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REMARKS ON JAMES'S DISTORTION THEOREMS II

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If a Banach space X contains a complemented subspace isomorphic to ℓ^1 and if $\varepsilon > 0$, then there exists a subspace Y of X and a projection P from X onto Y such that Y is $(1+\varepsilon)$-isometric to ℓ^1 and $\|P\| \leqslant 1+\varepsilon$. A stronger result for c_0 is proved for Banach spaces whose dual unit ball is weak* sequentially compact.

1. INTRODUCTION

In [4], the authors prove that if a Banach space X contains a complemented copy of ℓ^1 (respectively, c_0) and if $\varepsilon > 0$, then X contains a complemented $(1+\varepsilon)$-isometric copy of ℓ^1 (respectively, c_0). This means that if X contains a complemented copy of ℓ^1 (respectively, c_0), then X contains almost isometric complemented copies of ℓ^1 (respectively, c_0). A natural question to ask is whether one can improve not only the quality of the copy of ℓ^1 or c_0, but whether one can also improve the quality of the projection? More precisely, if a Banach space X contains a complemented copy of ℓ^1 (respectively, c_0) and if $\varepsilon > 0$, does there exist a subspace Z of X and a projection P from X onto Z such that Z is $(1+\varepsilon)$-isometric to ℓ^1 (respectively, c_0) and $\|P\| \leqslant 1+\varepsilon$? The aim of this note is to show that the answer is yes in the ℓ^1 case. We also show that if a Banach space X contains a copy of c_0 and if the unit ball of X^* is weak* sequentially compact then, for each $\varepsilon > 0$, there exists a subspace Z of X and a projection P from X onto Z such that X is $(1+\varepsilon)$-isometric to c_0 and $\|P\| \leqslant 1+\varepsilon$. This can be viewed as an extension of a classical result of Sobczyk [**11**].

2. THE RESULTS

The first result in this section provides a method for recognising when a Banach space has a complemented copy of ℓ^1 with a good projection constant. Our notation is standard and we refer the reader to the texts of Diestel [**3**] and Lindenstrauss and Tzafriri [**8**] for any unexplained terms.

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Lemma 1. *Let $0 < \varepsilon < 1$ and let $(e_n)_n$ denote the standard unit vector basis of ℓ^1. Suppose that X is a Banach space, that $T : X \to \ell^1$ is a bounded linear operator with $\|T\| \leqslant 1$, and that $(x_n)_n$ is a sequence in the unit ball of X satisfying $\|Tx_n - e_n\| < \varepsilon$ for all $n \in \mathbb{N}$. Then there exists a subspace Z of X and a projection P from X onto Z such that Z is $1/(1-\varepsilon)$-isometric to ℓ^1 and $\|P\| \leqslant 1/(1-\varepsilon)$.*

Proof: Let $(a_n)_n \in \ell^1$. Since $\|x_n\| \leqslant 1$ for all $n \in \mathbb{N}$,

$$\left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leqslant \sum_{n=1}^{\infty} |a_n| \|x_n\| \leqslant \sum_{n=1}^{\infty} |a_n|.$$

Also, since $\|T\| \leqslant 1$, we have

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} a_n x_n\right\| &\geqslant \left\|T\left(\sum_{n=1}^{\infty} a_n x_n\right)\right\| \\ &= \left\|\sum_{n=1}^{\infty} a_n T x_n\right\| \\ &\geqslant \left\|\sum_{n=1}^{\infty} a_n e_n\right\| - \left\|\sum_{n=1}^{\infty} a_n (Tx_n - e_n)\right\| \\ &\geqslant \sum_{n=1}^{\infty} |a_n| - \sum_{n=1}^{\infty} |a_n| \|Tx_n - e_n\| \\ &\geqslant (1-\varepsilon) \sum_{n=1}^{\infty} |a_n|.\end{aligned}$$

Thus $(1-\varepsilon)\sum_{n=1}^{\infty} |a_n| \leqslant \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leqslant \sum_{n=1}^{\infty} |a_n|$ for all $(a_n)_n \in \ell^1$. Therefore the Banach space $Z = \overline{\operatorname{span}}\{x_n : n \in \mathbb{N}\}$ is $1/(1-\varepsilon)$-isometric to ℓ^1.

Define a bounded linear operator $S : \ell^1 \to X$ by $S\big((a_n)_n\big) = \sum_{n=1}^{\infty} a_n x_n$ for all $(a_n)_n \in \ell^1$. Clearly $\|S\| \leqslant 1$, and so the operator $TS : \ell^1 \to \ell^1$ satisfies $\|TS\| \leqslant 1$. Also, for $(a_n)_n \in \ell^1$,

$$\left\|TS\big((a_n)_n\big)\right\| = \left\|T\left(\sum_{n=1}^{\infty} a_n x_n\right)\right\| = \left\|\sum_{n=1}^{\infty} a_n T x_n\right\| \geqslant (1-\varepsilon) \sum_{n=1}^{\infty} |a_n|.$$

Therefore TS is an isomorphism and $\left\|(TS)^{-1}\right\| \leqslant 1/(1-\varepsilon)$. Note also, with I denoting the identity map on ℓ^1, that since $\|I - TS\| < 1$, $(TS)^{-1} = \sum_{n=0}^{\infty} (I - TS)^n$. This implies that the domain of $(TS)^{-1}$ is all of ℓ^1. Define $P : X \to X$ by $P = S(TS)^{-1}T$.

It is easily seen that P is a projection of X onto Z and $\|P\| \leqslant \|S\| \|(TS)^{-1}\| \|T\| \leqslant 1/(1-\varepsilon)$. This completes the proof. ∎

For our next result, Theorem 5, we need the following three ingredients.

LEMMA 2. [1] *Let $(x_n)_n$ be a basic sequence in an infinite dimensional Banach space X. Then there is a block basic sequence $(y_n)_n$ of (x_n) and a sequence of functionals $(y_n^*)_n$ in X^* which form a unit biorthogonal system of X. That is, for each $n \in \mathbb{N}$, $\|y_n\| = \|y_n^*\| = y_n^*(y_n) = 1$ and $y_n^*(y_m) = 0$ for all $m \neq n$.*

THEOREM 3. [3, 7, 8] *Let X be a separable infinite dimensional Banach space. If $(x_n^*)_n$ is a weak* null normalised sequence in X^*, then $(x_n^*)_n$ has a subsequence $(y_n^*)_n$ which is a weak* basic sequence.*

THEOREM 4. [5] *Let X be a Banach space and let X_0 be a separable subspace of X. Then there exists a separable subspace Z of X which contains X_0, and an isometric embedding $J : Z^* \to X^*$ such that $\big(J(z^*)\big)(z) = z^*(z)$ for all $z \in Z$ and $z^* \in Z^*$.*

THEOREM 5. *Let X be a Banach space which contains a complemented subspace isomorphic to ℓ^1 and let $\varepsilon > 0$. Then there exists a subspace Y of X and a projection P from X onto Y such that Y is $(1+\varepsilon)$-isometric to ℓ^1 and $\|P\| \leqslant 1+\varepsilon$.*

PROOF: We shall consider the case where X is a separable Banach space containing a complemented subspace isomorphic to ℓ^1. Then, by the Bessaga-Pełczyński Theorem [3], X^* contains a subspace isomorphic to c_0. Let $\delta = \varepsilon/(1+2\varepsilon)$. By the James Distortion Theorem [3, 6, 8], there is a sequence $(x_n^*)_n$ in X^* such that

$$(1-\delta)\sup_n |a_n| \leqslant \left\| \sum_n a_n x_n^* \right\| \leqslant \sup_n |a_n| \quad \text{for all } (a_n)_n \in c_0.$$

Since $(x_n^*)_n$ is a basic sequence in X^*, there is a block basic sequence $(y_n^*)_n$ of $(x_n^*)_n$ and a sequence $(y_n^{**})_n$ in X^{**} such that for each $n \in \mathbb{N}$, $\|y_n^*\| = \|y_n^{**}\| = y_n^{**}(y_n^*) = 1$ and $y_n^{**}(y_m^*) = 0$ for all $m \neq n$, by Lemma 2. Hence there is a strictly increasing sequence of integers, $(k_n)_{n=0}^\infty$ with $k_0 = 0$, and scalars, $\alpha_j^{(n)}$, where $k_{n-1}+1 \leqslant j \leqslant k_n$ for $n \in \mathbb{N}$, so that

$$y_n^* = \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} x_j^* .$$

Since $\|y_n^*\| = 1$, we get that $|\alpha_j^{(n)}| \leqslant 1/(1-\delta)$, for all $n \in \mathbb{N}$ and $k_{n-1} < j \leqslant k_n$. For each $n \in \mathbb{N}$, define $z_n^* = (1-\delta)y_n^*$. Then, for all $(a_n)_n \in c_0$,

$$\begin{aligned}\Big\|\sum_n a_n z_n^*\Big\| &= (1-\delta)\Big\|\sum_n a_n y_n^*\Big\| \\ &= (1-\delta)\Big\|\sum_n \sum_{j=k_{n-1}+1}^{k_n} a_n \alpha_j^{(n)} x_j^*\Big\| \\ &\leqslant (1-\delta)\sup_n \sup_{k_{n-1}<j\leqslant k_n} |a_n \alpha_j^{(n)}| \\ &\leqslant \sup_n |a_n|.\end{aligned}$$

Since $(x_n^*)_n$ is equivalent to the unit vector basis of c_0 and since $(y_n^*)_n$ is a block basis of $(x_n^*)_n$, $(y_n^*)_n$ is equivalent to the unit vector basis of c_0. Hence $(y_n^*)_n$ is a weak* null normalised sequence in X^*. By Theorem 3, $(y_n^*)_n$ has a subsequence (again denoted by $(y_n^*)_n$) which is weak* basic. By the (assumed) separability of X and the construction of this subsequence [**8**, pages 11-12], there is a bounded linear operator $T : X \to \big(\overline{\text{span}}\{y_n^* : n \in \mathbb{N}\}\big)^*$, given by $T(x)(y^*) = y^*(x)$, for all $x \in X$ and $y^* \in \overline{\text{span}}\{y_n^* : n \in \mathbb{N}\}$. We note that $\|T\| \leqslant 1$. Moreover, this operator has the property that for each $\eta > 0$ and $y^{**} \in \text{span}\{y_n^{**} : n \in \mathbb{N}\}$ with $\|y^{**}\| = 1$, there is $x \in X$ with $\|x\| = 1$ and $\|Tx - y^{**}\| < \eta$. In particular, for each $n \in \mathbb{N}$, there is $x_n \in X$ with $\|x_n\| = 1$ and $\|Tx_n - y_n^{**}\| < \big(\varepsilon^2/(1+\varepsilon)(1+2\varepsilon)\big)$.

Define an operator $\Theta : \big(\overline{\text{span}}\{y_n^* : n \in \mathbb{N}\}\big)^* \to \ell^1$ by $\Theta(y^{**}) = \big(y^{**}(z_n^*)\big)_n$. Note that for each $m \in \mathbb{N}$ we have $\Theta(y_m^{**}) = \big(y_m^{**}(z_n^*)\big)_n = (1-\delta)\big(y_m^{**}(y_n^*)\big)_n = (1-\delta)e_m$. Also

$$\begin{aligned}\|\Theta\| &= \sup\Big\{\|\Theta(y^{**})\| : y^{**} \in \big(\overline{\text{span}}\{y_n^* : n \in \mathbb{N}\}\big)^*,\ \|y^{**}\| = 1\Big\} \\ &= \sup\Big\{\sum_{n=1}^{\infty} |y^{**}(z_n^*)| : \|y^{**}\| = 1\Big\} \\ &= \sup\Big\{\sum_{n\in\Delta} |y^{**}(z_n^*)| : \|y^{**}\| = 1 \text{ and } \Delta \text{ is a finite subset of } \mathbb{N}\Big\} \\ &= \sup\Big\{\Big\|\sum_{n\in\Delta} \theta_n z_n^*\Big\| : \Delta \text{ is a finite subset of } \mathbb{N} \text{ and } |\theta_n| = 1 \text{ for all } n \in \Delta\Big\} \\ &\leqslant 1.\end{aligned}$$

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Define $S : X \to \ell^1$ by $S = \Theta T$. Then $\|S\| \leqslant \|\Theta\|\|T\| \leqslant 1$. Moreover, for each $n \in \mathbb{N}$

$$\begin{aligned}\|Sx_n - e_n\| &= \|\Theta(Tx_n) - e_n\| \\ &= \|\Theta(Tx_n - y_n^{**}) + \Theta(y_n^{**}) - e_n\| \\ &\leqslant \|\Theta\|\|Tx_n - y_n^{**}\| + \|\Theta(y_n^{**}) - e_n\| \\ &< \frac{\varepsilon^2}{(1+\varepsilon)(1+2\varepsilon)} + \|(1-\delta)e_n - e_n\| \\ &= \frac{\varepsilon^2}{(1+\varepsilon)(1+2\varepsilon)} + \delta = \frac{\varepsilon}{1+\varepsilon}.\end{aligned}$$

The proof is complete by an application of Lemma 1.

For the general case, since X contains a complemented copy of ℓ^1, X^* contains a copy of c_0. Therefore, by the James Distortion Theorem, there is a sequence $(x_n^*)_n$ in X^* such that

$$(1-\delta)\sup_n |a_n| \leqslant \left\|\sum_n a_n x_n^*\right\| \leqslant \sup_n |a_n| \quad \text{for all } (a_n)_n \in c_0,$$

where $\delta = \varepsilon/(1+2\varepsilon)$. Define $Z = \overline{\operatorname{span}}\{x_n^* : n \in \mathbb{N}\}$. Then Z is a separable subspace of X^*. Let $\{z_n : n \in \mathbb{N}\}$ be a countable dense subset of the unit ball of Z. For each $n \in \mathbb{N}$, choose a sequence $(x_{n,k})_k$ in the unit ball of X such that $\|z_n\| = \lim_{k\to\infty} z_n(x_{n,k})$. Define $Y = \overline{\operatorname{span}}\{x_{n,k} : n, k \in \mathbb{N}\}$. Clearly Y is a separable subspace of X and hence, by Theorem 4, there is a separable subspace Y_1 of X which contains Y and there is an isometric embedding $J : Y_1^* \to X^*$ satisfying $(Jy^*)(y) = y^*(y)$ for all $y^* \in Y_1^*$ and $y \in Y_1$.

Clearly from the construction of Y_1, Z is isometric to a subspace of Y_1^*. Hence Y_1^* contains an isomorphic copy of c_0. Therefore, since Y_1 is separable, the first part of the proof says there is an operator $S : Y_1 \to \ell^1$ with $\|S\| \leqslant 1$ and there is a sequence $(y_n)_n$ in the unit ball of Y_1 with $\|Sy_n - e_n\| < \varepsilon$ for each $n \in \mathbb{N}$. By [**3**, page 114], there is a weakly unconditionally Cauchy series $\sum_n y_n^*$ in Y_1^* so that $S(y) = (y_n^*(y))_n$ for all $y \in Y_1$. Moreover,

$$\|S\| = \sup\left\{\left\|\sum_{n\in\Delta} \theta_n y_n^*\right\| : \Delta \text{ is a finite subset of } \mathbb{N} \text{ and } |\theta_n| = 1 \text{ for all } n \in \Delta\right\}.$$

For each $n \in \mathbb{N}$, define $x_n^* = J(y_n^*)$. Since J is an isometric embedding, $\sum_n x_n^*$ is a weakly unconditionally Cauchy series in X^* and the operator $S_0 : X \to \ell^1$ defined by $S_0(x) = (x_n^*(x))_n$, for all $x \in X$, satisfies $\|S_0\| = \|S\| \leqslant 1$. Also, for each $y \in Y_1$,

$$S_0(y) = (x_n^*(y))_n = (J(y_n^*)(y))_n = (y_n^*(y))_n = S(y).$$

Therefore, $\|S_0(y_n) - e_n\| < \varepsilon$ for all $n \in \mathbb{N}$, and so an application of Lemma 1 completes the proof. ∎

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REMARK. [**4,** Corollary 3] states that if a dual Banach space contains a subspace isomorphic to ℓ^∞, then it contains almost isometric copies of ℓ^∞. This result is true in general Banach spaces by a result of Partington [**10**]. However, the proof of the result in [**4**] is incorrect, but an application of Theorem 5 can be used to "fix" the proof. The authors wish to thank Dirk Werner for informing them of the error in the proof of [**4,** Corollary 3].

Sobczyk's Theorem states that if a separable Banach space X contains a subspace Y isometric to c_0, then there exists a projection of norm less than or equal to 2 from X onto Y [**8,** Theorem 2.f.5]. In view of a result of Taylor [**12,** p. 547] that every projection from c onto its subspace c_0 has norm at least two, the projection constant in Sobczyk's Theorem cannot in general be improved. However, if one considers Sobczyk's and Taylor's results in the spirit of Theorem 5, it is reasonable to ask, given a separable Banach space X that contains a subspace isometric to c_0, whether there exist other subspaces of X isometric to c_0 on which the norms of the projections are less than 2. The second statement in the next result gives as nice a positive answer to this question as one could hope for.

THEOREM 6. *Let X be a Banach space whose dual unit ball is weak* sequentially compact and let $\varepsilon > 0$. If X contains a subspace isomorphic to c_0, then there exists a subspace Z of X and a projection P from X onto Z such that Z is $(1+\varepsilon)$-isometric to c_0 and $\|P\| \leqslant 1+\varepsilon$. Moreover, if X contains a subspace isometric to c_0, then there exists a subspace Z of X and a projection P from X onto Z such that Z is isometric to c_0 and $\|P\| = 1$.*

PROOF: Let $\delta = \varepsilon/(1+\varepsilon)$. Since X contains a subspace isomorphic to c_0, the James Distortion Theorem [**3, 6, 8**] says there is a sequence $(x_n)_n$ in X such that

$$(1-\delta)\sup_n |a_n| \leqslant \left\| \sum_n a_n x_n \right\| \leqslant \sup_n |a_n|, \qquad \text{for all } (a_n)_n \in c_0.$$

Let $Y = \overline{\operatorname{span}}\{x_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, define $y_n^* \in Y^*$ by

$$y_n^*\left(\sum_k a_k x_k\right) = a_n.$$

Then $\|y_n^*\| \geqslant |y_n^*(x_n)| = 1$, since $\|x_n\| \leqslant 1$, and

$$\begin{aligned}\|y_n^*\| &= \sup\left\{ \left| y_n^*\left(\sum_k a_k x_k\right)\right| : \left\|\sum_k a_k x_k\right\| \leqslant 1 \right\} \\ &= \sup\left\{ |a_n| : \left\|\sum_k a_k x_k\right\| \leqslant 1 \right\}\end{aligned}$$

$$\leqslant \sup\left\{|a_n| : \sup_k |a_k| \leqslant \frac{1}{1-\delta}\right\}$$
$$\leqslant \frac{1}{1-\delta} = 1+\varepsilon.$$

By the Hahn-Banach theorem we extend y_n^* to an element $x_n^* \in X^*$ with $\|x_n^*\| = \|y_n^*\| \leqslant 1+\varepsilon$. Since the unit ball of X^* is weak* sequentially compact, $(x_n^*)_n$ has a subsequence $\left(x_{n_k}^*\right)_k$ converging weak* in X^*. Define $w_k^* = \left(x_{n_{2k+1}}^* - x_{n_{2k}}^*\right)/2$ and $w_k = x_{n_{2k+1}} - x_{n_{2k}}$, for each $k \in \mathbb{N}$. Clearly, $(w_k^*)_k$ is a weak* null sequence in X^* with $\|w_k^*\| \leqslant 1+\varepsilon$ and $w_k^*(w_k) = 1$ for all $k \in \mathbb{N}$ and $w_k^*(w_m) = 0$ for all $m \neq k$.

Define an operator $J : c_0 \to X$ by $J\big((a_n)_n\big) = \sum\limits_{n=1}^{\infty} a_n w_n$, for all $(a_n)_n \in c_0$. Define an operator $Q : X \to c_0$ by $Q(x) = (w_n^*(x))_n$ for all $x \in X$. It is easily checked that QJ is the identity operator on c_0 and so the operator $P = JQ : X \to X$ is a projection. Note that the range of JQ is $Z = \overline{\operatorname{span}}\{w_n : n \in \mathbb{N}\}$. By the construction of Z, it is easily seen that Z is $(1+\varepsilon)$-isometric to c_0. Finally, note that

$$\|J\| = \sup\left\{\left\|\sum_n a_n w_n\right\| : \sup_n |a_n| \leqslant 1\right\} \leqslant 1$$

and

$$\|Q\| = \sup\left\{\sup_n |w_n^*(x)| : \|x\| \leqslant 1\right\} \leqslant \sup_n \|w_n^*\| \leqslant 1+\varepsilon,$$

and hence $\|P\| \leqslant \|J\|\|Q\| \leqslant 1+\varepsilon$. Thus Z and P have all of the advertised properties.

For the case where X contains an isometric copy of c_0, repeat the above proof with $\varepsilon = 0$. □

REMARK. To the extent that Sobczyk's Theorem (stated prior to Theorem 6) guarantees that every separable Banach space containing a subspace isometric to c_0 contains a complemented isometric copy of c_0, Theorem 6 extends and improves Sobczyk's Theorem since it applies to a larger class of spaces and guarantees projections of norm 1. However, it is more accurate to think of Theorem 6 as a variation on Sobczyk's theme rather than as an actual extension or improvement of Sobczyk's Theorem.

A result similar to Theorem 6 was proved by Díaz and Fernández [**2**] in the setting of Banach spaces not containing a copy of ℓ^1. An extension of Sobczyk's Theorem was proved by Moltó [**9**].

We finish with two natural questions:

QUESTION 1. *If a Banach space X contains a complemented copy of c_0 and if $\varepsilon > 0$, does there exist a subspace Z of X and a projection P from X onto Z such that Z is $(1+\varepsilon)$-isometric to c_0 and $\|P\| \leqslant 1+\varepsilon$?*

QUESTION 2. *If a Banach space X contains a complemented subspace isometric to ℓ^1, does there exist a subspace Z of X and a projection P from X onto Z such that Z is isometric to ℓ^1 and $\|P\| = 1$?*

REFERENCES

[1] B.J. Cole, T.W. Gamelin and W.B. Johnson, 'Analytic disks in fibers over the unit ball of a Banach space', *Michigan Math. J.* **39** (1992), 551–569.

[2] S. Díaz and A Fernández, 'Reflexivity in Banach lattices', *Arch. Math.* **63** (1994), 549–552.

[3] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics **92** (Springer-Verlag, Berlin, Heidelberg, New York, 1984).

[4] P.N. Dowling, N. Randrianantoanina and B. Turett, 'Remarks on James's distortion theorems', *Bull. Austral. Math. Soc.* **57** (1998), 49–54.

[5] S. Heinrich and P. Mankiewicz, 'Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces', *Studia Math.* **73** (1982), 225–251.

[6] R.C. James, 'Uniformly non-square Banach spaces', *Ann. of Math.* **80** (1964), 542–550.

[7] W.B. Johnson and H.P. Rosenthal, 'On ω^* basic sequences and their applications to the study of Banach spaces', *Studia Math.* **43** (1972), 77–92.

[8] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Ergebnisse der Mathematik und Ihrer Grenzgebiete **92** (Springer-Verlag, Berlin, Heidelberg, New York, 1977).

[9] A. Moltó, 'On a theorem of Sobczyk', *Bull. Austral. Math. Soc.* **43** (1991), 123–130.

[10] J.R. Partington, 'Equivalent norms on spaces of bounded functions', *Israel J. Math.* **35** (1980), 205–209.

[11] A. Sobczyk, 'Projection of the space m on its subspace c_0', *Bull. Amer. Math. Soc.* **47** (1941), 938–947.

[12] A.E. Taylor, 'The extension of linear functionals', *Duke Math. J.* **5** (1939), 538–547.

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