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On Sylow intersections

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Let G be a finite group, p a prime divisor of $|G|$, and T a p-subgroup of G. Define $\sigma(T)$ to be the number of Sylow p-subgroups of G containing T. Call T a central p-Sylow intersection if for some $\Sigma \subseteq \mathrm{Syl}_p(G)$, $T = \bigcap\{S \mid S \in \Sigma\}$, and if, in addition, T contains the center of a Sylow p-subgroup of G. This work is inspired and motivated by work of G. Stroth [*J. Algebra* 37 (1975), 111-120]. Generalizing an argument of his we describe finite groups in which every central p-Sylow intersection T with $p\text{-rank}(T) > 2$ satisfies $\sigma(T) \le p$.

Related methods yield the description of finite groups in which every central p-Sylow intersection T with $p\text{-rank}(T) \ge 2$ satisfies $\sigma(T) \le 2p$.

1. Introduction

Let G be a finite group, p a prime divisor of the order of G, and T a p-subgroup of G. Define $\sigma(T)$ to be the number of Sylow p-subgroups of G containing T, and $p\text{-rank}(T)$ to be the maximal number n such that T contains an elementary abelian subgroup of order p^n. We call T a p-Sylow intersection if for some $\Sigma \subseteq \mathrm{Syl}_p(G)$, $T = \bigcap\{S \mid S \in \Sigma\}$, and we call T a central p-Sylow intersection if, in addition, T contains the center of a Sylow p-subgroup of G.

In a previous paper [9] we proved

THEOREM 1. *Let every central p-Sylow intersection T satisfy*

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$\sigma(T) \le 2p$. *Then there exists a non identity abelian subgroup, strongly closed in a Sylow* p*-subgroup of* G *with respect to* G .

In that paper, [9], we also characterized those groups G in which $\sigma(T) \le 6$ for every central 2-Sylow intersection T .

This work is inspired and motivated by Stroth [10]. In that paper, Stroth gives a detailed characterization of finite groups in which every intersection of two distinct Sylow 2-subgroups is of 2-rank ≤ 2 , and in which there exists such an intersection of 2-rank = 2 . Generalizing an argument of his we prove:

THEOREM 2. *Let every central* p*-Sylow intersection* T *with* $p\text{-rank}(T) > 2$ *satisfy* $\sigma(T) \le p$. *Let* S *be a Sylow* p*-subgroup of* G . *Then either*

(i) $\Omega(Z(S))$ *is strongly closed in* S *with respect to* G *, or*

(ii) $p\text{-rank}(S) = 2$ *, or*

(iii) *there exists some* $x \in S$ *with* $C_S(x)$ *elementary abelian of order* p^2 .

We remark that by Lemma 6 below, the condition of Theorem 2 forces every central p-Sylow intersection T with $p\text{-rank}(T) > 2$ to belong to $\text{Syl}_p(G)$. We also remark that p-groups satisfying conclusion *(iii)* are discussed in [7], Kapitel III, §14. For $p = 2$, a subgroup S satisfying *(iii)* is dihedral or semidihedral by Lemma 4 of [11]. Thus we get

COROLLARY 3. *Let every central* 2*-Sylow intersection* $T \notin \text{Syl}_2(G)$ *satisfy* $2\text{-rank}(T) \le 2$. *Let* S *be a Sylow* 2*-subgroup of* G . *Then either*

(i) $\Omega(Z(S))$ *is strongly closed in* S *with respect to* G *, or*

(ii) $2\text{-rank}(S) = 2$.

We remark that for finite simple groups the conclusions of Corollary 3 are in fact equivalent to its assumptions. These finite simple groups were already listed in [10], namely: $L_2(q)$, $U_3(q)$, $Sz(q)$, q even, $L_2(q)$, $L_3(q)$, $U_3(q)$, q odd, A_7, M_{11} , and simple groups of Janko-Ree type.

In this paper we also prove a generalization of Theorem 1.

THEOREM 4. *Let every central p-Sylow intersection T with $p\text{-rank}(T) \geq 2$ satisfy $\sigma(T) \leq 2p$. Let S be a Sylow p-subgroup of G. Then either*

(i) there exists a non identity abelian subgroup, strongly closed in S with respect to G, or

(ii) there exists some $x \in S$ with $C_S(x)$ elementary abelian of order p^2.

Again, for $p = 2$, we get

COROLLARY 5. *Let every central 2-Sylow intersection T with $2\text{-rank}(T) \geq 2$ satisfy $\sigma(T) \leq 4$. Let S be a Sylow 2-subgroup of G. Then either*

(i) there exists a non identity abelian subgroup, strongly closed in S with respect to G, or

(ii) S is dihedral or semidihedral.

Those finite simple groups satisfying the hypothesis of Corollary 5 are: $L_2(q)$, $U_3(q)$, $Sz(q)$, q even, $L_2(q)$, $q \equiv 3, 5 \pmod 8$, and simple groups of Janko-Ree type, as can be verified by [4] and by Remark 8 of [9].

2. Preliminary results

LEMMA 6. *(i) If T is a p-subgroup of G then $\sigma(T) \equiv 1 \pmod p$.*

(ii) Let T and T' be p-Sylow intersections in G. If $T \subseteq T'$, then $\sigma(T) \geq \sigma(T')$, and if $T \subset T'$, then $\sigma(T) > \sigma(T')$.

(iii) If T is a p-Sylow intersection satisfying $\sigma(T) = 1$, then $T \in \mathrm{Syl}_p(G)$.

(iv) If T is a p-Sylow intersection satisfying $\sigma(T) = 1 + p$, then $N_S(T)/T$ is cyclic of order p, for every Sylow p-subgroup S of G containing T.

Proof. Assertion *(i)* is Lemma 6 of [8]. Assertion *(ii)* is trivial once we notice that any p-Sylow intersection T is the intersection of those Sylow p-subgroups containing it. Assertion *(iii)* is also trivial.

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Denote by Ω the set of $1 + p$ Sylow p-subgroups of G containing T, and take any $S \in \Omega$. The subgroup $N_S(T)$ acts by conjugation on $\Omega' \equiv \Omega \setminus \{S\}$. If $g \in N_S(T)$ stabilizes some $R \in \Omega'$, then being a p-element, $g \in N_S(T) \cap R \subseteq S \cap R$. But $S \supset S \cap R \supseteq T$ forces $S \cap R = T$ by *(i)*, *(ii)*, and *(iii)*, so that $N_S(T)/T$ acts faithfully on Ω'. In fact, every $g \in \left(N_S(T)/T\right)^{\#}$ acts fixed point freely on Ω', whence $|\Omega'| = p$ yields assertion *(iv)*.

The following result is due to Alperin. The first part is Theorem 5.2 of [1], and the second is a strengthening of the Corollary in [2], achieved by self suggestive changes in its proof.

THEOREM 7 (Alperin). *Let x and y be elements of $S \in \mathrm{Syl}_p(G)$, such that x is conjugate to y in G. Then there exist central p-Sylow intersections $H_i \subseteq S$, $i = 1, \ldots, n$, and elements $t_i \in N_G(H_i)$, $i = 1, \ldots, n$, such that $N_S(H_i) \in \mathrm{Syl}_p(N_G(H_i))$, $i = 1, \ldots, n$, that t_i is a p-element if $H_i \subset S$, and that, setting $x_1 = x$, $x_2 = x^{t_1}$, $\ldots$, $x_{n+1} = x^{t_1 t_2 \ldots t_n}$, we get $x_i \in H_i$, $i = 1, \ldots, n$, and $x_{n+1} = y$.*

Moreover, if $C_S(y) \in \mathrm{Syl}_p(C_G(y))$, then we can assure in addition that

(i) $C_S(x_i) \subseteq H_i$, $i = 1, \ldots, n$, *and that*

(ii) $|C_S(x_1)| \le |C_S(x_2)| \le \ldots \le |C_S(x_{n+1})|$.

Let S be a Sylow p-subgroup of G. Denote by J the set of elements in $S \setminus Z(S)$, which are conjugate in G to an element of $\Omega(Z(S))$. Denote by J^* the set of those elements $j \in J$ which are conjugate to an element of $\Omega(Z(S))$ in $N_G(C_S(j))$.

LEMMA 8. *(i) $J = \emptyset$ if and only if $\Omega(Z(S))$ is strongly closed in S with respect to G.*

(ii) For every $j \in J$ there exists some $S' \in \mathrm{Syl}_p(C_G(j)) \subseteq \mathrm{Syl}_p(G)$,

such that $C_S(j) = S \cap S'$ $\left(\textit{whence}\ \sigma(C_S(j)) > p\right)$.

(iii) If $J \neq \emptyset$ *there exist* $j \in J$ *and* g *, a p-element of* $N_G(C_S(j))$ *, such that:*

(1) $j \in \Omega(Z(S^g))$;

(2) $C_S(j) = S \cap S^g$; *and*

(3) $N_S(C_S(j)) \in \mathrm{Syl}_p(N_G(C_S(j)))$;

in particular, $J \neq \emptyset$ *implies* $J^* \neq \emptyset$.

Proof. Assertion *(i)* is obvious. To satisfy assertion *(ii)*, any $S' \in \mathrm{Syl}_p(C_G(j))$ containing $C_S(j)$ will do. To prove *(iii)* choose some $x \in J$, and some $y \in \Omega(Z(S))$ such that x is conjugate in G to y . Now quote Theorem 7. As $|C_S(y)| > |C_S(x)|$, the set $\{k \mid |C_S(x_k)| < |C_S(x_{k+1})|\}$ is not empty; let i_0 be its maximal element, and set $j = x_{i_0}$, $g = t_{i_0}^{-1}$, and $H \equiv H_{i_0}$. Clearly $j \in J$, $j^{g^{-1}} \in \Omega(Z(S))$, and $C_S(j) \subseteq H$. Now j is conjugate in $N_G(H)$ to $j^{g^{-1}}$ which is an element of $H \cap Z(S) \subseteq Z(H)$. Thus $H \subseteq C_S(j)$ and we are done.

COROLLARY 9. *Let* S *be a Sylow p-subgroup of* G *. If* $\sigma(T) \leq p$ *for every central p-Sylow intersection* T *with* $p\text{-rank}(T) \geq 2$ *, then* $\Omega(Z(S))$ *is strongly closed in* S *with respect to* G *.*

Proof. If $j \in J$, then $T \equiv C_S(j)$ is a central p-Sylow intersection with $p\text{-rank}(T) \geq 2$ and $\sigma(T) > p$ by *(ii)* of Lemma 8. Thus $J = \emptyset$ and we are through by *(i)* of Lemma 8.

We remark that the results of Herzog and Shult [6] and those of Gomi [5] follow from Corollary 9 and Goldschmidt [4].

3. Proof of Theorem 2

By Lemma 8 *(i)* and *(iii)*, either conclusion *(i)* of our theorem holds, or or $J^* \neq \emptyset$. Thus we may assume the existence of $i \in \Omega(Z(S))$ and $j \in J$ such that i is conjugate to j in $N_G(C_S(j))$. By Lemma 8 *(ii)*, $\sigma(C_S(j)) > p$; hence, $C_S(j)$ being a central p-Sylow intersection, $p\text{-rank}(C_S(j)) = 2$ by the assumption of the theorem. As $Z(C_S(j))$ contains the subgroup $\langle i, j\rangle$, which is elementary abelian of order p^2 , it follows that

$$(*) \qquad \Omega(C_S(j)) = \langle i, j\rangle ,$$

and that $\Omega(Z(S)) = \langle i\rangle$.

As $C_S(\langle i, j\rangle) = C_S(j)$, and $N_S(\langle i, j\rangle) \supseteq N_S(C_S(j))$, we have that $N_S(C_S(j))/C_S(j)$ is a subgroup of $N_S(\langle i, j\rangle)/C_S(\langle i, j\rangle)$, which is isomorphic to a subgroup of $GL(2, p)$ - the automorphism group of an elementary abelian group of order p^2 . Thus, being a non-trivial p-group, $N_S(C_S(j))/C_S(j)$ is cyclic of order p .

Now $N_G(C_S(j))$ acts by conjugation on Ω , the set of all cyclic subgroups of $\langle i, j\rangle$, and we claim that this action is transitive. Indeed, take any $g \in N_S(C_S(j))\backslash C_S(j)$. As g is a p-element which does not centralize any element of $\langle i, j\rangle\backslash\langle i\rangle$, it follows that g acts fixed point freely on $\Omega\backslash\{\langle i\rangle\}$. Hence $|\Omega| = p + 1$ yields that g acts transitively on $\Omega\backslash\{\langle i\rangle\}$. Concluding the fact that i is conjugate to j in $N_G(C_S(j))$ implies that $\langle i\rangle$ is conjugate to $\langle j\rangle$ in $N_G(C_S(j))$ and our claim is proved.

Proceeding with our proof, let us assume first that $N_S(C_S(j)) \subset S$. Take any $g \in N_S(N_S(C_S(j)))\backslash N_S(C_S(j))$ to get $C_S(j) \neq C_S(j^g) \subseteq N_S(C_S(j))$, whence $j^g \notin \langle i, j\rangle$ implies that $\langle j^g\rangle^\# \subseteq N_S(C_S(j))\backslash C_S(j)$. As $N_S(C_S(j))/C_S(j)$ is cyclic of order p , no element of $\langle j^g\rangle^\#$ has any p-roots in $C_S(j^g)$. Hence, by the preceding paragraph, no element of $\langle i, j\rangle^\#$ has any p-roots in $C_S(j)$. It follows by (*), that

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$C_S(j) = \langle i, j\rangle$, and conclusion *(iii)* holds.

Assume then that $N_S(C_S(j)) = S$, so that $|S : C_S(j)| = p$. Let K be an elementary abelian subgroup of S of maximal order. If $|K \cap C_S(j)| \geq p^2$, then (*) forces $j \in K \cap C_S(j)$, whence $K \subseteq C_S(j)$, and (*) yields $|K| \leq p^2$. If $|K \cap C_S(j)| \leq p$, then $|K : K \cap C_S(j)| \leq p$ implies that again $|K| \leq p^2$. Anyhow, conclusion *(ii)* holds and we are done.

4. Proof of Theorem 4

If $p\text{-rank}(Z(S)) > 1$, then by our assumption $\sigma(Z(S)) \leq 2p$, whence conclusion *(i)* holds by Theorem 1. Hence we may assume that $p\text{-rank}(Z(S)) = 1$, whence $\Omega(Z(S))$ is cyclic of order p, say, $\Omega(Z(S)) = \langle i\rangle$.

If $j \in J$, then $p\text{-rank}(C_S(j)) \geq 2$. Thus, being a central p-Sylow intersection, $p < \sigma(C_S(j)) \leq 2p$. Hence Lemma 6 *(i)* yields that $\sigma(C_S(j)) = 1 + p$, and Lemma 8 *(iii)* that $N_S(C_S(j))/C_S(j)$ is cyclic of order p.

We claim that

(*) if $j_1 \in J^*$, $j_2 \in J$, and $|C_S(j_1) : C_S(j_1) \cap C_S(j_2)| = p$, then *(ii)* holds.

Indeed, $C_S(j_1) \cap C_S(j_2)$ is a central p-Sylow intersection strictly contained in $C_S(j_1)$. Thus $1 + p = \sigma(C_S(j)) < \sigma(C_S(j_1) \cap C_S(j_2))$ by Lemma 6 *(ii)*. Hence, by Lemma 6 *(i)*, $2p < \sigma(C_S(j_1) \cap C_S(j_2))$, so that $p\text{-rank}(C_S(j_1) \cap C_S(j_2)) = 1$, and $\Omega(C_S(j_1) \cap C_S(j_2)) = \Omega(Z(S))$. Now $\langle j_1\rangle^{\#} \cap \Omega(Z(S)) = \emptyset$, so that by the assumption of (*), no element of $\langle j_1\rangle^{\#}$ has any p-roots in $C_S(j_1)$. As $j_1 \in J^*$, so is the case with the elements of $\Omega(Z(S))^{\#}$. Thus, the fact that $\Omega(C_S(j_1) \cap C_S(j_2)) = \Omega(Z(S))$

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implies that $C_S(j_1) \cap C_S(j_2) = \Omega(Z(S)) = \langle i \rangle$, so that $C_S(j_1)$ is elementary abelian of order p^2, and claim (*) is proved.

By Lemma 8 *(i)* and *(iii)*, either conclusion *(i)* of our theorem holds, or $J^* \neq \emptyset$. Thus we may assume the existence of some $j_0 \in J^*$. If $C_S(j_0)$ is not normal in S, take $g \in N_S(N_S(C_S(j_0))) \backslash N_S(C_S(j_0))$. Then $j_0^g \in J$, and $C_S(j_0) \neq C_S\left(j_0^g\right) \subseteq N_S(C_S(j_0))$. As $N_S(C_S(j_0))/C_S(j_0)$ is cyclic of order p, it follows that $\left|C_S(j_0) : C_S(j_0) \cap C_S\left(j_0^g\right)\right| = p$, and *(ii)* holds by (*). Hence we may assume that $C_S(j_0)$ is normal in S, so that $|S : C_S(j_0)| = p$. By the same argument we may assume now that $C_S(j_0)$ is normal in $N_G(S)$, for otherwise we take any $g \in N_G(S)\backslash N_G(C_S(j_0))$, and repeat the process, to show that *(ii)* holds. We may also assume that $J^* \subseteq C_S(j_0)$. If this is not the case, take in (*) any $j \in J^*\backslash C_S(j_0)$ as j_1, and j_0 as j_2, and conclude that *(ii)* holds.

We claim now, that $\Omega(Z(C_S(j_0)))$ is strongly closed in S with respect to G. To prove it, suppose that there exist $j' \in S\backslash\Omega(Z(C_S(j_0)))$, and $k' \in \Omega(Z(C_S(j_0)))$, such that j' is conjugate to k' in G. By Theorem 7 we may assume that there exists a central p-Sylow intersection H and elements $j \in H\backslash\Omega(Z(C_S(j_0)))$ and $k \in H \cap \Omega(Z(C_S(j_0)))$, such that j is conjugate to k in $N_G(H)$.

Assume first that $k \notin \Omega(Z(S))$. Then $\langle k, \Omega(Z(S))\rangle \subseteq H \cap C_S(j_0) \subseteq H$ yields that $1 \leq \sigma(H) \leq \sigma(H \cap C_S(j_0)) \leq 2p$ by the assumption of the theorem. Thus, by Lemma 6, either $\sigma(H) = 1$, whence $H = S$, or $\sigma(H) = \sigma(H \cap C_S(j_0)) = \sigma(C_S(j_0))$, and $H = C_S(j_0)$. But $\Omega(Z(C_S(j_0)))$ is normal in $\langle N_G(S), N_G(C_S(j_0))\rangle$, a contradiction.

Thus we may assume that $k \in \Omega(Z(S))$ so that $j \in J$. Moreover, $\langle j, \Omega(Z(S))\rangle \subseteq H \cap C_S(j) \subseteq H$ yields as before that either $H = S$ or

$H = C_S(j)$. The first case is impossible as $j \notin \Omega(Z(S))$ and $k \in \Omega(Z(S))$.

Thus $j \in J^*$, and as $J^* \subseteq C_S(j_0)$, we have

$$\langle j, \Omega(Z(C_S(j_0)))\rangle \subseteq C_S(j) \cap C_S(j_0) \subseteq C_S(j) .$$

Using again the above argument yields $C_S(j) = C_S(j_0)$, so that $j \in \Omega(Z(C_S(j_0)))$, a contradiction. Thus our claim is proved and conclusion *(i)* of our theorem holds.

We remark that the condition $\sigma(T) \le 2p$ in the assumption of the theorem is needed only to assure that if $T \subset T' \subseteq S \in \text{Syl}_p(G)$, where T' is a p-Sylow intersection, then

(i) $N_S(T)/T$ is cyclic of order p , and

(ii) $T' = S$ (see Lemma 6).

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