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0-DISTRIBUTIVE SEMILATTICES

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Abstract. Sufficient conditions for a semilattice to be a 0-distributive are obtained. Some equivalent formulations of 0-distributivity in a semilattice are given. Further, disjunctive 0-distributive semilattices are also characterized.

1. Introduction. Let S be a meet semilattice with 0. Let a, b, c in S be such that whenever $b \vee c$ exists, $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$, then S is called a 0-distributive semilattice. 0-distributive lattices discussed by Varlet [4] and Hoffman-Keimal [3] are also 0-distributive semilattices. Additional examples of 0-distributive semilattices are pseudocomplemented semilattices, bounded implicative semilattices and prime semilattices (with 0) investigated by Balbes [2]. It may be recalled that on account of Theorem 5 (to follow) our definition of a 0-distributive semilattice coincides with that given by Varlet [5].

The Hasse diagram given below is of a 0-distributive semilattice.

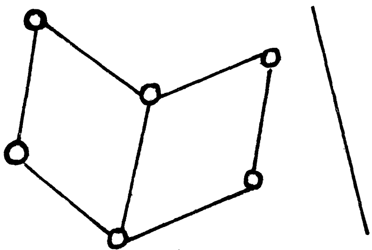

Fig. 1

It may be observed that the following diagram is an example of a 0-distributive semilattice which is not a prime semilattice.

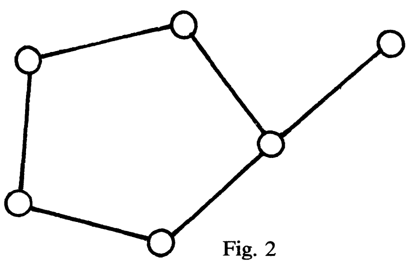

Fig. 2

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The next diagram shows that a 0-distributive semilattice need not be distributive.

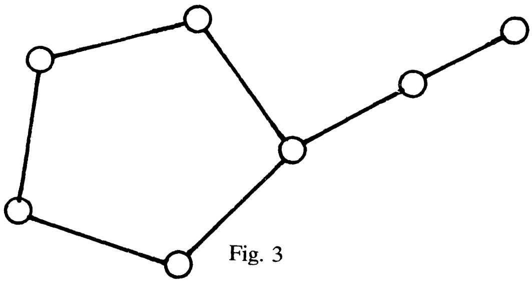

Fig. 3

Denote the set of all disjoint elements of $A \subseteq S$ in S by $A^{\perp}$ i.e. $A^{\perp} = \{x \in S : x \wedge a = 0$ for every a in $A\}$. An ideal I in S is a non-empty subset of S such that $a \leq b$, $b \in I$ implies $a \in I$ and whenever $a \vee b$ exists for a, b in I then $a \vee b \in I$. (see Venkatanarasimhan [6]) A proper ideal I in S is called prime if $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$. It can be verified that in a 0-distributive semilattice $S, \{a\}^{\perp}$, $a \in S$ is an ideal of S.

It is well known that a pseudocomplemented semilattice S is characterized by the property that for any element a in S, the subset of elements that are disjoint from a is a principal ideal. We show in the course of our investigation that our 0-distributive semilattices are characterized by the property that the set of all elements that are disjoint from a given element forms an ideal. Besides this we have obtained several equivalent formulations of 0-distributive semilattices. We also characterize disjunctive 0-distributive semilattices.

2. Characterizations. We begin with a rather elementary result the easy proof of which is omitted.

THEOREM 1. *A* 0-*distributive semilattice is pseudocomplemented if and only if* $\{a\}^{\perp}$ *is a principal ideal of S, for every a in S.*

A sufficient condition for a semilattice to be 0-distributive is stated in the following theorem.

THEOREM 2. *If the intersection of all prime ideals of a semilattice S (with* 0) *is* $\{0\}$ *then S is* 0-*distributive.*

Proof. Let a, b, c be in S such that $a \wedge b = 0$, $a \wedge c = 0$, and $b \vee c$ exist. For any prime ideal P of S, $a \in P$ or $a \notin P$. If $a \in P$ then $a \wedge (b \vee c) \leq a$ implies that $a \wedge (b \vee c) \in P$. Again if $a \notin P$ then by primeness of P, $b \in P$ and $c \in P$. As $b \vee c$ exists, $b \vee c \in P$, which in turn implies that $a \wedge (b \vee c) \in P$. Thus $a \wedge (b \vee c)$ is in every prime ideal P of S and hence $a \wedge (b \vee c) = 0$, proving that S is 0-distributive.

Let us now obtain one more sufficient condition involving the lattice of filters of a semilattice.

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THEOREM 3. *A semilattice S with* 0 *is* 0-*distributive if the lattice of filters of S is distributive.*

Proof. Let $a \wedge b = 0$, $a \wedge c = 0$, and $b \vee c$ exist. By distributivity of lattice of filters of S, the principal filter generated by $a \wedge (b \vee c)$ is $[a \wedge (b \vee c)) = [a \wedge b) U [a \wedge c)$ which by assumption is $[0)$, hence $a \wedge (b \vee c) = 0$, proving the 0-distributivity of S.

In fact, this theorem very comfortably leads us to the anticipated conclusion that every bounded distributive semilattice is 0-distributive. For, a semilattice S with 1 is distributive if and only if the lattice of filters of S is distributive (see [2]).

We now state a lemma that is needed to characterize 0-distributive semilattices. In fact, the property stated by Adams [1] of maximal filters in a lattice with 0 also holds in a semilattice which is bounded below. This is proved in the following

LEMMA 4. *Let S be a Semi-lattice with* 0. *A proper filter M in S is maximal if and only if* (*) *for any element* $a \notin M (a \in S)$ *there exists an element* $b \in M$ *with* $a \wedge b = 0$.

Proof. Suppose that for all b in M, $a \wedge b \neq 0$. Consider the set $M^1 = \{y \in S : y \geq a \wedge b, b \in M\}$. Clearly M^1 is a filter of S and is proper as $0 \notin M^1$. Further $M \subseteq M^1$ contradicts the maximality of M. Hence there must exist some b in M such that $a \wedge b = 0$.

Conversely, if M is not maximal, then as $0 \in S$, there exists a maximal filter M^1 properly containing M. For any element $a \in M^1 \backslash M$ there exists, by (*), an element b in M with $a \wedge b = 0$. Hence $a \in M^1$, $b \in M^1$ imply that $0 = a \wedge b \in M^1$; which is a contradiction. Thus M must be a maximal filter.

Let $I(S)$ denote the lattice of all ideals of a semilattice S with 0. Characterizations of a 0-distributive semilattice are given in the following

THEOREM 5. *Following are equivalent in S*

(1) *S is* 0-*distributive.*

(2) $\{a\}^{\perp}$ *is an ideal for all* $a \in S$.

(3) $A^{\perp}$ *is an ideal for all* $A \subseteq S$.

(4) $I(S)$ *is pseudocomplemented.*

(5) $I(S)$ *is* 0-*distributive.*

(6) *Every maximal filter is prime.*

Proof. (1)⇒(2)⇒(3) follow easily. For any ideal I of S, by the definition $I^{\perp}$ will be the pseudocomplement of I in $I(S)$ if $I^{\perp} \in I(S)$; we get $I(S)$ to be pseudocomplemented and hence (3)⇒(4). As every pseudocomplemented lattice is 0-distributive we get (4)⇒(5). Let F be a maximal filter and $f \notin F$, $g \notin F$. By Lemma 4 we get $a \wedge f = 0$ and $b \wedge g = 0$ for some $a, b \in F$. Hence

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$(f]\cap(a\wedge b]=(0]$ and $(g]\cap(a\wedge b]=(0]$. If $f\vee g$ exists in S then $(f\vee g]\cap(a\wedge b]=[(f]\cup(g]]\cap(a\wedge b]=(0]$, by 0-distributivity of $I(S)$. Hence $(f\vee g)\wedge(a\wedge b)=0$. As $(f\vee g)\wedge(a\wedge b)\notin F$ and $(a\wedge b)\in F$ we have $f\vee g\notin F$. Hence F is prime, completing the proof of (5)⇒(6). Now for (6)⇒(1) let $a\wedge b=0$, $a\wedge c=0$, and $b\vee c$ exist. If $a\wedge(b\vee c)\neq 0$ then $a\wedge(b\vee c)\in F$ for some maximal filter F of S. As F is prime, by assumption, $a\in F$ and $b\in F$ or $c\in F$ i.e. $a\wedge b\in F$ or $a\wedge c\in F$. Thus $0\in F$, which leads to the contradiction and hence the result.

It may be mentioned that a characterization of 0-distributivity in terms of (6) was obtained by Varlet [5].

We shall use the property (2) from the above characterizations to obtain a rather interesting result.

THEOREM 6. *In a 0-distributive semilattice S, if* $\{0\}\neq A$ *is the intersection of all non-zero ideals of S, then* $A^{\perp}=D$ *where* $D=\{x\in S:\{x\}^{\perp}\neq\{0\}\}$.

Proof. As $A\neq\{0\}$, we get for any x in $A^{\perp}, \{x\}^{\perp}\neq\{0\}$ i.e. $x\in D$. Hence $A^{\perp}\subseteq D$.

Conversely, as S is 0-distributive, $\{d\}^{\perp}$ is a non-zero ideal of S for every d in D. Therefore $A\subseteq\{d\}^{\perp}$, $d\in D$ implies that $A^{\perp}\supseteq\{d\}^{\perp\perp}$ i.e. $d\in A^{\perp}$, which in turn gives that $D\subseteq A^{\perp}$. Hence $A^{\perp}=D$ proving the result.

A necessary and sufficient condition for a semilattice S with 0, to be 0-distributive is given in the following

THEOREM 7. *Let S be a semilattice with* 0. *S is* 0-*distributive if and only if for any filter F disjoint with* $\{x\}^{\perp}$*(x in S), there exists a prime filter containing F and disjoint with* $\{x\}^{\perp}$.

Proof. Consider the family $\mathfrak{F}$ in S of all filters containing F and disjoint with $\{x\}^{\perp}$; clearly $\mathfrak{F}$ is non-empty. By Zorn's lemma there exists a maximal element, say Q, in $\mathfrak{F}$. We claim that $x\in Q$. For if $x\notin Q$ then the filter generated by Q and x intersects $\{x\}^{\perp}$. Hence there exists an element y in S such that $y\geq q\wedge x$ for some q in Q and $y\wedge x=0$. But this gives that $q\wedge x=0$ i.e. $q\in\{x\}^{\perp}$, which is a contradiction since $Q\cap\{x\}^{\perp}=\varnothing$. To prove Q is prime, let $z\in S$ such that $z\notin Q$. As the filter generated by $Q\cup\{z\}$ intersects $\{x\}^{\perp}$ there exists an element y in $\{x\}^{\perp}$ such that $y\geq f\wedge z$ for some $f\in Q$. Now $0=y\wedge x\geq f\wedge z\wedge x$ gives $f\wedge z\wedge x=0$. But by $f\in Q$ and $x\in Q$ we have $f\wedge x\in Q$ with $z\wedge(f\wedge x)=0$. Hence by Lemma 4, Q is prime.

Conversely, let $x\wedge y=0, x\wedge z=0$, and $y\vee z$ exist such that $x\wedge(y\vee z)\neq 0$ i.e. $y\vee z\notin\{x\}^{\perp}$. As $[(y\vee z))\cap\{x\}^{\perp}=\varnothing$, there exists a prime filter Q containing $[(y\vee z))$ and disjoint with $\{x\}^{\perp}$. As y and z are in $\{x\}^{\perp}$, $y\vee z\notin Q$, Q being a prime filter; which in turn implies that $[(y\vee z)\not\subseteq Q$, a contradiction. Hence $x\wedge(y\vee z)=0$, proving that S is 0-distributive.

COROLLARY. *Any two distinct elements a, b for which $a \wedge b \neq 0$ are separated by a prime filter in a 0-distributive semilattice.*

The result of Theorem 7 is very close to being Stone type theorem for 0-distributive semilattices, where we have selected special types of ideals. Hence it is reasonable to conjecture,

"*A semilattice S with 0 is 0-distributive if and only if for any filter F and any ideal I such that $F \cap I = \varnothing$ there exists a prime filter containing F and disjoint with I*".

Let S be a 0-distributive semilattice and f be the map $S \to \{\{a\}^{\perp\perp} : a \in S\}$ given by $f(a) = \{a\}^{\perp\perp}$.

This map is a meet homomorphism. We now state a simple lemma,

LEMMA 8. *For a 0-distributive semilattice S, $f(a) = 0$ if and only if $a = 0$. Moreover $f(\{a\}^{\perp}) = \{f(a)\}^{\perp}$.*

Proof. If $f(a) = 0$ then $\{a\}^{\perp\perp} = \{0\}$ will imply that $\{a\}^{\perp\perp\perp} = \{0\}^{\perp} = S$ i.e. $\{a\}^{\perp} = S$. Hence $a \wedge s = 0$ for every s in S which in turn will imply that $a = 0$. When $a = 0$, $f(0) = \{0\}^{\perp\perp} = S^{\perp} = \{0\}$, the reverse implication follows. As f is a semilattice homomorphism we get $a \wedge b = 0$ if and only if $f(a \wedge b) = f(a) \wedge f(b) = \{a\}^{\perp\perp} \cap \{b\}^{\perp\perp} = \{0\}$. Thus we have, $f(\{a\}^{\perp}) = \{\{b\}^{\perp\perp} : a \wedge b = 0\} = \{\{b\}^{\perp\perp} : \{a\}^{\perp\perp} \cap \{b\}^{\perp\perp} = \{0\}\} = \{f(a)\}^{\perp}$ (see also Hoffman–Keimal [3] p.93).

Let us also have the *Definition* (*see* [6]): A semilattice S with 0 is called disjunctive if $a \neq b$ implies that either $\{a\}^{\perp} \setminus \{b\}^{\perp} \neq \varnothing$ or $\{b\}^{\perp} \setminus \{a\}^{\perp} \neq \varnothing$.

As can be easily seen this definition is equivalent to the following

If $a < b$ then there is $x \in \{a\}^{\perp}$ such that $x \wedge b \neq 0$.

We shall set ourselves to obtain the following characterizations of a disjunctive 0-distributive semilattice.

THEOREM 9. *In a 0-distributive semilattice S, following are equivalent.*

(1) *$f : S \to \{\{a\}^{\perp\perp} : a \in S\}$ defined by $f(a) = \{a\}^{\perp\perp}$ is injective.*

(2) *$\{a\}^{\perp} = \{b\}^{\perp}$ (in $I(S)$) implies $a = b$ for all a, b in S.*

(3) *S is disjunctive.*

Proof. We shall prove this assertion by exhibiting the equivalence of (1) and (2) and (2) and (3).

(1)⇒(2). In view of lemma-8 we need to consider the case of a, b both non-zero. If $\{a\}^{\perp} = \{b\}^{\perp}$ for $a \neq b$ then $f(a) \neq f(b)$ implies that $\{a\}^{\perp\perp} \neq \{b\}^{\perp\perp}$. Hence there is an x in $\{a\}^{\perp\perp}$ such that $x \notin \{b\}^{\perp\perp}$. But $x \notin \{b\}^{\perp\perp}$ means that for some z in $\{b\}^{\perp}$ $x \wedge z \neq 0$. As $\{a\}^{\perp} = \{b\}^{\perp}$, we have $x \wedge z \neq 0$ for some z in $\{a\}^{\perp}$; i.e. $x \notin \{a\}^{\perp\perp}$, which is a contradiction. Hence $\{a\}^{\perp} = \{b\}^{\perp}$ implies $a = b$.

(2)⇒(1). Obvious.

(2)⇒(3). Let $a<b$. On account of (2) we must have $\{a\}^\perp \supset \{b\}^\perp$. Hence there exists x in $\{a\}^\perp$ such that $x \notin \{b\}^\perp$ which in turn implies that S is disjunctive.

(3)⇒(2). Let $a \neq b$ then surely either $a \wedge b < a$ or $a \wedge b < b$. Assume $a \wedge b < a$. As S is disjunctive, by definition, there exists x in $\{a \wedge b\}^\perp$ such that $x \wedge a \neq 0$. Thus we have $x \wedge a \in \{b\}^\perp$ and $x \wedge a \notin \{a\}^\perp$ i.e. $\{a\}^\perp \neq \{b\}^\perp$.

3. Remarks. REMARK 1. Let S be a 0-distributive semilattice. The Stone's space for S is obtained by considering the hull-Kernal topology on the set $\mathfrak{P}$ of all prime filters of S. It can be easily verified that the Stone's space is compact and T_0. If $\mathfrak{M}$ denotes the set of all maximal filters of S with the induced topology of $\mathfrak{P}$, $\mathfrak{M}$ is compact and T_1. Note, that closure of $\mathfrak{M}$ in $\mathfrak{P}$ is the hull of the set of dense elements of S. All these considerations follow verbatim from the considerations of Venkatanaramsimhan [6].

Let us now recall that Venkatanarasimhan [6] has characterized the Stone's space $\mathfrak{P}$ of a pseudocomplemented lattice as: $\mathfrak{P}$ is normal if and only if L is an S-lattice, where S-lattice is a pseudocomplemented lattice in which $a \vee a^* = 1$ for every a in L.

Since 0-distributive lattices are the generalizations of the pseudo-complemented lattices, it will be interesting to obtain analogus characterization for 0-distributive lattices (or semilattices).

REMARK 2. The relation "$\equiv$" defined by $a \equiv b$ if and only if $a \wedge x = 0$ is equivalent to $b \wedge x = 0$ in a lattice L is a congrence relation if L is 0-distributive (see [4]). The quotient lattice $\hat{L}$ of L with respect to this congruence relation is also 0-distributive. Following Venkatanarasimhan [6] one easily obtains the following results:

RESULT 1. There is a one–one reversible correspondence between the set of all maximal filters of L and the set of all prime filters of $\hat{L}$.

RESULT 2. $\mathfrak{M}$ is homeomorphic to $\mathfrak{P}$ where $\mathfrak{P}$ is the Stone's space of prime dual ideals of $\hat{L}$.

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REFERENCES

1. D. Adams, *Prime and maximal ideals in lattices*, Publicationes Mathematicae **17,** (1970), 57–59.

2. R. Balbes, *A representation theory for prime and implicative semilattices*, Trans. Amer. Math. Soc. **136,** (1969), 261–267.

3. Hoffman-Keimel, *A general character theory for partially ordered sets and lattices*, Mem. Amer. Math. Soc. **122,** (1972).

4. J. C. Varlet, *A generalization of the notion of pseudocomplementedness*, Bull. Soc. Roy. Liège, **36,** (1968), 149–158.

5. J. C. Varlet, *Distributive semilattices and Boolean Lattices*, Bull. Soc. Roy. Liège, **41,** (1972), 5–10.

https://doi.org/10.4153/CMB-1978-080-6 Published online by Cambridge University Press

6. P. V. Venkatanarasimhan, *Stone's topology for pseudocomplemented and bi-complemented lattices*, Trans. Amer. Math. Soc. **170,** (1972), 57–70.

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