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LEFT AND RIGHT EIGENVECTORS OF A VARIANT OF THE SYLVESTER–KAC MATRIX

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Abstract

As an extension of Sylvester's matrix, a tridiagonal matrix is investigated by determining both left and right eigenvectors. Orthogonality relations between left and right eigenvectors are derived. Two determinants of the matrices constructed by the left and right eigenvectors are evaluated in closed form.

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1. Introduction and motivation

Tridiagonal matrices are an important class of matrices in mathematics and physics (see [1, 8, 13, 17, 18, 21]). One particular case whose determinant evaluation was conjectured (without proof) by Sylvester [19, page 305] is

$$\det_{0\le i,j\le m}[\tau_{i,j}] = \prod_{k=0}^{m}(x+m-2k),$$

where the matrix entries are given by

$$\tau_{i,j} = \begin{cases} x & \text{if } i=j, \\ j & \text{if } j-i=1, \\ m-j & \text{if } i-j=1, \\ 0 & \text{if } |i-j|>1. \end{cases}$$

For this elegant result, there exist a number of generalisations and applications (see, for example, [2, 4, 9–11, 14–16, 20]). However, eigenvectors have only been found for a few related tridiagonal matrices (see [3, 6, 7, 12]).

The first proof of Sylvester's determinant formula is attributed by Muir [17, page 442] to Francesco Mazza in 1866. However, Kac [14] was perhaps the first to give a complete proof of the formula claimed by Sylvester and provided a polynomial characterisation of the eigenvectors through the generating function approach. Therefore, the matrix $[\tau_{i,j}]$ is also called the Sylvester–Kac matrix.

The first author [5] examined the following extended matrix (here, y has been replaced by $2y$ to avoid rational expressions for the eigenvalues). For two free variables $x,\ y$, we define the tridiagonal matrix of order $m+1$ by

$$\Omega_m(x,y) = [\sigma_{i,j}(x,y)]_{0\le i,j\le m},$$

where

$$\sigma_{i,j}(x,y) = \begin{cases} x+2jy & \text{if } i-j=0,\\ m-j & \text{if } i-j=1,\\ j & \text{if } j-i=1,\\ 0 & \text{if } |i-j|>1. \end{cases}$$

For instance, $\Omega_5(x,y)$ is illustrated as follows (where zeros are omitted).

$$\Omega_5(x,y) = \begin{bmatrix} x & 1 & & & & \\ 5 & x+2y & 2 & & & \\ & 4 & x+4y & 3 & & \\ & & 3 & x+6y & 4 & \\ & & & 2 & x+8y & 5 \\ & & & & 1 & x+10y \end{bmatrix}$$

The eigenvalues of $\Omega_m(x,y)$ were explicitly determined in [5]): that is,

$$\mathcal{P}_{2n+\delta}(x,y) = \Big\{\lambda_k := x + (2n+\delta)y + (2k-\delta)\sqrt{1+y^2}\Big\}_{k=-n}^{n+\delta},$$

where we write $m = \delta + 2n$ with $\delta = 0,\ 1$, in accordance with the parity of m.

In mathematics, physics and applied sciences, it is important to determine, not only determinants and eigenvalues, but also eigenvectors for certain classes of matrices. The aim of the present paper is to determine the left and right eigenvectors of $\Omega_m(x,y)$. Our findings may potentially serve as testing samples (in the sense of [8]) to assess the numerical accuracy of algorithms for computations on similar matrices.

The paper is organised as follows. In the next section, the eigenvectors of $\Omega_m(x,y)$ are determined explicitly by following the same approach as in [6]. In Section 3, we prove orthogonality relations between the left and right eigenvectors. Finally in Section 4, we evaluate, in closed form, the two determinants constructed, respectively, by the left eigenvectors and the right ones.

2. Left and right eigenvectors

For the sake of brevity, define the algebraic function ρ by

$$\rho = y + \sqrt{1+y^2}.$$

Then the eigenvectors of $\Omega_m(x, y)$ are determined by the following theorem.

THEOREM 2.1. *Let λ_k be an eigenvalue of $\Omega_m(x, y)$ with $-n \le k \le n+\delta$. Then the following two statements hold.*

(a) *The vector $\boldsymbol{u}_k = (u_k(0), u_k(1), \dots, u_k(m))$ is a left eigenvector corresponding to the eigenvalue λ_k, where $u_k(j)$ is defined by the binomial sum*

$$u_k(j) = \sum_{\ell=0}^{j}(-1)^{j+\ell}\binom{j}{\ell}\binom{\delta+2n-j}{n+k-\ell}\rho^{2\ell-j}.$$

(b) *The vector $\boldsymbol{v}_k = (v_k(0), v_k(1), \dots, v_k(m))$ is a right eigenvector corresponding to the eigenvalue λ_k, where $v_k(i)$ is defined by the binomial sum*

$$v_k(i) = \sum_{\ell=0}^{i}(-1)^{i+\ell}\binom{n+k}{\ell}\binom{\delta+n-k}{i-\ell}\rho^{2\ell-i}.$$

PROOF. We begin by showing that $\mathbf{u}_k$ is the left eigenvector corresponding to the eigenvalue λ_k. It suffices to prove that, for each pair (k, j),

$$\begin{aligned}\lambda_k u_k(j) &= \sum_{i=0}^{m}\sigma_{i,j}(x, y)u_k(i)\\ &= \sigma_{j,j}(x, y)u_k(j) + \sigma_{j-1,j}(x, y)u_k(j-1) + \sigma_{j+1,j}(x, y)u_k(j+1),\end{aligned}$$

which can be, equivalently, restated as

$$(\lambda_k - x - 2jy)u_k(j) = ju_k(j-1) + (m-j)u_k(j+1). \tag{2.1}$$

Observing the functional relations

$$\rho = y + \sqrt{1+y^2} \quad \rightleftharpoons \quad y = \frac{\rho^2-1}{2\rho},$$

we can manipulate the expression

$$\begin{aligned}\lambda_k - x - 2jy &= (2n-2j+\delta)y + (2k-\delta)\sqrt{1+y^2}\\ &= (2n-2j-2k+2\delta)y + (2k-\delta)\rho,\end{aligned}$$

which leads to the useful relation

$$\lambda_k - x - 2jy = (n-j+k)\rho - (n-j-k+\delta)\rho^{-1}. \tag{2.2}$$

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According to the definition, the left-hand side of (2.1) can be written as

$$(\lambda_k - x - 2jy)u_k(j) = \mathrm{P} - \mathrm{Q},$$

where

$$\mathrm{P} = (n - j + k)\sum_{\ell=0}^{j}(-1)^{j+\ell}\binom{j}{\ell}\binom{\delta + 2n - j}{n + k - \ell}\rho^{2\ell-j+1},$$

$$\mathrm{Q} = (n - j - k + \delta)\sum_{\ell=0}^{j}(-1)^{j+\ell}\binom{j}{\ell}\binom{\delta + 2n - j}{n + k - \ell}\rho^{2\ell-j-1}.$$

Consequently, the equality (2.1) can be restated as

$$\mathrm{P} - ju_k(j-1) = \mathrm{Q} + (\delta + 2n - j)u_k(j+1). \tag{2.3}$$

The two sums on the left can be combined as

$$\begin{aligned}
\mathrm{P} - ju_k(j-1) &= \sum_{\ell=0}^{j}(-1)^{j+\ell}\binom{j}{\ell}\binom{\delta + 2n - j}{n + k - \ell}\rho^{2\ell-j+1} \\
&\quad\times\left\{(n - j + k) + \frac{(j-\ell)(1 + \delta + 2n - j)}{1 + \delta + n - j - k + \ell}\right\} \\
&= \sum_{\ell=0}^{j}(-1)^{j+\ell}\binom{j}{\ell}\binom{\delta + 2n - j}{n + k - \ell}\rho^{2\ell-j+1}\frac{(n + k - \ell)(1 + \delta + n - k)}{1 + \delta + n - j - k + \ell} \\
&= (1 + \delta + n - k)\sum_{\ell=0}^{j}(-1)^{j+\ell}\binom{j}{\ell}\binom{\delta + 2n - j}{n + k - \ell - 1}\rho^{2\ell-j+1},
\end{aligned}$$

while, similarly, for the two sums on the right,

$$\begin{aligned}
\mathrm{Q} + (\delta + 2n - j)u_k(j+1) &= \sum_{\ell=0}^{j+1}(-1)^{j+\ell}\binom{j}{\ell}\binom{\delta + 2n - j}{n + k - \ell}\rho^{2\ell-j-1} \\
&\quad\times\left\{(n - j - k + \delta) - \frac{(j+1)(\delta + n - j - k + \ell)}{1 + j - \ell}\right\} \\
&= \sum_{\ell=0}^{j+1}(-1)^{j+\ell-1}\binom{j}{\ell}\binom{\delta + 2n - j}{n + k - \ell}\rho^{2\ell-j-1}\frac{\ell(1 + \delta + n - k)}{1 + j - \ell} \\
&= (1 + \delta + n - k)\sum_{\ell=0}^{j+1}(-1)^{j+\ell-1}\binom{j}{\ell - 1}\binom{\delta + 2n - j}{n + k - \ell}\rho^{2\ell-j-1}.
\end{aligned}$$

Shifting forward the summation index $\ell \to 1 + \ell$ for the last sum, we see that the above two expressions for both sides of equation (2.3) coincide. This confirms item (a).

Likewise, (b) will be confirmed if we can show that $\mathbf{v}_k$ is the right eigenvector corresponding to λ_k. This can be done by showing, for each pair (k, i), that

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$$\begin{aligned}\lambda_k v_k(i) &= \sum_{j=0}^{m} \sigma_{i,j}(x,y) v_k(j)\\ &= \sigma_{i,i-1}(x,y)v_k(i-1) + \sigma_{i,i}(x,y)v_k(i) + \sigma_{i,i+1}(x,y)v_k(i+1),\end{aligned}$$

which can be, equivalently, restated as

$$(\lambda_k - x - 2iy)v_k(i) = (m-i+1)v_k(i-1) + (i+1)v_k(i+1). \tag{2.4}$$

Keeping in mind (2.2), we can express the left-hand side of (2.4) as

$$(\lambda_k - x - 2iy)v_k(i) = \mathcal{P} - \mathcal{Q},$$

where

$$\mathcal{P} = (n-i+k)\sum_{\ell=0}^{i}(-1)^{i+\ell}\binom{n+k}{\ell}\binom{\delta+n-k}{i-\ell}\rho^{2\ell-i+1},$$

$$\mathcal{Q} = (n-i-k+\delta)\sum_{\ell=0}^{i}(-1)^{i+\ell}\binom{n+k}{\ell}\binom{\delta+n-k}{i-\ell}\rho^{2\ell-i-1}.$$

Therefore, (2.4) can be reformulated as

$$\mathcal{P} - (\delta+2n-i+1)v_k(i-1) = \mathcal{Q} + (i+1)v_k(i+1).$$

According to the definitions, we first simplify the left-hand side of this equation,

$$\begin{aligned}&\mathcal{P} - (\delta+2n-i+1)v_k(i-1)\\ &= \sum_{\ell=0}^{i}(-1)^{i+\ell}\binom{n+k}{\ell}\binom{\delta+n-k}{i-\ell}\rho^{2\ell-i+1} \times \Big\{(n-i+k) - \frac{(\delta+2n-i+1)(i-\ell)}{\delta+n-k-i+\ell+1}\Big\}\\ &= \sum_{\ell=0}^{i}(-1)^{i+\ell}\binom{n+k}{\ell}\binom{\delta+n-k}{i-\ell}\rho^{2\ell-i+1}\frac{(n+k-\ell)(1+\delta+n-k)}{\delta+n-k-i+\ell+1}\\ &= (n+k)\sum_{\ell=0}^{i}(-1)^{i+\ell}\binom{n+k-1}{\ell}\binom{1+\delta+n-k}{i-\ell}\rho^{2\ell-i+1},\end{aligned}$$

and then the right-hand side of the same equation,

$$\begin{aligned}&\mathcal{Q} + (i+1)v_k(i+1)\\ &= \sum_{\ell=0}^{i+1}(-1)^{i+\ell}\binom{n+k}{\ell}\binom{\delta+n-k}{i-\ell}\rho^{2\ell-i-1} \times \Big\{(n-i-k+\delta) - \frac{(i+1)(\delta+n-i-k+\ell)}{1+i-\ell}\Big\}\\ &= \sum_{\ell=0}^{i+1}(-1)^{i+\ell-1}\binom{n+k}{\ell}\binom{\delta+n-k}{i-\ell}\rho^{2\ell-i-1}\frac{\ell(1+\delta+n-k)}{1+i-\ell}\\ &= (n+k)\sum_{\ell=0}^{i+1}(-1)^{i+\ell-1}\binom{n+k-1}{\ell-1}\binom{1+\delta+n-k}{1+i-\ell}\rho^{2\ell-i-1}.\end{aligned}$$

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The last two expressions become identical when the replacement $\ell \to 1+\ell$ is made in the latter one. This shows that $\mathbf{v}_k$ is indeed the right eigenvector of the matrix $M_{\delta+2n}(x,y)$ corresponding to λ_k. □

3. Orthogonality relations

For two vectors $\mathbf{u}$ and $\mathbf{v}$ of the same dimension, denote their usual scalar product by $\langle \mathbf{u}, \mathbf{v}\rangle$. The next theorem highlights orthogonality relations between the left and right eigenvectors of $\Omega_m(x,y)$.

THEOREM 3.1. *Assume $\boldsymbol{u}_i$ and $\boldsymbol{v}_j$ as in Theorem 2.1. Then the following orthogonality relations hold for all i and j subject to $-n \le i, j \le n+\delta$.*

$$\langle \boldsymbol{u}_i, \boldsymbol{v}_j\rangle = 2^m \frac{(1+y\rho)^m}{\rho^{2m-2n-2k}} \chi(i=j),$$

where χ is the logical function defined by $\chi(true) = 1$ and $\chi(false) = 0$.

PROOF. To prove these orthogonality relations, write the scalar product as the triple sum

$$\begin{aligned}\langle \mathbf{u}_i, \mathbf{v}_j\rangle &= \sum_{k=0}^{m} u_i(k) v_j(k) \\ &= \sum_{k=0}^{\delta+2n} \sum_{\imath=0}^{k} (-1)^{k+\imath} \binom{k}{\imath} \binom{\delta+2n-k}{n+i-\imath} \rho^{2\imath-k} \sum_{\jmath=0}^{k} (-1)^{k+\jmath} \binom{n+j}{\jmath} \binom{\delta+n-j}{k-\jmath} \rho^{2\jmath-k}. \qquad (3.1)\end{aligned}$$

Observing that

$$\begin{aligned}\mathbf{u}_i \Omega_m(x,y) \mathbf{v}_j &= \langle \mathbf{u}_i \Omega_m(x,y), \mathbf{v}_j\rangle = \lambda_i \langle \mathbf{u}_i, \mathbf{v}_j\rangle \\ &= \langle \mathbf{u}_i, \Omega_m(x,y)\mathbf{v}_j\rangle = \lambda_j \langle \mathbf{u}_i, \mathbf{v}_j\rangle,\end{aligned}$$

immediately, we have

$$\langle \mathbf{u}_i, \mathbf{v}_j\rangle = 0 \quad \text{when } i \ne j \quad \text{for } \lambda_i \ne \lambda_j.$$

When $i = j = \ell$, the scalar product (3.1) can be reformulated, by first making the replacement $\jmath \to k - \jmath$ and then interchanging the order of the triple sum, as

$$\begin{aligned}\langle \mathbf{u}_\ell, \mathbf{v}_\ell\rangle &= \sum_{k=0}^{\delta+2n} \sum_{\imath=0}^{k} (-1)^{k+\imath} \binom{k}{\imath} \binom{\delta+2n-k}{n+\ell-\imath} \rho^{2\imath-k} \sum_{\jmath=0}^{k} (-1)^{\jmath} \binom{n+\ell}{k-\jmath} \binom{\delta+n-\ell}{\jmath} \rho^{k-2\jmath} \\ &= \sum_{\imath,\jmath=0}^{\delta+2n} (-1)^{\imath+\jmath} \binom{\delta+n-\ell}{\jmath} \rho^{2\imath-2\jmath} \sum_{k=\max\{\imath,\jmath\}}^{\delta+2n} (-1)^k \binom{k}{\imath} \binom{\delta+2n-k}{n+\ell-\imath} \binom{n+\ell}{k-\jmath}.\end{aligned}$$

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The inner sum with respect to k can be reformulated under $k \to \kappa + \jmath$ and then evaluated in closed form as

$$\sum_{\kappa=0}^{\delta+2n-\jmath} (-1)^{\kappa+\jmath} \binom{n+\ell}{\jmath} \binom{\kappa+\jmath}{\imath} \binom{\delta+2n-\kappa-\jmath}{n+\ell-\imath} = (-1)^{\jmath} \binom{n+\ell}{\imath},$$

since the last sum results substantially, apart from an alternating sign, in the finite differences of order $n+\ell$ for a polynomial of the same degree $n+\ell$.

Therefore, the triple sum is reduced to a double one and can be evaluated further by the binomial theorem: that is,

$$\langle \mathbf{u}_\ell, \mathbf{v}_\ell \rangle = \sum_{\imath=0}^{\delta+2n} \binom{n+\ell}{\imath} \rho^{2\imath} \sum_{\jmath=0}^{\delta+2n} \binom{\delta+n-\ell}{\jmath} \rho^{-2\jmath} = (1+\rho^2)^{n+\ell}(1+\rho^{-2})^{\delta+n-\ell}.$$

This is equivalent to the expression

$$\langle \mathbf{u}_\ell, \mathbf{v}_\ell \rangle = \frac{(1+\rho^2)^m}{\rho^{2\delta+2n-2\ell}} = \frac{(2+2y\rho)^m}{\rho^{2m-2n-2\ell}}. \qquad \square$$

4. Two determinantal evaluations

Finally, we prove the two determinantal identities as in the following theorem.

THEOREM 4.1. *Letting $\mathbf{u}_i$ and $\mathbf{v}_j$ be as in Theorem 2.1, define the matrices*

$$U_m = [u_{k-n}(j)]_{0\le k,j\le m} \quad \text{and} \quad V_m = [v_{k-n}(i)]_{0\le i,k\le m}.$$

Their determinants are evaluated in closed form as

$$\det U_m = \det V_m = (4+4y^2)^{m(m+1)/4}.$$

PROOF. We first evaluate the determinant $\det U_m$. By making use of (2.2), we rewrite the three-term relation (2.1) as

$$\frac{(2k-m)(\rho+\rho^{-1})}{2(m-j)} u_{k-n}(j) = u_{k-n}(j+1) + \frac{j}{m-j} u_{k-n}(j-1) + \frac{(2j-m)(\rho-\rho^{-1})}{2(m-j)} u_{k-n}(j). \tag{4.1}$$

For the determinant of the matrix

$$\det U_m = \det_{0\le k\le m} [\mathbf{u}_{k-n}] = \det_{0\le j,k\le m} [u_{k-n}(j)],$$

perform the following three operations:

- add $j/(m-j)$ times the $(j-1)$th row to the $(j+1)$th row;
- add $(2j-m)(\rho-\rho^{-1})/(2m-2j)$ times the jth row to the $(j+1)$th row; and
- after the above two operations, the entry at position $(j+1,k)$ in the $(j+1)$th row becomes $((2k-m)(\rho+\rho^{-1})/(2m-2j))u_{k-n}(j)$, in view of (4.1).

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Repeating this operation upwards for all the rows except the first one and then pulling out the common factors in rows, we get the expression (with k indicating the kth column)

$$\det U_n = \prod_{j=0}^{m-1} \frac{(\rho+\rho^{-1})}{2(m-j)} \times \det_{0\le k\le m} \begin{bmatrix} & u_{k-n}(0) \\ (2k-m) & u_{k-n}(0) \\ (2k-m) & u_{k-n}(1) \\ (2k-m) & u_{k-n}(2) \\ \cdots & \cdots \ \cdots \\ (2k-m) & u_{k-n}(m-1) \end{bmatrix}.$$

By carrying out the same operations in the above matrix (except for the first two rows), we can further reduce the determinant: that is,

$$\det U_m = \prod_{i=1}^{2} \prod_{j=0}^{m-i} \frac{(\rho+\rho^{-1})}{2(m-j)} \times \det_{0\le k\le m} \begin{bmatrix} & u_{k-n}(0) \\ (2k-m) & u_{k-n}(0) \\ (2k-m)^2 & u_{k-n}(0) \\ (2k-m)^2 & u_{k-n}(1) \\ \cdots & \cdots \ \cdots \\ (2k-m)^2 & u_{k-n}(m-2) \end{bmatrix}.$$

Iterating the same procedure m times and extracting the common factors $u_{k-n}(0)$ in the columns of the resulting matrix, we derive the simplified expression

$$\det U_n = \prod_{i=1}^{m} \prod_{j=0}^{m-i} \frac{(\rho+\rho^{-1})}{2(m-j)} \times \det_{0\le j,k\le m} [(2k-m)^j] \prod_{k=0}^{m} u_{k-n}(0).$$

Keeping in mind that $u_{k-n}(0) = \binom{m}{k}$ and the determinant in the middle is of Vandermonde type, we can evaluate separately

$$\prod_{i=1}^{m} \prod_{j=0}^{m-i} \frac{(\rho+\rho^{-1})}{2(m-j)} = \prod_{i=1}^{m} \frac{(i-1)!}{m!} \Big(\frac{\rho+\rho^{-1}}{2}\Big)^{m-i+1} = \Big(\frac{\rho+\rho^{-1}}{2}\Big)^{\binom{m+1}{2}} \prod_{i=1}^{m} \frac{(i-1)!}{m!},$$

$$\det_{0\le j,k\le m} [(2k-m)^k] = \prod_{0\le j<k\le m} (2k-2j) = 2^{\binom{m+1}{2}} \prod_{k=1}^{m} (k!\,),$$

$$\prod_{k=0}^{m} u_{k-n}(0) = \prod_{k=0}^{m} \binom{m}{k} = \prod_{k=1}^{m} \frac{m!}{k!\,(k-1)!}.$$

Multiplying these together, we find the closed formula

$$\det U_m = (\rho+\rho^{-1})^{\binom{m+1}{2}} = (4+4y^2)^{m(m+1)/4}.$$

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Finally, to evaluate the other determinant $\det V_m$, rewrite, analogously, the three-term relation (2.4) as

$$\frac{(2k-m)(\rho+\rho^{-1})}{2(i+1)}v_{k-n}(i) = v_{k-n}(i+1) + \frac{m-i+1}{i+1}v_{k-n}(i-1) + \frac{(2i-m)(\rho-\rho^{-1})}{2(i+1)}v_{k-n}(i). \tag{4.2}$$

Similarly, for the determinant

$$\det V_n = \det_{0\le k\le n}[\mathbf{v}_k] = \det_{0\le i,k\le n}[v_k(i)],$$

make the following three operations:

- add $(m-i+1)/(i+1)$ times the $(i-1)$th row to the $(i+1)$th row;
- add $(2i-m)(\rho-\rho^{-1})/(2i+2)$ times the ith row to the $(i+1)$th row; and
- after the above two operations, the entry at position $(i+1,k)$ in the $(i+1)$th row becomes $((2k-m)(\rho+\rho^{-1})/(2i+2))v_{k-n}(i)$, taking into account (4.2).

Repeating this operation upwards for all the rows except the first one and then pulling out the common factors in rows, we get the expression (with k indicating the kth column)

$$\det V_m = \prod_{i=0}^{m-1}\frac{\rho+\rho^{-1}}{2(i+1)} \times \det_{0\le k\le n}\begin{bmatrix} & v_{k-n}(0) \\ (2k-m) & v_{k-n}(0) \\ (2k-m) & v_{k-n}(1) \\ (2k-m) & v_{k-n}(2) \\ \cdots & \cdots\ \cdots \\ (2k-m) & v_{k-n}(n-1) \end{bmatrix}.$$

By carrying out the same operations in the above matrix (except for the first two rows), we can reduce this further: that is,

$$\det V_m = \prod_{j=1}^{2}\prod_{i=0}^{m-j}\frac{\rho+\rho^{-1}}{2(i+1)} \times \det_{0\le k\le n}\begin{bmatrix} & v_k(0) \\ (2k-m) & v_{k-n}(0) \\ (2k-m)^2 & v_{k-n}(0) \\ (2k-m)^2 & v_{k-n}(1) \\ \cdots & \cdots\ \cdots \\ (2k-m)^2 & v_{k-n}(m-2) \end{bmatrix}.$$

Iterating the same procedure m times and extracting the common factors $v_k(0)$ in the columns of the resulting matrix, we find the following simplified expression.

$$\det V_n = \prod_{j=1}^{m}\prod_{i=0}^{m-j}\frac{\rho+\rho^{-1}}{2(i+1)} \times \det_{0\le i,k\le n}[(2k-m)^i]\prod_{k=0}^{m} v_{k-n}(0).$$

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Keeping in mind that $v_{k-n}(0) \equiv 1$ and then evaluating the product

$$\prod_{j=1}^{m}\prod_{i=0}^{m-j}\frac{\rho+\rho^{-1}}{2(i+1)} = \prod_{j=1}^{m}\frac{1}{(m-j+1)!}\Big(\frac{\rho+\rho^{-1}}{2}\Big)^{m-j+1} = \Big(\frac{\rho+\rho^{-1}}{2}\Big)^{\binom{m+1}{2}}\prod_{j=1}^{m}\frac{1}{j!},$$

we derive, after substitution, the second determinantal evaluation

$$\det V_m = (\rho+\rho^{-1})^{\binom{m+1}{2}} = (4+4y^2)^{m(m+1)/4}. \qquad \square$$

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