{ onto } Y\}.$$

Put $d_{\mathrm{cb}}(X, Y) = \infty$ if X and Y are not completely isomorphic.

Lemma 3.2 (Lemma 3.4 from [15]). *For every $C' > 0$, there is a finite-dimensional subspace $F \subset \max(\mathbb{B}(H))$ such that $d_{\mathrm{cb}}(F, G) > C'$ for all n and $G \subset \max(\mathbb{M}_n)$.*

Taking the dual of the inclusion $F \subset \max(\mathbb{B}(H))$ we obtain a complete metric surjection $q\colon \min(S_1) \to E$, where $E = F^*$. By the above lemma and a small perturbation argument, we obtain the following lemma.

Lemma 3.3. *For every $C' > 0$ there is a finite-rank complete metric surjection $q\colon \min(S_1) \to E$ such that $d_{\mathrm{cb}}(E, F/(\ker q \cap F)) > C'$ for all finite-dimensional subspaces $F \subset \min(S_1)$.*

We now prove the following lemma.

Lemma 3.4. *For every $C > 0$ there is a Q-space which is not C-locally reflexive.*

Proof. Fix $C' > C$ and take a finite-rank complete metric surjection $q\colon \min(S_1) \to E$ as in Lemma 3.3. Let $\{F_n\}$ be an increasing sequence of finite-dimensional subspaces of $\min(S_1)$ such that $\overline{\bigcup F_n} = \min(S_1)$ and $q(F_1) = E$. Let $E_n = F_n/(\ker q \cap F_n)$ and let $\varphi_n\colon E_n \to E$ be the complete contraction induced by $q|_{F_n}\colon F_n \to E$. By Lemma 3.3, we

https://doi.org/10.1017/S0013091599000978 Published online by Cambridge University Press

have $\|\varphi_n^{-1}\|_{\mathrm{cb}} > C'$ for all n. On the other hand, it can be seen that $\lim_{n\to\infty}\|\varphi_n^{-1}\|_k = 1$ for all $k \in \mathbb{N}$. Finally, let X be an operator space defined by

$$X = \left\{(x_n) \in \left(\prod E_n\right)_{\ell_\infty} : \lim_{n\to\infty} \varphi_n(x_n) \text{ exists in } E\right\}.$$

Since all the E_n are Q-spaces, X is also a Q-space. We will show that X is not C-locally reflexive. There is a natural map $\varphi\colon X \to E$ defined by $\varphi((x_n)) = \lim \varphi_n(x_n)$. For each n, define $\psi_n\colon E \to X$ by

$$\psi_n(x) = (0, \dots, 0, \varphi_n^{-1}(x), \varphi_{n+1}^{-1}(x), \dots).$$

Since $\varphi_m^{-1} \circ \varphi_n\colon E_n \to E_m$ is completely contractive for all $m \geqslant n$, we have

$$\lim_{n\to\infty} \|\psi_n\|_k = \lim_{n\to\infty} \|\varphi_n^{-1}\|_k = 1$$

for all $k \in \mathbb{N}$. Let $\psi\colon E \to X^{**}$ be a cluster point of the sequence $\{\psi_n\}_n$ in the point-weak* topology. Then, by the previous argument, we have $\|\psi\|_{\mathrm{cb}} \leqslant 1$. Since $\varphi \circ \psi_n = \mathrm{id}_E$ for all n, we have $\varphi^{**} \circ \psi = \mathrm{id}_E$. Now, suppose that X is C-locally reflexive. Since $\varphi\colon X \to E$ is of finite rank, applying the local reflexivity to the complete contraction $\psi\colon E \to X^{**}$, we obtain a map $\theta\colon E \to X$ with $\|\theta\|_{\mathrm{cb}} < C'$ such that $\varphi \circ \theta = \mathrm{id}_E$. Let $\theta_n\colon E \to E_n$ be the 'nth coordinate' of θ. Then, by the definition of φ, we have

$$\lim_{n\to\infty} \varphi_n \circ \theta_n(x) = x$$

for all $x \in E$. Since E is finite dimensional, we have

$$\limsup_{n\to\infty} \|\varphi_n^{-1}\|_{\mathrm{cb}} \leqslant \limsup_{n\to\infty} (\|\theta_n\|_{\mathrm{cb}} + \dim(E)\|\varphi_n^{-1} - \theta_n\|) \leqslant \|\theta\|_{\mathrm{cb}} < C'.$$

This contradicts the choice of E. □

Theorem 3.5. *There is a Q-space which is not locally reflexive.*

Proof. For each n, there is a Q-space X_n which is not n-locally reflexive by Lemma 3.4. Define a Q-space Y by $Y = (\bigoplus X_n)_{c_0}$. It is easy to see that Y is not locally reflexive. □

Problem 3.6. In the proof of Lemma 3.4, it can be seen that $\mathrm{ex}(X) = \sup \mathrm{ex}(E_n)$. Can we control this value?

4. Crude representability and local reflexivity

Let $(\bigoplus X_n)_{\ell_1}$ be the ℓ_1-direct sum of a sequence $\{X_n\}$ of operator spaces. We equip $(\bigoplus X_n)_{\ell_1}$ with the natural operator space structure (see pp. 34–36 in [**18**]). $(\bigoplus X_n)_{\ell_1}$ is then an operator space with the following properties. If $\varphi_n\colon X_n \to \mathbb{B}(H)$ is a complete contraction for all n, then $\varphi\colon (\bigoplus X_n)_{\ell_1} \ni (x_n) \mapsto \sum \varphi_n(x_n) \in \mathbb{B}(H)$ is a complete contraction. We have a completely isometric identity $(\bigoplus X_n)_{\ell_1}^* = (\prod X_n^*)_{\ell_\infty}$; and if $Y_n \subset X_n$ for all n, then we have $(\bigoplus Y_n)_{\ell_1} \subset (\bigoplus X_n)_{\ell_1}$ completely isometrically. When $X_n = X$ for all n, we simply denote $(\bigoplus X_n)_{\ell_1}$ by $\ell_1(X)$.

https://doi.org/10.1017/S0013091599000978 Published online by Cambridge University Press

Lemma 4.1. *Let X be a separable operator space and let $\{E_n\}$ be an increasing sequence of finite-dimensional subspaces of X such that $\overline{\bigcup E_n} = X$. If $Y = (\bigoplus E_n)_{\ell_1}$ is C-locally reflexive, then so is X.*

Proof. We follow the construction due to Johnson [**9**]. Let $q\colon Y \to X$ be a complete metric surjection defined by $q((x_n)) = \sum_{n=1}^{\infty} x_n$. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define a map $r\colon Y^* \to X^*$ by

$$\langle r(f), x\rangle = \lim_{\mathcal{U}} \langle f_n, x\rangle$$

for all $f = (f_n) \in Y^* = (\prod E_n^*)_{\ell_\infty}$ and $x \in \bigcup E_n$. Then, r is a well-defined complete contraction and $r \circ q^* = \mathrm{id}_{X^*}$. To prove that X is C-locally reflexive, we give ourselves finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$ and $\varepsilon > 0$. Let $\tilde{E} = r^*(E) \subset Y^{**}$ and let $\tilde{F} = q^*(F) \subset Y^*$. Since Y is C-locally reflexive, there is a map $\varphi\colon \tilde{E} \to Y$ with $\|\varphi\|_{\mathrm{cb}} \leqslant C+\varepsilon$ such that $\langle \varphi(\tilde{e}), \tilde{f}\rangle = \langle \tilde{e}, \tilde{f}\rangle$ for all $\tilde{e} \in \tilde{E}$ and $\tilde{f} \in \tilde{F}$. Now, define $\psi\colon E \to X$ by $\psi = q \circ \varphi \circ (r^*|_E)$. Then, we have $\|\psi\|_{\mathrm{cb}} \leqslant C + \varepsilon$ and

$$\begin{aligned}\langle \psi(e), f\rangle &= \langle \varphi(r^*(e)), q^*(f)\rangle \\ &= \langle r^*(e), q^*(f)\rangle \\ &= \langle e, r \circ q^*(f)\rangle \\ &= \langle e, f\rangle\end{aligned}$$

for all $e \in E$ and $f \in F$. This completes the proof. □

Lemma 4.2 (Theorem 4.3 in [7]). *An operator space X is locally reflexive if every separable subspace of X is locally reflexive.*

Proof. The 'isometric' version of this lemma has been proved in [**7**]. Observe that the assumption implies that there is a constant C so that every separable subspace of X is C-locally reflexive. Now the proof of C-local reflexivity of X is almost same as the proof of Theorem 4.3 in [**7**]. □

Let Z and X be operator spaces. We say X is crudely representable in Z if there is a constant C such that for any finite-dimensional subspace E of X, there is a subspace F of Z with $d_{\mathrm{cb}}(F, E) < C$.

Theorem 4.3. *Let Z be an operator space such that Z contains a completely isomorphic copy of $\ell_1(Z)$. Assume that Z is locally reflexive. If X is an operator space which is crudely representable in Z, then X is locally reflexive.*

Proof. By Lemma 4.2, we may assume that X is separable. Take an increasing sequence $\{E_n\}$ of finite-dimensional subspaces of X with $\overline{\bigcup E_n} = X$. Since X is crudely representable in Z, $Y = (\bigoplus E_n)_{\ell_1}$ can be embedded into $\ell_1(Z)$ completely isomorphically. Since a subspace of locally reflexive operator space is also locally reflexive, by Lemma 4.1, we are done. □

https://doi.org/10.1017/S0013091599000978 Published online by Cambridge University Press

Remark 4.4. In [**10**], Junge has proved that the operator space S_1 of trace class operators satisfies the assumption of Theorem 4.3 and the consequence is already known [**6**]. There is a locally reflexive operator space X such that $\ell_1(X)$ is not locally reflexive. Indeed, if V is a separable operator space which is not locally reflexive and $\{E_n\}$ is an increasing sequence of finite-dimensional subspaces with $\overline{\bigcup E_n} = V$, then $X = (\bigoplus E_n)_{c_0}$ is locally reflexive, but $\ell_1(X)$ is not locally reflexive. This answers a question raised by Le Merdy (personal communication). There is an 'ℓ_∞-version' of Theorem 4.3 (use Lusky's construction [**14**] at Lemma 4.1), but it seems that for only few operator spaces X, $\ell_\infty(X)$ is locally reflexive.

Problem 4.5. Are exact operator spaces necessarily locally reflexive?

It has been shown in Corollary 4.8 in [**7**] that every 1-exact operator space is 1-locally reflexive.

Acknowledgements. The author is supported by JSPS. The author thanks Professor Gilles Pisier for introducing him to the problem and for a stimulating conversation.

References

1. D. Blecher, The standard dual of an operator space, *Pacific J. Math.* **153** (1992), 15–30.
2. D. Blecher and V. Paulsen, Tensor products of operator spaces, *J. Funct. Analysis* **99** (1991), 262–292.
3. J. De Cannière and U. Haagerup, Multipliers of the Fourier algebras of some Lie groups and their discrete subgroups, *Am. J. Math.* **107** (1985), 455–500.
4. E. Effros and Z.-J. Ruan, A new approach to operator spaces, *Can. Math. Bull.* **34** (1991), 329–337.
5. E. Effros and Z.-J. Ruan, Mapping spaces and liftings for operator spaces, *Proc. Lond. Math. Soc.* **69** (1994), 171–197.
6. E. Effros, M. Junge and Z.-J. Ruan, Integral mappings and the principle of local reflexivity for non-commutative L^1-spaces, *Ann. Math.* (2) **151** (2000), 59–92.
7. E. Effros, N. Ozawa and Z.-J. Ruan, On injectivity and nuclearity for operator spaces, *Duke Math. J.*, in press.
8. U. Haagerup and G. Pisier, Bounded linear operators between C^*-algebras, *Duke Math. J.* **71** (1993), 889–925.
9. W. B. Johnson, A complementary universal conjugate Banach space and its relation to the approximation problem, *Israel J. Math.* **13** (1972), 301–310.
10. M. Junge, *Factorization theory for spaces of operators* (Habilitationsschrift, Universität Kiel, 1996).
11. E. Kirchberg, On non-semisplit extensions, tensor products and exactness of group C^*-algebras, *Invent. Math.* **112** (1993), 449–489.
12. E. Kirchberg, On subalgebras of the CAR-algebra, *J. Funct. Analysis* **129** (1995), 35–63.
13. C. Lance, On nuclear C^*-algebras, *J. Funct. Analysis* **12** (1973), 157–176.
14. W. Lusky, A note on Banach spaces containing c_0 or C_∞, *J. Funct. Analysis* **62** (1985), 1–7.
15. T. Oikhberg, Subspaces of maximal operator spaces, preprint.
16. G. Pisier, Remarks on complemented subspaces of von Neumann algebras, *Proc. R. Soc. Edinb.* A **121** (1992), 1–4.

17. G. Pisier, Exact operator spaces, in Recent Advances in Operator Algebras, Orléans 1992, *Astérisque Soc. Math. France* **232** (1995), 159–186.
18. G. Pisier, Non-commutative vector valued L_p-spaces and completely p-summing maps, *Astérisque Soc. Math. France* **247** (1998), vi + 131 pp.
19. A. G. Robertson, A non-extendible positive map on the reduced C^*-algebra of a free group, *Bull. Lond. Math. Soc.* **18** (1986), 389–391.
20. A. G. Robertson and R. Smith, Lifting and extension of maps on C^*-algebras, *J. Operat. Theory* **21** (1989), 117–131.
21. R. Smith, Completely bounded maps between C^*-algebras, *J. Lond. Math. Soc.* **27** (1983), 157–166.