Glasgow Math. J. **46** (2004) 529–536. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504001971. Printed in the United Kingdom

ON DISCONTINUOUS FUNCTIONAL VOLTERRA INTEGRAL EQUATIONS AND IMPULSIVE DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

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(Received 3 October, 2003; accepted 14 January, 2004)

Abstract. In this paper we apply fixed point results in ordered spaces to derive existence and comparison results for discontinuous functional integral equations of Volterra type in ordered Banach spaces. The results obtained are then applied to first order impulsive differential equations.

2000 *Mathematics Subject Classification.* 45D05, 45N05, 34A37.

1. Introduction. In this paper we prove existence and comparison results for the functional Volterra integral equation

$$u(t) = h(t, u) + \int_{t_0}^{t} f(t, s, u(s), u)\, ds, \qquad t \in J,$$

where J is a real interval with t_0 as its left endpoint, $f : \Lambda \times E \times L^p(J, E) \to E$, $\Lambda = \{(t, s) \in J \times J \mid a \leq s \leq t\}$ and $h : J \times L^p(J, E) \to E$, $1 \leq p \leq \infty$. We assume that E is a lattice-ordered Banach space which has the following properties.

(E0) Bounded and monotone sequences of E have weak limits.

(E1) The mapping $E \ni x \mapsto x^+ := \sup\{0, x\}$ is demicontinuous, and $\|x^+\| \leq \|x\|$ for all $x \in E$.

We shall also study cases where ordinary iterative methods are applicable. As an application we prove an existence and comparison result for a first order impulsive initial value problem involving discontinuities and functional dependencies.

The main features of this paper are:

– The functions h and f may be discontinuous in all their arguments.

– Many hypotheses common in papers dealing with equations in ordered Banach spaces, such as normality, (full) regularity and/or solidity of their order cones, or the existence of upper and lower solutions, are not assumed.

2. Preliminaries. Let J be a real interval and $E = (E, \leq, \|\cdot\|)$ a lattice-ordered Banach space having properties (E0) and (E1). Denote by $L^p(J, E)$, $1 \leq p \leq \infty$, the space of all strongly measurable functions $u : J \to E$ for which $t \mapsto \|u(t)\|$ belongs to

$L^p(J, \mathbb{R})$. Identifying a.e. equal functions, then $L^p(J, E)$ is an ordered Banach space with respect to the p-norm and the partial ordering

$$u \leq v \quad \text{if and only if } u(t) \leq v(t) \text{ for a.e. } t \in J. \tag{2.1}$$

Given $p \in [1, \infty]$, $w \in L^p(J, \mathbb{R}_+)$ and $a, b \in L^p(J, E)$, denote

$$\begin{cases} P = \{u \in L^p(J, E) \mid \|u(t)\| \leq w(t) \quad \text{for a.e. } t \in J\}, \\ [a, b] = \{u \in L^p(J, E) \mid a \leq u \leq b\}. \end{cases} \tag{2.2}$$

The hypothesis (E1) implies that the mapping $v^+ = \sup\{0, v\} = t \mapsto \sup\{0, v(t)\}$ belongs to $L^p(J, E)$ for each $v \in L^p(J, E)$ by [**7**, Corollary 3.1], and $\|v^+(t)\| \leq \|v(t)\|$ for all $t \in J$. These properties ensure that $v^+ = \sup\{0, v\}$, and hence also $v^- = \sup\{0, -v\}$ and $\inf\{0, v\} = -v^-$ belong to P for each $v \in P$.

We say that a mapping $G : P \to P$ is *increasing* if $Gu \leq Gv$ whenever $u, v \in P$ and $u \leq v$. Given a subset W of P, we say that $u \in W$ is the *least fixed point* of G in W if $u = Gu$, and if $u \leq v$ whenever $v \in W$ and $v = Gv$. The greatest fixed point of G in W is defined similarly, by reversing the inequality. A fixed point u of P is called *minimal* if $v \in P$, $v = Gv$ and $v \leq u$ imply $v = u$, and *maximal* if $v \in P$, $v = Gv$ and $u \leq v$ imply $v = u$.

Notice that if $G : P \to P$ is increasing, then the relations

$$G_- u = \inf\{0, Gu\}, \quad G_+ u = \sup\{0, Gu\}, \quad u \in P \tag{2.3}$$

define increasing mappings $G_\pm : P \to P$.

Our main result is based on the following fixed point theorem.

THEOREM 2.1. *Let E be a lattice-ordered Banach space with properties* (*E0*) *and* (*E1*), *let P be given by* (2.2) *with $w \in L^p(J, \mathbb{R}_+)$ and $p \in [1, \infty]$, and assume that $G : P \to P$ is an increasing mapping.*

(a) *G has minimal and maximal fixed points.*

(b) *G has least and greatest fixed points in $[a_-, b_+] \cap P$, where a_- is any minimal fixed point of G_- and b_+ is any maximal fixed point of G_+.*

(c) *G has least and greatest fixed points u_* and u^* in $[a, b] \cap P$, where a is the greatest fixed point of G_- and b is the least fixed point of G_+.*

Moreover, u_* and u^* are increasing with respect to G.

Proof. The results follow from [**7**], Theorems 5.1, 5.3 and 5.4 when L is the identity mapping and N is chosen to be G, G_- and G_+, respectively.

3. Existence and comparison results for discontinuous functional integral equations. Let E be an ordered Banach space and J a real interval with t_0 its left endpoint. In this section we study the functional Volterra integral equation

$$u(t) = h(t, u) + \int_{t_0}^{t} f(t, s, u(s), u)\, ds, \quad t \in J, \tag{3.1}$$

where $h : J \times L^p(J, E) \to E$, $f : \Lambda \times E \times L^p(J, E) \to E$, $1 \leq p \leq \infty$ and $\Lambda = \{(t, s) \in J \times J \mid a \leq s \leq t\}$.

Assuming that $L^p(J, E)$ is equipped with a.e. pointwise ordering (2.1), we impose the following hypotheses on the functions h and f.

(h0) $h(t, \cdot)$ is increasing for a.e. $t \in J$, $h(\cdot, u)$ is Bochner integrable for all $u \in L^p(J, E)$, and there exists an $\alpha \in L^p(J, \mathbb{R})$ such that $\|h(t, u)\| \leq \alpha(t)$ for a.e. $t \in J$ and all $u \in L^p(J, E)$.

(f0) The mappings $f(t, \cdot, u(\cdot), u)$, $t \in J$, and $t \mapsto \int_{t_0}^t f(t, s, u(s), u)\, ds$ are strongly measurable for each $u \in L^p(J, E)$.

(f1) $f(t, s, z, u)$ is increasing with respect to z and u for a.e. $(t, s) \in \Lambda$.

(f2) $\|f(t, s, x, u)\| \leq g(t, s, \|x\|)$ for a.e. $(t, s) \in \Lambda$ and all $x \in E, u \in L^p(J, E)$, where $g : \Lambda \times \mathbb{R}_+ \to \mathbb{R}_+$, $g(t, s, r)$ is increasing in r for a.e. $(t, s) \in \Lambda$, the functions $g(t, \cdot, w(\cdot))$ and $t \mapsto \int_{t_0}^t g(t, s, w(s))\, ds$ are Lebesgue integrable for each $w \in L^p(J, \mathbb{R})$ and the integral equation

$$w(t) = \beta(t) + \int_{t_0}^t g(t, s, w(s))\, ds, \qquad t \in J \tag{3.2}$$

has for each $\beta \in L^p(J, \mathbb{R}_+)$ the greatest solution in $L^p(J, \mathbb{R}_+)$.

Assuming also that E is lattice ordered and has properties (E0) and (E1) we prove in subsection 3.1 existence and comparison results for (3.1). In subsection 3.2 we study the cases where solutions of (3.1) can be obtained by iterative methods.

3.1. Existence and comparison results for (3.1). In this subsection we derive existence and comparison results for the integral equation (3.1) under the hypotheses given above.

THEOREM 3.1. *Let E be a lattice-ordered Banach space with properties* (*E0*) *and* (*E1*), *and assume that the hypotheses* (*f0*), (*f1*), (*f2*) *and* (*h0*) *are satisfied. Then the equation* (3.1) *has*

(a) *minimal and maximal solutions in* $L^p(J, E)$;

(b) *least and greatest solutions in* $[a_-, b_+]$, *where a_- is any minimal solution of the integral equation*

$$u(t) = -\left(h(t, u) + \int_{t_0}^t f(t, s, u(s), u)\, ds\right)^-, \qquad t \in J, \tag{3.3}$$

and b_+ is any maximal solution of the integral equation

$$u(t) = \left(h(t, u) + \int_{t_0}^t f(t, s, u(s), u)\, ds\right)^+, \qquad t \in J \tag{3.4}$$

in $L^p(J, E)$;

(c) *least and greatest solutions u_* and u^* in* $[a, b]$, *where a is the greatest solution of* (3.3) *and b is the least solution of* (3.4) *in* $L^p(J, E)$.

Moreover, the solutions u_* and u^* are increasing with respect to h and f.

Proof. Let P be given by (2.2), where $w \in L^p(J, \mathbb{R}_+)$ is the greatest solution of (3.2) with $\beta = \alpha$. We shall first show that the relation

$$Gu(t) = h(t, u) + \int_{t_0}^t f(t, s, u(s), u)\, ds, \qquad t \in J, \tag{3.5}$$

https://doi.org/10.1017/S0017089504001971 Published online by Cambridge University Press

defines a mapping $G: P \to P$. If $u \in P$, then $\|u(t)\| \le w(t)$ for a.e. $t \in J$. Applying the hypotheses (h0) and (f2) we obtain

$$\|Gu(t)\| \le \|h(t,u)\| + \int_{t_0}^{t} \|f(t,s,u(s),u)\|\,ds \le \alpha(t) + \int_{t_0}^{t} g(t,s,w(s))\,ds = w(t)$$

for a.e. $t \in J$. This result implies that G maps P into P. The hypotheses (f1) and (h0) imply that if $u,\ v \in L^p(J,E)$ and $u \le v$, then

$$Gu(t) = h(t,u) + \int_{t_0}^{t} f(t,s,u(s),u)\,ds \le h(t,v) + \int_{t_0}^{t} f(t,s,v(s),v)\,ds = Gv(t)$$

for a.e. $t \in J$. This proves that G is increasing.

Thus the hypotheses of Theorem 2.1 hold for $G: P \to P$, defined by (3.5). Assume now that u is a fixed point of G in $L^p(J,E)$, and let $\overline{w}$ denote the greatest solution of (3.2) with $\beta(t) = \max\{\|u(t)\|, \alpha(t)\}$. Then

$$\begin{aligned}\|u(t)\| &\le \|h(t,u)\| + \int_{t_0}^{t} \|f(t,s,u(s),u)\|\,ds \le \alpha(t) + \int_{t_0}^{t} g(t,s,\|u(s)\|)\,ds\\ &\le \alpha(t) + \int_{t_0}^{t} g(t,s,\overline{w}(s))\,ds \le \beta(t) + \int_{t_0}^{t} g(t,s,\overline{w}(s))\,ds = \overline{w}(t), \quad t \in J.\end{aligned}$$

Thus, denoting $\underline{w} = t \mapsto \|u(t)\|$, the relation

$$Qv(t) = \alpha(t) + \int_{t_0}^{t} g(t,s,v(s))\,ds, \quad t \in J \tag{3.6}$$

defines an increasing mapping Q from the order interval $[\underline{w}, \overline{w}]$ of $L^p(J,\mathbb{R})$ into itself. It follows from [**8**, Theorem 1.2.3 and Proposition 5.8.9] that Q has a fixed point in $[\underline{w}, \overline{w}]$. But w, as the greatest solution of (3.2), is the greatest fixed point of Q, whence $\|u(t)\| = \underline{w}(t) \le w(t)$ for a.e. $t \in J$. This proves that all the solutions of (3.1) are contained in P. Because of the property (E1) of E the similar reasoning shows that all the solutions of (3.3) and (3.4) belong to P. Noticing also that fixed points of G defined by (3.5) are solutions of (3.1) and vice versa, the assertions follow from Theorem 2.1.

3.2. Applicability of iterative methods. In this subsection we consider a case when the extremal solutions of the integral equation (3.1) can be obtained by successive approximations.

PROPOSITION 3.1. *Let E be a lattice-ordered Banach space with properties* (*E0*) *and* (*E1*). *Assume that the hypotheses* (*f0*), (*f1*), (*f2*), *and* (*h0*) *hold, and that*

(B) $h(t,u_n) \rightharpoonup h(t,u)$ *for a.e.* $t \in J$ *and* $f(t,s,u_n(s),u_n) \rightharpoonup f(t,s,u(s),u)$ *for a.e.* $(t,s) \in \Lambda$ *if* (u_n) *is a monotone sequence in* $L^p(J,E)$ *and* $u_n(s) \rightharpoonup u(s)$ *for a.e.* $s \in J$.

Then the successive approximations:

(a) $a_{n+1}(t) = -(h(t,a_n) + \int_{t_0}^{t} f(t,s,a_n(s),a_n)\,ds)^-, \quad t \in J, \quad a_0 = 0,$
converge weakly a.e. pointwise to the greatest solution a of (3.3);

(b) $b_{n+1}(t) = (h(t,b_n) + \int_{t_0}^{t} f(t,s,b_n(s),b_n)\,ds)^+, \quad t \in J, \quad b_0 = 0,$
converge weakly a.e. pointwise to the least solution b of (3.4);

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(c) $u_{n+1}(t) = h(t, u_n) + \int_{t_0}^t f(t, s, u_n(s), u_n)\, ds, \quad t \in J, \quad u_0 = a,$
converge weakly a.e. pointwise to the least solution u_* of (3.1) in $[a, b]$;

(d) $v_{n+1}(t) = h(t, v_n) + \int_{t_0}^t f(t, s, v_n(s), v_n)\, ds, \quad t \in J, \quad v_0 = b,$
converge weakly a.e. pointwise to the greatest solution u^* of (3.1) *in* $[a, b]$.

Proof. It is easy to see that the sequences (b_n) and (u_n) are increasing, and that the sequences (a_n) and (v_n) are decreasing. Moreover, all these are contained in P, whence they are a.e. pointwise bounded. Thus it follows from the hypothesis (E0) that all these sequences possess asserted a.e. pointwise weak limits a, u_*, b and u^*. In view of (2.2), [**7**, (3.1), Corollary 3.1 and Proposition 3.2] these limits belong to P. The hypothesis (B) implies that

$$\varphi(f(t, s, u_n(s), u_n)) \to \varphi(f(t, s, u_*(s), u_*)) \quad \text{for a.e. } (t, s) \in \Lambda \text{ and for all } \varphi \in E'.$$

This result and the Dominated Convergence Theorem imply that

$$\begin{aligned}\varphi\left(\int_{t_0}^t f(t, s, u_n(s), u_n)\, ds\right) &= \int_{t_0}^t \varphi(f(t, s, u_n(s), u_n))\, ds \to \int_{t_0}^t \varphi(f(t, s, u_*(s), u_*))\, ds \\ &= \varphi\left(\int_{t_0}^t f(t, s, u_*(s), u_*)\, ds\right)\end{aligned}$$

for a.e. $t \in J$ and for all $\varphi \in E'$. In view of this result and the hypothesis (B) we have

$$h(t, u_n) \rightharpoonup h(t, u_*) \text{ and } \int_{t_0}^t f(t, s, u_n(s), u_n)\, ds \rightharpoonup \int_{t_0}^t f(t, s, u_*(s), u_*)\, ds \quad \text{for a.e. } t \in J.$$

It then follows from (c) as $n \to \infty$ that u_* is a solution of (3.1). Similar reasoning shows that u^* is also a solution of (3.1), that a is a solution of (3.3), and that b_- is a solution of (3.4). By standard arguments one can show that a is the greatest solution of (3.3), that b is the least solution of (3.4), and that u_* and u^* are least and greatest solutions of (3.1) in $[a, b]$.

Remarks 3.1. The hypothesis (B) is required to hold only for those iteration sequences which are defined in Proposition 3.1.

If the values of h and f are contained in the order cone E_+ of E, then in Theorem 3.1 and in Proposition 3.1 $u^* = b$ is the least solution of (3.1). Similarly, if the values of h and f are in $-E_+$, then $u_* = a$ is the greatest solution of (3.1). Thus the lower and upper bounds a and b of the solutions u_* and u^* cannot be improved, in general.

4. An application to an impulsive IVP. The result of Theorem 3.1 will now be applied to the following impulsive initial value problem (IIVP)

$$\begin{cases} u'(t) + p(t)u(t) = F(t, u(t), u) \text{ a.e. on } J = [t_0, t_1], \\ u(t_0) = x_0, \ \Delta u(\lambda) = H(\lambda, u), \ \lambda \in W, \end{cases} \tag{4.1}$$

where $p \in L^1(J, \mathbb{R})$, $F : J \times E \times L^1(J, E) \to E$, $x_0 \in E$, $\Delta u(\lambda) = u(\lambda + 0) - u(\lambda)$, $H : W \times L^1(J, E) \to E$, and W is a well-ordered (and hence countable) subset of (t_0, t_1).

Denoting $W^{<t} = \{\lambda \in W \mid \lambda < t\}$, $t \in J$, and by $AC(J, E)$ the set of all absolutely continuous functions $v : J \to E$, we say that $u : J \to E$ is a solution of the IIVP (4.1) if it satisfies the equations of (4.1), and if it is contained in the set

$$V = \left\{ u : J \to E \mid \sum_{\lambda \in W} \|\Delta u(\lambda)\| < \infty \quad \text{and} \quad t \mapsto u(t) - \sum_{\lambda \in W^{<t}} \Delta u(\lambda) \in AC(J, E) \right\}.$$

It is easy to verify that V is a subset of $L^1(J, E)$.

The following result, which is proved in [**3**], allows us to convert the IIVP (4.1) to a Volterra integral equation.

LEMMA 4.1. ([**3**, Lemma 3.1]) *If* $p \in L^1(J, \mathbb{R})$, $q \in L^1(J, E)$, $x_0 \in E$ *and* $c : W \to E$, *and if* $\sum_{\lambda \in W} \|c(\lambda)\| < \infty$, *then problem*

$$\begin{cases} u'(t) + p(t)u(t) = q(t) \ \text{ a.e. on } \ J, \\ u(t_0) = x_0, \ \ \Delta u(\lambda) = c(\lambda), \ \ \lambda \in W, \end{cases} \tag{4.2}$$

has a unique solution u. This solution can be represented as

$$u(t) = e^{-\int_{t_0}^t p(s)ds} x_0 + \sum_{\lambda \in W^{<t}} e^{-\int_\lambda^t p(s)ds} c(\lambda) + \int_{t_0}^t e^{-\int_s^t p(\tau)d\tau} q(s)\, ds, \qquad t \in J. \tag{4.3}$$

We shall impose the following hypotheses on the functions H and F.

(H0) $H(\lambda, \cdot)$ is increasing for all $\lambda \in W$, and there exists an $M > 0$ such that $\sum_{\lambda \in W} \|H(\lambda, u)\| \le M$ for all $u \in L^1(J, E)$.

(F0) The mapping $F(\cdot, u(\cdot), u)$ is Bochner integrable for each $u \in L^1(J, E)$.

(F1) $F(s, z, u)$ is increasing with respect to z and u for a.e. $s \in J$.

(F2) $\|F(s, x, u)\| \le q(s)\psi(\|x\|)$ for a.e. $s \in J$ and all $x \in E$, $u \in L^1(J, E)$, where $q \in L^1(J, \mathbb{R}_+)$, $\psi : \mathbb{R}_+ \to (0, \infty)$ is increasing, and $\int_0^\infty \frac{dx}{\psi(x)} = \infty$.

THEOREM 4.1. *Let E be a lattice-ordered Banach space with properties* (*E*0) *and* (*E*1), *and assume that the hypotheses* (*F*0), (*F*1), (*F*2) *and* (*H*0) *are satisfied. Then the IIVP* (4.1) *has for each* $x_0 \in E$ *and* $p \in L^1(J, \mathbb{R})$,

(a) *minimal and maximal solutions*;

(b) *least and greatest solutions in* $[a_-, b_+]$, *where* a_- *is any minimal solution of the integral equation*

$$u(t) = -\left(e^{-\int_{t_0}^t p(s)ds} x_0 + \sum_{\lambda \in W^{<t}} e^{-\int_\lambda^t p(s)ds} H(\lambda, u) + \int_{t_0}^t e^{-\int_s^t p(\tau)d\tau} F(s, u(s), u)\, ds \right)^-, \tag{4.4}$$

and b_+ *is any maximal solution of the integral equation*

$$u(t) = \left(e^{-\int_{t_0}^t p(s)ds} x_0 + \sum_{\lambda \in W^{<t}} e^{-\int_\lambda^t p(s)ds} H(\lambda, u) + \int_{t_0}^t e^{-\int_s^t p(\tau)d\tau} F(s, u(s), u) ds \right)^+; \tag{4.5}$$

(c) *least and greatest solutions* u_* *and* u^* *in* $[a, b]$, *where a is the greatest solution of* (4.4) *and b is the least solution of* (4.5).

Moreover, the solutions u_* *and* u^* *are increasing with respect to* x_0, *H and F*.

Proof. The given hypotheses ensure that for each $x_0 \in E$ the relations

$$\begin{cases} h(t,u) = e^{-\int_{t_0}^t p(s)ds} x_0 + \sum_{\lambda \in W^{<t}} e^{-\int_\lambda^t p(s)ds} H(\lambda, u), & t \in J,\ u \in L^1(J,E), \\ f(t,s,x,u) = e^{-\int_s^t p(\tau)d\tau} F(s,x,u), & (t,s) \in \Lambda,\ u \in L^1(J,E) \end{cases} \tag{4.6}$$

define mappings $h : J \times L^1(J,E) \to E$ and $f : \Lambda \times E \times L^1(J,E)$. Denoting

$$\begin{cases} K = e^{\int_{t_0}^{t_1} |p(s)|ds}, \quad \alpha(t) = (\|x_0\| + M)K, \quad t \in J, \\ g(t,s,r) = Kq(s)\psi(r),\ (t,s) \in \Lambda,\ r \geq 0, \end{cases} \tag{4.7}$$

it follows that the hypotheses (h0), (f0), (f1), and also (f2) hold with the exception that $\beta \in L^1(J, \mathbb{R}_+)$ is now replaced in (3.2) by a constant $w_0 \geq 0$. This replacement and (4.7) imply that (3.2) can be rewritten as

$$w(t) = w_0 + \int_{t_0}^t Kq(s)\psi(w(s))\, ds, \quad t \in J. \tag{4.8}$$

The hypothesis (F2) and [**2**, Lemma B.7.1] ensure that (4.8) has a unique solution in $AC(J, \mathbb{R})$. This is enough because in the proof of Theorem 3.1 we used the functions $\beta = \alpha$, which is now constant by (4.7), and $\beta(t) = \max\{\alpha(t), \|u(t)\|\}$, $t \in J$, where u is a fixed point of G, i.e. a solution of (3.1), which by (4.6) can be rewritten as

$$u(t) = e^{-\int_{t_0}^t p(s)ds} x_0 + \sum_{\lambda \in W^{<t}} e^{-\int_\lambda^t p(s)ds} H(\lambda, u) + \int_{t_0}^t e^{-\int_s^t p(\tau)d\tau} F(s, u(s), u)\, ds. \tag{4.9}$$

Thus

$$\|u(t)\| \leq w_0 := (\|x_0\| + M)K + \int_{t_0}^{t_1} K\|F(s, u(s), u)\|\, ds, \quad t \in J,$$

whence we can replace the function $\beta(t) = \max\{\alpha(t), \|u(t)\|\}$, $t \in J$ in the proof of Theorem 3.1 by w_0. Consequently, the results of Theorem 3.1 hold for (3.1), or equivalently, for (4.9), which implies the assertions because by Lemma 4.1 the solutions of the IIVP (4.1) are the same as the solutions of the integral equation (4.9).

REMARKS 4.1. The result of Proposition 3.1 implies that some solutions of the IIVP (4.1) can be obtained via successive approximations if H and F satisfy also the following hypothesis.

(A) $H(\lambda, u_n) \rightharpoonup H(\lambda, u)$ for all $\lambda \in W$ and $F(s, u_n(s), u_n) \rightharpoonup F(s, u(s), u)$ for a.e. $s \in J$ if (u_n) is a monotone sequence in $L^1(J,E)$ and $u_n(s) \rightharpoonup u(s)$ for a.e. $s \in J$.

The functional dependence on the last argument u of h, f, H and F can be formed, e.g., by bounded, linear and positive operators, such as integral operators of Volterra and/or Fredholm type with nonnegative kernels. Thus the results derived in this paper can be applied also to integro-differential equations.

Weakly complete Banach lattices have properties (E0) and (E1) (cf. [**10**]). Examples of such spaces are, for instance, UMB-lattice defined in [**1**, XV, 14], the spaces $\mathbb{R}^m$, $m = 1, 2, \ldots$, and l^p, $p \in [1, \infty)$, ordered coordinatewise and normed by p-norm, and spaces $L^p(\Omega)$, where $p \in [1, \infty)$ and $\Omega = (\Omega, \mathcal{A}, \mu)$ is a measure space, equipped with p-norm and a.e. pointwise ordering. Moreover, the Sobolev spaces $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$, $p \in (1, \infty)$ ordered a.e. pointwise, where Ω is a bounded domain in $\mathbb{R}^m$, posses properties

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(E0) and (E1) (cf. [**2**]). In particular, we can choose E to be one of these spaces in the above considerations. As for other existence results for integral equations in abstract spaces, see, e.g., [**4**, **5**, **9**, **11**, **12**]. Existence results for explicit and implicit discontinuous IIVP's are derived in [**3**] in the case when the order cone of E is regular.

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