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The interruption phenomenon for generalized continued fractions

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Dedicated to K. Mahler on the occasion of his 75th birthday

After defining a generalized C-fraction (a kind of Jacobi-Perron algorithm) for an n-tuple of formal power series over $\mathbb{C}$ $(n \geq 2)$, the connection between interruptions in the algorithm and linear dependence over $\mathbb{C}[x]$ of the power series is studied.

Examples will be given showing that the algorithm behaves in a way similar to the Jacobi-Perron algorithm for an n-tuple of real numbers (the gcd-algorithm): there do exist n-tuples of formal power series $f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ with a C-n-fraction without interruptions but for which $1, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ is nevertheless linearly dependent over $\mathbb{C}[x]$.

Moreover an example will be given of algebraic functions f, g of degree n over $\mathbb{C}[x]$ (formally defined) for which the C-n-fraction for $f, f^2, \ldots, f^n$ has just one interruption and that for $g, g^2, \ldots, g^n$ none, while of course $1, f, f^2, \ldots, f^n$ and $1, g, g^2, \ldots, g^n$ admit (only) one dependence relation over $\mathbb{C}[x]$.

1. Introduction

It is well known that there exists a one-to-one correspondence between

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formal power series in an indeterminate x with complex coefficients

(1) $$f(x) = \sum_{\nu=0}^{\infty} c_\nu x^\nu \quad (c_0 \neq 0) ,$$

and so-called C-fractions, a terminating or non-terminating continued fraction of the form

(2) $$b_0 + \frac{a_1 x^{r_1}|}{|1} + \frac{a_2 x^{r_2}|}{|1} + \dots$$
$$(b_0, a_1, a_2, \dots \in \mathbb{C}\backslash\{0\};\ r_1, r_2, \dots \in \mathbb{N}) .$$

In the case that we admit the power series in (1) to have a vanishing constant term, that is, $f(x) = \sum_{\nu=k}^{\infty} c_\nu x^\nu$ $(c_k \neq 0,\ k \in \mathbb{N})$, the correspondence still holds if we replace b_0 in (2) by $c_k x^k$.

An important property of the one-to-one correspondence is

(3) f in (1) is the MacLaurin series of a rational function $(f \in \mathbb{C}(x))$ if and only if the C-fraction (2) corresponding to f terminates,

which (for the sequel) is rephrased into

(4) $1, f$ linearly dependent over $\mathbb{C}[x]$ if, and only if, the C-fraction for f terminates.

For the theory of C-fractions the reader can consult the basic texts Perron [8], Wall [10].

As the reader immediately realizes, (4) is the analogue of a similar assertion for the ordinary continued fraction for a real number (replace $\mathbb{C}[x]$ by $\mathbb{Z}$).

This just mentioned continued fraction for a real number has been generalized in many ways to an algorithm for an n-tuple of real numbers. Of these generalizations we only mention one that is closely connected to the greatest common divisor algorithm, see Perron [7], the so-called Jacobi-Perron algorithm.

Algebraic properties have been studied amongst others by Bernstein [1] and metrical properties by Schweiger [9].

There is, however, a loss compared to the ordinary continued fraction algorithm: a generalization of (4) does not hold for $n \geq 3$ (then there exist n-tuples of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ which have a Jacobi-Perron algorithm without interruptions but for which $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ is nevertheless linearly dependent over $\mathbb{Z}$); the case $n = 2$ is not yet clear for the algorithm from [7].

Now there are different ways of generalizing the C-fraction algorithm to n-tuples of formal power series. For instance, see the work of Dubois [5] and Paysant Le Roux [6].

They use the well known non-archimedean valuation on the field of formal power series to define the notions "distance" and "integer" and thereby reach an algorithm that satisfies a modified version of (4) with $\mathbb{C}[x]$ replaced by the set of "integers". It is then possible to prove, see [5], that the number of independent dependence relations is equal to the number of interruptions in the algorithm.

In this paper another generalization is considered which behaves very much like the ordinary C-fraction algorithm and which is also connected with the sequence of Padé approximants on the main stepline in a generalized Padé table, see de Bruin [2], [3], [4].

2. The C-n-fraction algorithm

Consider an n-tuple of formal power series in an indeterminate x with complex coefficients

$$(5) \qquad f_0^{(i)}(x) = \sum_{\nu=0}^{\infty} c_\nu^{(i)} x^\nu \quad \left(i = 1, 2, \ldots, n; \; c_0^{(n)} = 1\right) .$$

There is no loss of generality in requiring $c_0^{(n)} = 1$ as will be pointed out in the sequel. The use of this condition lies mainly in the fact that it enables one to recover many results for ordinary C-fractions as they appear in [8], [10] by simply taking $n = 1$ in the general theory, see [2], [3].

Also, for the sake of simplicity and because we are otherwise led to a

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rather trivial situation, we assume that $f_0^{(1)}$ is not a monomial nor identically zero.

Let $b_{i,0}x^{r(i,0)}$ $(i = 1, 2, \ldots, n)$ be the first non-zero term in $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$, respectively (so $r(n, 0) = 0$, $b(n, 0) = 1$) and $a_{1,1}x^{r(1,1)}$ the second non-zero term in $f_0^{(1)}$.

Then

$$f_0^{(1)}(x) = b_{1,0}x^{r(1,0)} + \left\{a_{1,1}x^{r(1,1)}\right\}/\left\{f_1^{(n)}(x)\right\} \tag{6a}$$

uniquely defines the formal power series $f_1^{(n)}$ with constant term equal to 1. This $f_1^{(n)}$ is then used to define the formal power series $f_1^{(1)}, f_1^{(2)}, \ldots, f_1^{(n-1)}$ (uniquely) by

$$f_0^{(i)}(x) = b_{i,0}x^{r(i,0)} + \left\{f_1^{(i-1)}(x)\right\}/\left\{f_1^{(n)}(x)\right\} \quad (i = 2, 3, \ldots, n) . \tag{6b}$$

Thus we get another n-tuple of formal power series, $f_1^{(1)}, f_1^{(2)}, \ldots, f_1^{(n)}$, of which the last one has constant term equal to one, and we can try to apply the method described in (6a, 6b) once more.

Now, however, we have two different situations: $f_1^{(1)}$ is a monomial or not; they will be considered separately.

Let the n-tuple of formal power series $f_\nu^{(1)}, f_\nu^{(2)}, \ldots, f_\nu^{(n)}$ ($f_\nu^{(n)}$ has constant term equal to 1) have been constructed for a certain $\nu \in \mathbb{N}$.

A. $f_\nu^{(1)}$ IS NOT A MONOMIAL NOR IDENTICALLY ZERO

Let $a_{2,\nu}x^{r(2,\nu)}$, respectively $a_{1,\nu+1}x^{r(1,\nu+1)}$, be the first, respectively second, non-zero term in the series $f_\nu^{(1)}$ (that is, $a_{2,\nu}, a_{1,\nu+1} \neq 0$). Then the formal power series $f_{\nu+1}^{(n)}$ with constant term

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equal to 1 is uniquely defined by

(7a) $$f_{\nu}^{(1)}(x) = a_{2,\nu}x^{r(2,\nu)} + \left\{a_{1,\nu+1}x^{r(1,\nu+1)}\right\}/\left\{f_{\nu+1}^{(n)}(x)\right\} .$$

After this, let $a_{i+1,\nu}x^{r(i+1,\nu)}$ be the first non-zero term in the series $f_{\nu}^{(i)}$ (or zero if $f_{\nu}^{(i)} \equiv 0$) for $i = 2, 3, \ldots, n-1$; we know that the first non-zero term in $f_{\nu}^{(n)}$ is the constant 1 . Then the formal power series $f_{\nu+1}^{(1)}, f_{\nu+1}^{(2)}, \ldots, f_{\nu+1}^{(n-1)}$ are uniquely defined by

(7b) $$\begin{cases} f_{\nu}^{(i)}(x) = a_{i+1,\nu}x^{r(i+1,\nu)} + \left\{f_{\nu+1}^{(i-1)}(x)\right\}/\left\{f_{\nu+1}^{(n)}(x)\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 2, 3, \ldots, n-1) , \\ f_{\nu}^{(n)}(x) = b_{\nu} + \left\{f_{\nu+1}^{(n-1)}(x)\right\}/\left\{f_{\nu+1}^{(n)}(x)\right\} \quad (b_{\nu} = 1) . \end{cases}$$

CONCLUSION. The rules (7a), (7b) construct, starting with an n-tuple of formal power series of which the last one has constant term equal to 1 , an n-tuple of the same kind.

B. $f_{\nu}^{(1)}$ is a monomial or identically zero

Let $f_{\nu}^{(1)}, f_{\nu}^{(2)}, \ldots, f_{\nu}^{(k)}$ all be monomials or identically zero and let $f_{\nu}^{(k+1)}$ be the first formal power series (regarding the superscript) which has at least two non-zero terms (if there is one).

This situation is called an *interruption of order* k *at index* ν ("Störung") and leads to a subdivision of Case B.

B1. $k = n$; that is, all formal power series have at the most one non-zero term.

Take $a_{i+1,\nu}x^{r(i+1,\nu)} \equiv f_{\nu}^{(i)}(x)$ $(i = 1, 2, \ldots, n-1)$, $b_{\nu} = 1$; $a_{1,\nu}x^{r(1,\nu)}$ already follows from Case A for $\nu - 1$.

The algorithm terminates. Calculating backwards using (7a), (7b), for

$\nu, \nu-1, \ldots, 2, 1$ and (6a); (6b), shows that $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ are the MacLaurin series of rational functions which are regular at the origin.

B2. $1 \le k \le n-1$

Take $a_{i+1,\nu}x^{r(i+1,\nu)} \equiv f_\nu^{(i)}(x)$ $(i = 1, 2, \ldots, k)$, $a_{i,\mu}x^{r(i,\mu)} \equiv 0$ $(i = 1, 2, \ldots, k;\ \mu \ge \nu+1)$ and define the formal power series $f_{\nu+1}^{(n)}$ with constant term equal to 1 using the first and second non-zero term in $f_\nu^{(k+1)}$; $f_{\nu+1}^{(n)}$ is unique:

$$\text{(8a)}\quad f_\nu^{(k+1)}(x) = a_{k+2,\nu}x^{r(k+2,\nu)} + \left\{a_{k+1,\nu+1}x^{r(k+1,\nu+1)}\right\}\Big/\left\{f_{\nu+1}^{(n)}(x)\right\} .$$

After this the formal power series $f_{\nu+1}^{(k+1)}, f_{\nu+1}^{(k+2)}, \ldots, f_{\nu+1}^{(n-1)}$ are uniquely determined as in (7b) by the first non-zero term in $f_\nu^{(k+2)}, f_\nu^{(k+3)}, \ldots, f_\nu^{(n)}$:

$$\text{(8b)}\quad \begin{cases} f_\nu^{(i)}(x) = a_{i+1,\nu}x^{r(i+1,\nu)} + \left\{f_{\nu+1}^{(i-1)}(x)\right\}\Big/\left\{f_{\nu+1}^{(n)}(x)\right\} \\ \qquad\qquad\qquad\qquad (i = k+2, k+3, \ldots, n-1) , \\ f_\nu^{(n)}(x) = b_\nu + \left\{f_{\nu+1}^{(n-1)}(x)\right\}\Big/\left\{f_{\nu+1}^{(n)}(x)\right\} \quad (b_\nu = 1) . \end{cases}$$

CONCLUSION. After an interruption of order k at index ν the rules (8a), (8b) construct, starting with the $(n-k)$-tuple of formal power series $f_\nu^{(k+1)}, f_\nu^{(k+2)}, \ldots, f_\nu^{(n)}$ of which the last one has constant term equal to 1 , an $(n-k)$-tuple of the same kind.

In what manner the algorithm has to be adapted if after an interruption of order k another interruption, say of order m at index μ , appears is now evident.

C. $f_\mu^{(k+1)}, f_\mu^{(k+2)}, \ldots, f_\mu^{(n)}$ have an interruption of order m at index μ .

The case $m = n - k$ is the "same" as Case B1; the algorithm terminates. Let now $m+k \le n-1$.

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Then take $a_{i+1,\mu}x^{r(i+1,\mu)} \equiv f_{\mu}^{(i)}(x)$ $(i = k+1,\ k+2,\ \ldots,\ k+m)$, $a_{i,\nu}x^{r(i,\nu)} \equiv 0$ $(i = 1,\ 2,\ \ldots,\ k+m;\ \nu \geq \mu+1)$, and define the formal power series $f_{\mu+1}^{(n)}$ with constant term equal to 1 using the first and second non-zero term in $f_{\mu}^{(k+m+1)}$; $f_{\mu+1}^{(n)}$ is unique:

$$(9a)\quad f_{\mu}^{(k+m+1)}(x) = a_{k+m+2,\mu}x^{r(k+m+2,\mu)} + \left\{a_{k+m+1,\mu+1}x^{r(k+m+1,\mu+1)}\right\}/\left\{f_{\mu+1}^{(n)}(x)\right\} .$$

Then $f_{\mu+1}^{(k+m+1)},\ f_{\mu+1}^{(k+m+2)},\ \ldots,\ f_{\mu+1}^{(n-1)}$ follow as in Cases A and B2:

$$(9b)\quad \begin{cases} f_{\mu}^{(i)}(x) = a_{i+1,\mu}x^{r(i+1,\mu)} + \left\{f_{\mu+1}^{(i-1)}(x)\right\}/\left\{f_{\mu+1}^{(n)}(x)\right\} \\ \qquad\qquad (i = k+m+2,\ k+m+3,\ \ldots,\ n-1) \\ f_{\mu}^{(n)}(x) = b_{\mu} + \left\{f_{\mu+1}^{(n-1)}(x)\right\}/\left\{f_{\mu+1}^{(n)}(x)\right\} \quad (b_{\mu} = 1) \end{cases}$$

(if $m + k = n - 1$, (9a), (9b) have to be replaced by the second line of (9b) only).

The C-n-fraction for an arbitrary n-tuple of formal power series now follows by applying the construction given above, at each step choosing Case A or Case B and once Case B has been chosen, choosing Case B or Case C.

The construction terminates or not; this matter will be treated in the next section.

For notational convenience only the coefficients and powers of x that appear when we apply the algorithm are given; we have then

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$$(10)\quad \begin{pmatrix} f_0^{(1)} \\ f_0^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ f_0^{(n)} \end{pmatrix} \underline{\underline{K}} \begin{bmatrix} & a_{1,1}x^{r(1,1)} & \cdots & a_{1,\nu}x^{r(1,\nu)} & \cdots \\ b_{1,0}x^{r(1,0)} & a_{2,1}x^{r(2,1)} & \cdots & a_{2,\nu}x^{r(2,\nu)} & \cdots \\ b_{2,0}x^{r(2,0)} & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ & a_{n,1}x^{r(n,1)} & \cdots & a_{n,\nu}x^{r(n,\nu)} & \cdots \\ b_{n,0}x^{r(n,0)} & 1 & \cdots & 1 & \cdots \end{bmatrix}$$

(with $r(n,\,0) = 0$, $b_{n,0} = 1$).

REMARK 1. The notation (10) shows how the algorithm has to be adapted if $f_0^{(n)}$ does not have a constant term equal to 1 : just put the first non-zero term in the series $f_0^{(n)}$ in the place of $b_{n,0}x^{r(n,0)}$.

Interruptions show up in (10) in the following way: an interruption of order k at index ν leads to zeros in the rows $1, 2, \ldots, k$ starting in the column number $\nu + 1$ (if the first column is given the number 0).

Using the right hand side of (10) it is possible to define $n + 1$ sequences of polynomials $A_\nu^{(1)}, A_\nu^{(2)}, \ldots, A_\nu^{(n)}$ (numerator polynomials) and B_ν (denominator polynomials) for $\nu \in \mathbb{N} \cup \{-n, -n+1, \ldots, -1, 0\}$ by

$$(11)\quad \begin{cases} A_{-j}^{(i)} = \delta_{i+j,n+1} \quad (i = 1, 2, \ldots, n) , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad B_{-j} = 0 \quad \text{for} \quad j = 1, 2, \ldots, n , \\ A_0^{(i)} = b_{i,0}x^{r(i,0)} \quad (i = 1, 2, \ldots, n) , \quad B_0 = 1 , \end{cases}$$

and the recurrence relation, the same for each of the sequences,

$$(12)\quad Y_\nu = Y_{\nu-1} + a_{n,\nu}x^{r(n,\nu)}Y_{\nu-2} + \ldots + a_{1,\nu}x^{r(1,\nu)}Y_{\nu-n-1} \quad (\nu \in \mathbb{N}) .$$

For detailed information concerning the sequences of polynomials and the sequences $A_\nu^{(i)}/B_\nu$ $(i = 1, 2, \ldots, n;\ \nu \in \mathbb{N})$, for instance the order relations for the exponents $r(i, \nu)$

$$\left(0 \le r(j, 0) < r(j, 1) < r(j-1, 2) < \ldots < r(1, j+1) \text{ , and } r(n, \nu) < r(n-1, \nu+1) < \ldots < r(1, \nu+n-1)\right) ,$$

the relations $f_0^{(i)} - A_\nu^{(i)}/B_\nu = d_{i,\nu}x^{\nu+1}$ plus higher powers, and so on; see [2].

For the sequel we only need the initial values (11), the recurrence relation (12), and

$$\text{(13)} \qquad \det\begin{pmatrix} A_\nu^{(n)} & \ldots & A_{\nu-n}^{(n)} \\ \vdots & & \vdots \\ A_\nu^{(1)} & \ldots & A_{\nu-n}^{(1)} \\ B_\nu & \ldots & B_{\nu-n} \end{pmatrix} = (-1)^{n(\nu+1)} \prod_{j=1}^{\nu} a_{1,j}x^{r(1,j)} \quad (\nu \in \mathbb{N}_0)$$

(an empty product has to be taken as 1 ; for the proof see [2]).

EXAMPLE 1. Let g be the unique formal power series in x with constant term equal to 1 that satisfies

$$\text{(14)} \qquad g^3 - g^2 - xg - x^2 \equiv 0 .$$

Take $f = g^2 - g$; because g satisfies an irreducible (over $\mathbb{C}[x]$) equation of degree 3 , the triple 1, f, g is linearly independent over $\mathbb{C}[x]$. Straightforward (formal) calculation shows $f(x) = x + x^2$ plus higher powers of x . Apply the C-2-fraction algorithm to f, g :

$$\begin{cases} f = x + f - x = x + g^2 - g - x = x + \left(g^3-g^2-xg\right)/g = x + \left(x^2/g\right) \\ g = 1 + g - 1 = 1 + \left(g^2-g\right)/g = 1 + (f/g) . \end{cases} \quad (g \text{ constant term } 1)$$

This shows that the C-2-fraction is purely periodic (period length 1) and has the form

$$\text{(15)} \qquad \begin{pmatrix} & x^2 & \ldots & x^2 & \ldots \\ x & x & \ldots & x & \ldots \\ 1 & 1 & \ldots & 1 & \ldots \end{pmatrix} .$$

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EXAMPLE 2. Let g be the unique formal power series in x with constant term equal to 1 satisfying

$$g^2 + (x-1)g + x^2 \equiv 0 . \tag{16}$$

Take $f = g^2 - g$; then $(1, f, g)$ is linearly dependent over $\mathbb{C}[x]$ $\{f + xg + x^2 \equiv 0\}$, but f, g are not rational while (16) is irreducible.

We have

$$\begin{cases} f = g^2 - g = -xg - x^2 = -x - x\{g+(x-1)\} \\ \qquad\qquad\qquad\qquad = -x - x\{g^2+(x-1)g\}/g = -x + (x^3/g) , \\ g = 1 + g - 1 = 1 + f/g . \end{cases}$$

Again period length 1 and (17) gives the C-2-fraction

$$\begin{pmatrix} f \\ g \end{pmatrix} \overset{K}{=} \begin{bmatrix} & x^3 & \dots & x^3 & \dots \\ -x & -x & \dots & -x & \dots \\ 1 & 1 & \dots & 1 & \dots \end{bmatrix} . \tag{18}$$

EXAMPLE 3. Let f be the unique formal power series with constant term equal to 1 satisfying

$$f^2 - f - x \equiv 0 . \tag{19}$$

Then f, f^2 are not rational and $(1, f, f^2)$ allows just "one" dependence relation over $\mathbb{C}[x]$ while (19) is irreducible.

This time an interruption comes up:

$$f = 1 + f - 1 = 1 + x/f \quad (f \text{ constant term } 1), \tag{20}$$

$$f^2 = 1 + (x(f+1))/f ;$$

$$x(f+1) = 2x + x(f-1) = 2x + x^2/f , \quad f = 1 + x/f ; \tag{21}$$

$$x \text{ is a monomial}, \quad f = 1 + x/f . \tag{22}$$

From (22) follows an interruption of order 1 at index 2 ; the function to go on with leads to a non-terminating algorithm $f = 1 + x/f$ *ad infinitum*. We have

$$(23) \qquad \begin{pmatrix} f \\ f^2 \end{pmatrix} \stackrel{K}{=\!=} \begin{bmatrix} & x & x^2 & 0 & \dots & 0 & \dots \\ 1 & 2x & x & x & \dots & x & \dots \\ 1 & 1 & 1 & 1 & \dots & 1 & \dots \end{bmatrix} .$$

EXAMPLE 4. The C-2-fraction for the functions $1/(1-x)$, $1/\left((1-x)^2\right)$:

$$(24) \qquad \begin{cases} 1/(1-x) = 1 + x/(1-x) \;, \quad 1/\left((1-x)^2\right) = 1 + \left(x(2-x)/(1-x)\right)/(1-x) \;; \\ \left(x(2-x)\right)/(1-x) = 2x + x^2/(1-x) \;, \quad 1 - x = 1 + \left(-x(1-x)\right)/(1-x) \;; \\ -x(1-x) = -x + x^2/1 \;, \quad 1 - x = 1 + (-x)/1 \;; \\ -x \text{ is a monomial, } 1 \text{ is a monomial.} \end{cases}$$

We have an interruption of order 2 at index 3 ; the C-2-fraction terminates

$$(25) \qquad \begin{pmatrix} 1/(1-x) \\ 1/(1-x)^2 \end{pmatrix} \stackrel{K}{=\!=} \begin{bmatrix} & x & x^2 & x^2 \\ 1 & 2x & -x & -x \\ 1 & 1 & 1 & 1 \end{bmatrix} .$$

3. Interruptions and linear dependence

In the sequel the following abbreviations will be used:

$$b_0^{(i)} = b_{i,0}x^{r(i,0)} \;, \quad a_\nu^{(i)} = a_{i,\nu}x^{r(i,\nu)} \quad (i = 1, 2, \dots, n;\ \nu \in \mathbb{N}) \;.$$

THEOREM 1. *Let the C-n-fraction for $f_0^{(1)}, f_0^{(2)}, \dots, f_0^{(n)}$ have an interruption of order k at index μ $(1 \le k \le n-1;\ \mu \ge 1)$; the next interruption (if any) appears at index $\tau \ge \mu+1$.*

Let $A_\nu^{(i)}$ $(i = 1, 2, \dots, n)$, B_ν $(\nu \in \mathbb{N} \cup \{-n, -n+1, \dots, -1, 0\})$ be defined as in (11), (12). *Then*

$$(26) \quad f_0^{(i)} = \left\{A_{\nu-1}^{(i)}f_\nu^{(n)} + A_{\nu-2}^{(i)}f_\nu^{(n-1)} + \dots + A_{\nu-n}^{(i)}f_\nu^{(1)} + A_{\nu-n-1}^{(i)}a_\nu^{(1)}\right\} \Big/ \left\{B_{\nu-1}f_\nu^{(n)} + B_{\nu-2}f_\nu^{(n-1)} + \dots + B_{\nu-n}f_\nu^{(1)} + B_{\nu-n-1}a_\nu^{(1)}\right\}$$
$$(i = 1, 2, \dots, n;\ 1 \le \nu \le \mu) \;,$$

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$$(27)\quad f_0^{(i)} = \left\{A_{\nu-1}^{(i)}f_\nu^{(n)} + A_{\nu-2}^{(i)}f_\nu^{(n-1)} + \ldots + A_{\nu-n+k}^{(i)}f_\nu^{(k+1)} + A_{\nu-n+k-1}^{(i)}a_\nu^{(k+1)}\right\} \Big/ \left\{B_{\nu-1}f_\nu^{(n)} + B_{\nu-2}f_\nu^{(n-1)} + \ldots + B_{\nu-n+k}f_\nu^{(k+1)} + B_{\nu-n+k-1}a_\nu^{(k+1)}\right\}$$

$$(i = 1, 2, \ldots, n;\ \mu+1 \le \nu \le \tau) .$$

Furthermore, for $\nu = 1, 2, \ldots, \tau$,

(28) *the columns of*

$$\begin{pmatrix} A_\nu^{(n)} & \ldots & A_{\nu-n+k}^{(n)} \\ \vdots & & \vdots \\ A_\nu^{(1)} & \ldots & A_{\nu-n+k}^{(1)} \\ B_\nu & \ldots & B_{\nu-n+k} \end{pmatrix}$$

are linearly independent over $\mathbb{C}[x]$.

Proof. For $\nu = 1$, formula (26) follows from (6a), (6b):

$$\begin{cases} f_0^{(1)} = b_0^{(1)} + \left\{a_1^{(1)}\right\}/\left\{f_1^{(n)}\right\} = \left\{A_0^{(1)}f_1^{(n)} + A_{-n}^{(1)}a_1^{(1)}\right\}/\left\{B_0f_1^{(n)}\right\} , \\ f_0^{(i)} = b_0^{(i)} + \left\{f_1^{(i-1)}\right\}/\left\{f_1^{(n)}\right\} = \left\{A_0^{(i)}f_1^{(n)} + A_{-(n+1-i)}^{(i)}f_1^{(i-1)}\right\}/\left\{B_0f_1^{(n)}\right\} \\ \qquad (i = 2, 3, \ldots, n) . \end{cases}$$

Let (26) hold for a certain ν , $1 \le \nu \le \mu-1$; then (7a), (7b) imply

$$f_0^{(i)} = \left\{A_{\nu-1}^{(i)}\left\{1+\left\{f_{\nu+1}^{(n-1)}/f_{\nu+1}^{(n)}\right\}\right\} + \sum_{j=1}^{n-2} A_{\nu-1-j}^{(i)}\left\{a_\nu^{(n-j+1)}+\left\{f_{\nu+1}^{(n-j-1)}/f_{\nu+1}^{(n)}\right\}\right\} + A_{\nu-n}^{(i)}\left\{a_\nu^{(2)}+\left\{a_{\nu+1}^{(1)}/f_{\nu+1}^{(n)}\right\}\right\} + A_{\nu-n-1}^{(i)}a_\nu^{(1)}\right\} \Big/ \left\{B_{\nu-1}\left\{1+\left\{f_{\nu+1}^{(n-1)}/f_{\nu+1}^{(n)}\right\}\right\} + \sum_{j=1}^{n-2} B_{\nu-1-j}\left\{a_\nu^{(n-j+1)}+\left\{f_{\nu+1}^{(n-j-1)}/f_{\nu+1}^{(n)}\right\}\right\} + B_{\nu-n}\left\{a_\nu^{(2)}+\left\{a_{\nu+1}^{(1)}/f_{\nu+1}^{(n)}\right\}\right\} + B_{\nu-n-1}a_\nu^{(1)}\right\}$$

$$= \left\{\left\{A_{\nu-1}^{(i)} + \sum_{j=1}^{n} a_\nu^{(n-j+1)}A_{\nu-1-j}^{(i)}\right\}f_{\nu+1}^{(n)} + A_{\nu-1}^{(i)}f_{\nu+1}^{(n-1)} + \ldots + A_{\nu-n+1}^{(i)}f_{\nu+1}^{(1)} + A_{\nu-n}^{(i)}a_{\nu+1}^{(1)}\right\} \Big/ \left\{\left\{B_{\nu-1} + \sum_{j=1}^{n} a_\nu^{(n-j+1)}B_{\nu-1-j}\right\}f_{\nu+1}^{(n)} + B_{\nu-1}f_{\nu+1}^{(n-1)} + \ldots + B_{\nu-n+1}f_{\nu+1}^{(1)} + B_{\nu-n}a_{\nu+1}^{(1)}\right\} ;$$

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but this is (26) for $\nu + 1$ according to (12).

Substitute (7a), (7b) with $\nu = \mu$ into (26) with $\nu = \mu$; this leads to (27) for $\nu = \mu + 1$. After that we can use the method of proof for (26) to prove (27) by induction. We only have to keep in mind the change in (12) because of the interruption of order k at index μ :

(29) $$Y_\nu = Y_{\nu-1} + a_\nu^{(n)} Y_{\nu-2} + \ldots + a_\nu^{(k+1)} Y_{\nu-n+k-1} \quad (\nu = \mu+1, \mu+2, \ldots, \tau) .$$

From (13) we derive

(30) $$\det \begin{pmatrix} A_\nu^{(n)} & \ldots & A_{\nu-n}^{(n)} \\ \vdots & & \vdots \\ A_\nu^{(1)} & \ldots & A_{\nu-n}^{(1)} \\ B_\nu & \ldots & B_{\nu-n} \end{pmatrix} = (-1)^{n(\nu+1)} \prod_{j=1}^{\nu} a_{1,j} x^{r(1,j)} \not\equiv 0$$

for $\nu = 1, 2, \ldots, \mu$,

while $a_{1,j} \neq 0$ $(j = 1, 2, \ldots, \mu)$ owing to the fact that the first interruption occurs at index μ .

Let

$$K_\nu = \left(A_\nu^{(n)}, A_\nu^{(n-1)}, \ldots, A_\nu^{(1)}, B_\nu\right)^T \quad (\nu \in \mathbb{N} \cup \{-n, -n+1, \ldots, -1, 0\})$$

be the general column of the matrix in (30).

Formula (30) implies that $K_\nu, K_{\nu-1}, \ldots, K_{\nu-n}$ are linearly independent over $\mathbb{C}[x]$, and so $K_\nu, K_{\nu-1}, \ldots, K_{\nu-n+k}$ too, for $\nu = 1, 2, \ldots, \mu$.

The remaining values of ν for which (28) has to be proved are treated using the recurrence relation (29) written down for the columns K_ν :

(31) $$K_\nu = K_{\nu-1} + a_\nu^{(n)} K_{\nu-2} + \ldots + a_\nu^{(k+1)} K_{\nu-n+k-1} \quad (\nu = \mu+1, \mu+2, \ldots, \tau) .$$

Because $a_\nu^{(k+1)} = a_{k+1,\nu} x^{r(k+1,\nu)}$ with $a_{k+1,\nu} \neq 0$ for $\mu+1 \le \nu \le \tau$, (31) can be used to prove, for $\mu+1 \le \nu \le \tau$,

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$K_{\nu-1}, K_{\nu-2}, \ldots, K_{\nu-n+k-1}$ linearly independent over $\mathbb{C}[x]$ implies $K_\nu, K_{\nu-1}, \ldots, K_{\nu-n+k}$ also linearly independent over $\mathbb{C}[x]$.

Starting with the independence of $K_\mu, K_{\mu-1}, \ldots, K_{\mu-n+k}$ (from (28) with $\nu = \mu$) induction shows that (28) holds for $\nu = \mu+1, \mu+2, \ldots, \tau$. □

The following theorem is now obvious and will be given without proof. The change in (27) at each index where an interruption occurs is dictated by the number of zeros in the C-n-fraction; or - what amounts to the same - the change that is described in Case C of the algorithm. The $a_\nu^{(i)}$ in the numerator must be the first one, regarding superfix, that is different from zero; then $f_\nu^{(n)}, f_\nu^{(n-1)}, \ldots, f_\nu^{(i)}$ precede this $a_\nu^{(i)}$ in the adapted (27). The difference from Theorem 1 is that Theorem 2 covers the situation that interruptions of order $k_1, k_2, \ldots, k_j, m$ with $k_1 + k_2 + \ldots + k_j + m = j$ (total order k) occur at indices $\nu_1, \nu_2, \ldots, \nu_j, \mu$.

THEOREM 2. *Let the C-n-fraction for* $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ *have interruptions of total order* k *, the last of which appear(s) at index* μ *; furthermore let the next interruption (if any) occur at index* $\tau \geq \mu+1$. *Then* (27), (28) *hold as stated in Theorem* 1. □

THEOREM 3. *Let the C-n-fraction for* $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ *have interruptions of total order* k . *Then there exist at least* k *relations of the form*

$$(32)\quad p^{(0)}(x) + p^{(1)}(x)f_0^{(1)}(x) + \ldots + p^{(n)}(x)f_0^{(n)}(x) \equiv 0 ,$$
$$p^{(0)}, p^{(1)}, \ldots, p^{(n)} \in \mathbb{C}[x] ,$$

which are linearly independent over $\mathbb{C}[x]$.

Proof. If $k = n$, the C-n-fraction terminates and thus $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)} \in \mathbb{C}(x)$ from which the assertion follows.

Let now $1 \leq k \leq n-1$ and let μ be the index at which the last of the interruptions occur(s).

According to Theorem 2 the rank of the matrix

$$\begin{pmatrix} A^{(n)}_{\mu+1} & \cdots & A^{(n)}_{\mu-n+k+1} \\ \vdots & & \vdots \\ A^{(1)}_{\mu+1} & \cdots & A^{(1)}_{\mu-n+k+1} \\ B_{\mu+1} & \cdots & B_{\mu-n+k+1} \end{pmatrix}$$

is $n - k + 1$; that is, it is possible to choose $n - k + 1$ rows which are linearly independent over $\mathbb{C}[x]$. For the sake of simplicity let it be the 1st, 2nd, ..., $(n-k+1)$st row:

$$(33) \qquad q(x) = \det \begin{pmatrix} A^{(n)}_{\mu+1} & \cdots & A^{(n)}_{\mu-n+k+1} \\ \vdots & & \vdots \\ A^{(k)}_{\mu+1} & \cdots & A^{(k)}_{\mu-n+k+1} \end{pmatrix} \not\equiv 0 .$$

Formula (27) with $\nu = \mu + 1$ implies that the following system of $n - k + 2$ homogeneous linear equations over $\mathbb{C}[x]$ in the $n - k + 2$ unknowns, $y_1, y_2, \ldots, y_{n-k+2}$,

$$\begin{cases} f^{(n)}_0 y_1 + A^{(n)}_{\mu} y_2 + A^{(n)}_{\mu-1} y_3 + \ldots + A^{(n)}_{\mu-n+k} y_{n-k+2} = 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ f^{(k)}_0 y_1 + A^{(k)}_{\mu} y_2 + A^{(k)}_{\mu-1} y_3 + \ldots + A^{(k)}_{\mu-n+k} y_{n-k+2} = 0 \\ f^{(i)}_0 y_1 + A^{(i)}_{\mu} y_2 + A^{(i)}_{\mu-1} y_3 + \ldots + A^{(i)}_{\mu-n+k} y_{n-k+2} = 0 \end{cases},$$

has the non-trivial solution

$$y_1 = -\left(B_\mu f^{(n)}_{\mu+1} + B_{\mu-1} f^{(n-1)}_{\mu+1} + \ldots + B_{\mu-n+k+1} f^{(k+1)}_{\mu+1} + B_{\mu-n+k} a^{(k+1)}_{\mu+1} \right) ,$$

$$y_2 = f^{(n)}_{\mu+1} ,$$

$$y_3 = f^{(n-1)}_{\mu+1}, \ldots, y_{n-k+1} = f^{(k+1)}_{\mu+1} ,$$

$$y_{n-k+2} = a^{(k+1)}_{\mu+1} \not\equiv 0 \quad \text{for} \quad i = 0, 1, \ldots, k-1 ,$$

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where the notation $A_\mu^{(0)} = B_\mu$ and $f_0^{(0)} = 1$ has been used. Thus

$$\det\begin{pmatrix} f_0^{(n)} & A_\mu^{(n)} & \cdots & A_{\mu-n+k}^{(n)} \\ \vdots & \vdots & & \vdots \\ f_0^{(k)} & A_\mu^{(k)} & \cdots & A_{\mu-n+k}^{(k)} \\ f_0^{(i)} & A_\mu^{(i)} & \cdots & A_{\mu-n+k}^{(i)} \end{pmatrix} \equiv 0 \quad (i = 0, 1, \ldots, k-1) .$$

Multiplication by $(-1)^{n-k+1}$ and expanding the determinant using the first column leads, for $i = 0, 1, \ldots, k-1$, to

$$(34)\quad \begin{cases} q(x).1 + 0.f_0^{(1)}(x) + 0.f_0^{(2)}(x) + \ldots + 0.f_0^{(k-1)}(x) \\ \qquad\qquad + \sum_{j=k}^{n} p_{1j}(x) f_0^{(j)}(x) \equiv 0 , \\ 0.1 + q(x).f_0^{(1)}(x) + 0.f_0^{(2)}(x) + \ldots + 0.f_0^{(k-1)}(x) \\ \qquad\qquad + \sum_{j=k}^{n} p_{2j}(x) f_0^{(j)}(x) \equiv 0 , \\ \vdots \\ 0.1 + 0.f_0^{(1)}(x) + \ldots + 0.f_0^{(k-2)}(x) + q(x).f_0^{(k-1)}(x) + \\ \qquad\qquad + \sum_{j=k}^{n} p_{kj}(x) f_0^{(j)}(x) \equiv 0 , \end{cases}$$

with $p_{kj} \in \mathbb{C}[x]$ $(i = 1, 2, \ldots, k;\ j = k, k+1, \ldots, n)$.

According to (33) the relations (34) constitute k (obviously) linearly independent relations between $1, f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$. □

COROLLARY 1. *Let* $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ *be an* n*-tuple of formal power series. Then the following statements are equivalent:*

(a) the C-n-fraction for the n-tuple terminates;

(b) the C-n-fraction for the n-tuple has interruptions of total order n ;

(c) *there exist* n *relations (linearly independent over* $\mathbb{C}[x]$ *) between* $1, f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ *with coefficients in* $\mathbb{C}[x]$ *;*

(d) $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ *are the MacLaurin series for an n-tuple of rational functions each of which is regular at the origin.*

Proof. (a) $\Leftrightarrow$ (b), (c) $\Leftrightarrow$ (d), and (a) $\Rightarrow$ (d) are trivial (Theorem 3 gives for $k = n$ at least n linearly independent relations: that there can not be more than n, follows from the fact that the existence of $n+1$ linearly independent relations for $n+1$ functions $1, f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ would imply that those functions are all identically zero, a contradiction). For a proof of (d) $\Rightarrow$ (a), see [2]. □

COROLLARY 2. *If the n-tuple of formal power series* $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ *has the property that* $\left\{1, f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}\right\}$ *is linearly independent over* $\mathbb{C}[x]$ *, then the C-n-fraction for the n-tuple has no interruptions.* □

Example 2 shows that there do exist pairs of formal power series f, g with $1, f, g$ linearly dependent over $\mathbb{C}[x]$ but for which the C-2-fraction has no interruptions: the absence of an interruption does not imply the absence of a dependence relation!

In Example 3 the pair of formal power series admits exactly one dependence relation while the C-2-fraction has exactly one interruption (two interruptions are not possible according to Corollary 1).

That this kind of behaviour is not restricted to the case $n = 2$ will be shown in the next section.

4. Interruptions versus linear dependence

In this section we restrict ourselves to the case $n \geq 2$ (for $n = 1$ the problem is completely solved by (4)). For the sequel we need two types of formal power series to construct examples.

DEFINITION 1. Let g be the unique formal power series in x with

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constant term equal to 1 that satisfies

$$(35)\quad Y^n + (bx^s-1)Y^{n-1} + \sum_{j=2}^{n-1} \left\{b^j x^{js} - \left(\sum_{k=1}^{j} b^{j-k} a_{n-k+1}\right) x^{(j-1)s}\right\} Y^{n-j} + b^n x^{ns} \equiv 0$$

with

$$(36)\quad \begin{cases} \text{(i)} & s \in \mathbb{N}, \\ \text{(ii)} & a_n = 1, \quad b, a_{n-1}, a_{n-2}, \ldots, a_2, \quad \sum_{k=1}^{n-1} b^{n-1-k} a_{n-k+1} \in \mathbb{C}\setminus\{0\}, \\ \text{(iii)} & \text{the equation (35) is irreducible over } \mathbb{C}[x]. \end{cases}$$

REMARK 2. From (36) (iii), it follows that g is algebraic of degree n over $\mathbb{C}[x]$. Actually (35) has n formal power series solutions

$$\begin{cases} 1 & \text{with } c_0 \neq 0, \\ n-2 & \text{with } c_0 = 0, \quad c_1 \neq 0, \\ 1 & \text{with } c_0 = c_1 = 0, \quad c_2 \neq 0. \end{cases}$$

The proof is left to the reader.

That it is possible to find a's and b that satisfy (36) follows from the next remark.

REMARK 3. For $s = b = a_{n-1} = \ldots = a_2 = 1$ we have, instead of (35),

$$(37)\quad Y^n + (x-1)Y^{n-1} + x(x-2)Y^{n-2} + x^2(x-3)Y^{n-3} + \ldots + x^{n-2}\{x-(n-1)\}Y + x^n \equiv 0 .$$

Now it is easy to prove that (37) is irreducible over $\mathbb{C}[x]$ and the unique solution of (37) with constant term equal to 1 is actually algebraic of degree n over $\mathbb{C}[x]$.

DEFINITION 2. Let f be the unique formal power series in x with constant term equal to 1 that satisfies

$$(38)\quad Y^n - Y^{n-1} - ax^r \equiv 0 \quad (a \in \mathbb{C}\setminus\{0\},\ r \in \mathbb{N}) .$$

REMARK 4. Again the proof that (38) is irreducible over $\mathbb{C}[x]$ is left to the reader (there exists a solution of the form $c_0 + c_1 x + \ldots + c_p x^p \Rightarrow a = c_p^n$, $r = np \Rightarrow p = 1$ and so on); that is, f is algebraic of degree n over $\mathbb{C}[x]$.

The number of (formal) solutions of (38) is different from that of (35):

$$\begin{cases} r = k(n-1) \text{ for some } k \in \mathbb{N}, & 1 \text{ with } c_0 \neq 0, \\ & n-1 \text{ with } c_0 = c_1 = \ldots = c_{k-1} = 0, \text{ and } c_k \neq 0; \\ r \neq k(n-1) \text{ for all } k \in \mathbb{N}, & 1 \text{ with } c_0 \neq 0. \end{cases}$$

THEOREM 4. *Let the n-tuple $f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ be given by*

$$(39)\quad f_0^{(n)} = g, \quad f_0^{(n+1-i)} = g^i - \left(g^{i-1} + xg^{i-2} + x^2 g^{i-3} + \ldots + x^{i-2} g\right) \qquad (i = 2, 3, \ldots, n),$$

with g satisfying (37), constant term equal to 1. Then there exists precisely one relation between $1, f_0^{(1)}, f_0^{(2)}, \ldots, f_0^{(n)}$ over $\mathbb{C}[x]$; namely:

$$(40)\qquad \sum_{j=1}^{n} x^{j-1} f_0^{(j)}(x) + x^n \equiv 0,$$

but nevertheless the n-tuple has a C-n-fraction without interruptions of the following form:

$$(41)\qquad \begin{bmatrix} & x^{n+1} & \ldots & x^{n+1} & \ldots \\ -(n-1)x^{n-1} & -(n-1)x^{n-1} & \ldots & -(n-1)x^{n-1} & \ldots \\ x^{n-2} & x^{n-2} & \ldots & x^{n-2} & \ldots \\ \vdots & \vdots & & \vdots & \\ x & x & \ldots & x & \\ 1 & 1 & \ldots & 1 & \end{bmatrix}.$$

Proof. Substitution of (39) in (40) leads to (37) with g instead of Y; so (40) holds. Now

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$$f_0^{(1)} = g^n - g^{n-1} - xg^{n-2} - \dots - x^{n-2}g$$
$$= -[xg^{n-1} + x(x-1)g^{n-2} + \dots + x^{n-2}\{x-(n-2)\}g + x^n] \text{ by (37)}$$
$$= -x[g^n + (x-1)g^{n-1} + \dots + x^{n-3}\{x-(n-2)\}g^2 + x^{n-1}g]/g$$
$$= -x[g^n + (x-1)g^{n-1} + \dots + x^{n-2}\{x-(n-1)\}g + x^n + (n-1)x^{n-2}g - x^n]/g$$
$$= -x[(n-1)x^{n-2}g - x^n]/g \text{ by (37)}$$
$$= -(n-1)x^{n-1} + (x^{n+1}/g) ;$$

thus $f_1^{(n)} = g = f_0^{(n)}$ while g has constant term equal to 1 .

Furthermore we have, for $i = 2, 3, \dots, n-1$,

$$f_0^{(i)} = g^{n+1-i} - g^{n-i} - xg^{n-i-1} - \dots - x^{n-i-1}g$$
$$= (g^{n+2-i} - g^{n+1-i} - xg^{n-i} - \dots - x^{n-i-1}g^2)/g$$
$$= \left[f_0^{(i-1)} + x^{n-i}g\right]/g$$
$$= x^{n-i} + \left(f_0^{(i-1)}\right)\Big/\left(f_1^{(n)}\right) ;$$

that is, $f_1^{(i)} = f_0^{(i)}$ $(i = 1, 2, \dots, n-2)$.

Finally

$$f_0^{(n)} = g = 1 + g - 1 = 1 + (g^2-g)/g = 1 + \left(f_0^{(n-1)}\right)\Big/\left(f_1^{(n)}\right) ;$$

thus $f_1^{(n-1)} = f_0^{(n-1)}$.

This shows that the C-n-fraction algorithm for $f_0^{(1)}, f_0^{(2)}, \dots, f_0^{(n)}$ is purely periodic with period length 1 and leads to the form (41).

That there is only one dependence relation for $1, f_0^{(1)}, f_0^{(2)}, \dots, f_0^{(n)}$ follows from the fact that g is algebraic of degree n over $\mathbb{C}[x]$. □

THEOREM 5. *Let* g *be the formal power series from Definition* 1 *and define*

(42) $$f = 1 + ax^r/g \quad (a \in \mathbb{C}\backslash\{0\},\ r \in \mathbb{N}) .$$

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Then the n-tuple $f, f^2, \ldots, f^n$ *(the powers of* f *) allows just one dependence relation for* $1, f, f^2, \ldots, f^n$ *over* $\mathbb{C}[x]$ *but has a C-n-fraction without interruptions of the following form*

$$(43)\quad \left(\begin{array}{cccccccccccc}
 & \binom{1}{1}ax^r & \binom{2}{2}a^2x^{2r} & \binom{3}{3}a^3x^{3r} & \cdots & \binom{n-1}{n-1}a^{n-1}x^{(n-1)r} & \binom{n}{n}a^nx^{nr} & b^{n+1}x^{(n+1)s} & \cdots & b^{n+1}x^{(n+1)s} & \cdots \\
1 & \binom{2}{1}ax^r & \binom{3}{2}a^2x^{2r} & \cdot & & \binom{n}{n-1}a^{n-1}x^{(n-1)r} & a_1x^{(n-1)s} & a_1x^{(n-1)s} & \cdots & a_1x^{(n-1)s} & \cdots \\
1 & \binom{3}{1}ax^r & \cdot & \cdot & \cdot & a_2x^{(n-2)s} & a_2x^{(n-2)s} & a_2x^{(n-2)s} & \cdots & a_2x^{(n-2)s} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \vdots & \vdots & \vdots & & \vdots & \\
\cdot & \cdot & \cdot & \binom{n}{3}a^3x^{3r} & & a_{n-3}x^{3s} & a_{n-3}x^{3s} & a_{n-3}x^{3s} & \cdots & a_{n-3}x^{3s} & \cdots \\
\cdot & \cdot & \binom{n}{2}a^2x^{2r} & a_{n-2}x^{2s} & \cdots & a_{n-2}x^{2s} & a_{n-2}x^{2s} & a_{n-2}x^{2s} & \cdots & a_{n-2}x^{2s} & \cdots \\
1 & \binom{n}{1}ax^r & a_{n-1}x^s & a_{n-1}x^s & \cdots & a_{n-1}x^s & a_{n-1}x^s & a_{n-1}x^s & \cdots & a_{n-1}x^s & \cdots \\
1 & 1 & 1 & 1 & & 1 & 1 & 1 & \cdots & 1 & \cdots
\end{array}\right).$$

Proof. That there is just one dependence relation is an immediate consequence of (36) (iii); see also Remark 2.

Define the formal power series $U_1, U_2, \ldots, U_{n-1}$ by

$$(44)\quad U_1 = a_1x^{(n-1)s} + \left(b^{n+1}x^{(n+1)s}\right)/g, \quad U_j = a_jx^{(n-j)s} + \left(U_{j-1}\right)/g \qquad (j = 2, 3, \ldots, n-1),$$

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and let $a_1 = -b \sum_{k=1}^{n-1} b^{n-1-k} a_{n-k+1}$ $\left(a_1 \neq 0\right.$ because of (36), (ii)$\left.\right)$.

The definition of f, (42), implies that the C-n-fraction algorithm for $f, f^2, \ldots, f^n$ begins in the following way (application number 0):

$$(45)\qquad \begin{cases} f = 1 + ax^r/g \quad \left(\text{that is } f_1^{(n)} = g\right) \\ f^2 = 1 + f^2 - 1 = 1 + \left(ax^r(1+f)\right)/g \\ \vdots \\ f^n = 1 + f^n - 1 = 1 + \left(ax^r\left(1 + f + f^2 + \ldots + f^{n-1}\right)\right)/g . \end{cases}$$

This shows that the first column and the entry $\binom{1}{1}ax^r$ on the top of the second column are correct in (43) and also

$$f_1^{(i)} = ax^r\left(1 + f + f^2 + \ldots + f^i\right) \quad (i = 1, 2, \ldots, n-1) , \quad f_1^{(n)} = g .$$

The next application of the algorithm (application number 1) leads to

$$(46)\qquad \begin{cases} f_1^{(1)} = ax^r(1+f) = 2ax^r + ax^r(f-1) = 2ax^r + \left(a^2x^{2r}\right)/g \\ \qquad\qquad\qquad\qquad\qquad\qquad \left(\text{that is } f_2^{(n)} = g\right), \\ f_1^{(i)} = (i+1)ax^r + \left(a^2x^{2r}\{f^{i-1} + 2f^{i-2} + 3f^{i-3} + \ldots + (i-1)f + i\}\right)/g \\ \qquad\qquad\qquad\qquad\qquad\qquad (i = 2, 3, \ldots, n-1) , \\ f_1^{(n)} = g = 1 + U_{n-1}/g . \end{cases}$$

Only the last line of (46) needs comment.

Multiplication of (35), with g substituted for Y, by $g - bx^s$ leads to

$$(47)\qquad g^{n+1} - g^n = \sum_{j=1}^{n-1} a_{n-j} x^{js} g^{n-j} + b^{n+1} x^{(n+1)s} .$$

Successive application of (44) for $j = n-1, n-2, \ldots, 2, 1$ yields, combined with (47),

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$$\cdot\cdot\left(U_{n-1}\right)/g = U_{n-1}g^{n-1}/g^n = \left[a_{n-1}x^s g^{n-1} + U_{n-2}g^{n-2}\right]/g^n = \ldots$$

$$\ldots = \left\{\sum_{j=1}^{n-1} a_{n-j}x^{js}g^{n-j} + b^{n+1}x^{(n+1)s}\right\}/g^n = g - 1 \;;$$

that is the last line of (46).

The result of (46) gives the second column and the entry $\binom{2}{2}a^2x^{2r}$ on the top of the third column of (43), and

$$f_2^{(i)} = a^2x^{2r}\{f^i + 2f^{i-1} + 3f^{i-2} + \ldots + if + (i+1)\} \,,$$

$$(i = 1, 2, \ldots, n-2) \,, \quad f_2^{(n-1)} = U_{n-1} \,, \quad f_2^{(n)} = g \,.$$

The remaining part of the proof is now relatively simple: each time the C-n-fraction algorithm is applied, another U appears until we get (after application number $n - 1$):

$$f_{n-1}^{(i)} = U_i \quad (i = 1, 2, \ldots, n-1) \,, \quad f_{n-1}^{(n)} = g \,.$$

After that application, the algorithm is purely periodic with period length 1; from (44) we find, for $\nu \geq n$,

$$\text{(48)} \qquad \begin{cases} f_\nu^{(i)} = U_i \quad (i = 1, 2, \ldots, n-1) \,, \\ f_\nu^{(n)} = g \,, \\ f_\nu^{(1)} = a_1x^{(n-1)s} + \left(b^{n+1}x^{(n+1)s}\right)/\left(f_{\nu+1}^{(n)}\right) \,, \\ f_\nu^{(i)} = a_ix^{(n-i)s} + \left(f_{\nu+1}^{(i-1)}\right)/\left(f_{\nu+1}^{(n)}\right) \quad (i = 2, 3, \ldots, n-1) \,, \\ f_\nu^{(n)} = 1 + \left(f_{\nu+1}^{(i-1)}\right)/\left(f_{\nu+1}^{(n)}\right) \,. \end{cases}$$

That the entries of (43) on the jth anti-diagonal (starting at the jth 1 of the first column, counted from the top entry) slanting upwards under $\pi/4$ rad. actually are the monomials that appear in the expansion of $\left(1+ax^r\right)^j$ by the binomial theorem, can easily be proved by induction (and endurance) using $\sum_{r=k}^{j-1}\binom{r}{k} = \binom{j}{k+1}$ $(j, k \in \mathbb{N}, j \geq k+1)$; this is left to the reader. □

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THEOREM 6. *Let* f *be the formal power series from Definition* 2. *Then there exists just one relation between* $1, f, f^2, \ldots, f^n$ *over* $\mathbb{C}[x]$ *(namely, equation* (38)) *and the C-n-fraction for* $f, f^2, \ldots, f^n$ *has just one interruption of order* 1 .

This C-n-fraction has the form

$$(49)\quad \left\{\begin{array}{cccccc}
 & \binom{1}{1}ax^r & \binom{2}{2}a^2x^{2r} & \binom{3}{3}a^3x^{3r} & \cdots & \binom{n-2}{n-2}a^{n-2}x^{(n-2)r} \\
1 & \binom{2}{1}ax^r & \binom{3}{2}a^2x^{2r} & . & . & \binom{n-1}{n-2}a^{n-2}x^{(n-2)r} \\
1 & \binom{3}{1}ax^r & . & . & . & \binom{n}{n-2}a^{n-2}x^{(n-2)r} \\
1 & . & . & \binom{n-1}{3}a^3x^{3r} & . & \binom{n-1}{n-3}a^{n-3}x^{(n-3)r} \\
. & . & \binom{n-1}{2}a^2x^{2r} & \binom{n}{3}a^3x^{3r} & . & \vdots \\
. & \binom{n-1}{1}ax^r & \binom{n}{2}a^2x^{2r} & \binom{n-1}{2}a^2x^{2r} & \cdots & \binom{n-1}{2}a^2x^{2r} \\
1 & \binom{n}{1}ax^r & \binom{n-1}{1}ax^r & \binom{n-1}{1}ax^r & \cdots & \binom{n-1}{1}ax^r \\
1 & 1 & 1 & 1 & \cdots & 1
\end{array}\right.$$

$$\begin{array}{cccc}
\binom{n-1}{n-1}a^{n-1}x^{(n-1)r} & \binom{n}{n}a^nx^{nr} & 0 & \cdots \\
\binom{n}{n-1}a^{n-1}x^{(n-1)r} & \binom{n-1}{n-1}a^{n-1}x^{(n-1)r} & \binom{n-1}{n-1}a^{n-1}x^{(n-1)r} & \cdots \\
\binom{n-1}{n-2}a^{n-2}x^{(n-2)r} & \binom{n-1}{n-2}a^{n-2}x^{(n-2)r} & \binom{n-1}{n-2}a^{n-2}x^{(n-2)r} & \cdots \\
\vdots & \vdots & \vdots & \\
\binom{n-1}{2}a^2x^{2r} & \binom{n-1}{2}a^2x^{2r} & \binom{n-1}{2}a^2x^{2r} & \cdots \\
\binom{n-1}{1}ax^r & \binom{n-1}{1}ax^r & \binom{n-1}{1}ax^r & \cdots \\
1 & 1 & 1 & \cdots
\end{array}$$

$$\begin{matrix} \cdots & 0 & \cdots \\ \cdots & \binom{n-1}{n-1}a^{n-1}x^{(n-1)r} & \cdots \\ \cdots & \binom{n-1}{n-2}a^{n-2}x^{(n-2)r} & \cdots \\ & \vdots & \\ \cdots & \binom{n-1}{2}a^{2}x^{2r} & \cdots \\ \cdots & \binom{n-1}{1}ax^{r} & \cdots \\ \cdots & 1 & \cdots \end{matrix}\Bigg) .$$

Proof. That there is just one dependence relation follows at once from Remark 4; this implies, by Theorem 3, that there is at most one interruption of order 1 . Application number 0 of the algorithm gives, combined with (38),

$$(50)\quad \begin{cases} f = 1 + f - 1 = 1 + \left(f^{n}-f^{n-1}\right)/f^{n-1} = 1 + ax^{r}/f^{n-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \left(\text{that is, } f_1^{(n)} = f^{n-1}\right), \\ f^{2} = 1 + f^{2} - 1 = 1 + \left(ax^{r}(1+f)\right)/f^{n-1} , \\ \vdots \\ f^{n} = 1 + f^{n} - 1 = 1 + \left(ax^{r}\left(1 + f + f^{2} + \ldots + f^{n-1}\right)\right)/f^{n-1} . \end{cases}$$

Thus $f_1^{(i)} = ax^{r}\left(1 + f + f^{2} + \ldots + f^{i}\right)$ $(i = 1, 2, \ldots, n-1)$, $f_1^{(n)} = f^{n-1}$. Application number 1 leads to

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$$(51)\quad \begin{cases} f_1^{(1)} = ax^r(1+f) = 2ax^r + a^2x^{2r}/f^{n-1} \quad \left(\text{that is,}\ f_2^{(n)} = f^{n-1}\right), \\ f_1^{(i)} = (i+1)ax^r \\ \qquad + \left(a^2x^{2r}\{f^{i-1} + 2f^{i-2} + 3f^{i-3} + \ldots + (i-1)f + i\}\right)/f^{n-1} \\ \qquad\qquad (i = 2, 3, \ldots, n-1)\ , \\ f_1^{(n)} = f^{n-1} = 1 + \left(ax^r(f^{n-2} + f^{n-3} + \ldots + f + 1)\right)/f^{n-1}\ , \end{cases}$$

which shows

$$(52)\quad \begin{cases} f_2^{(i)} = a^2x^{2r}\{f^i + 2f^{i-1} + 3f^{i-2} + \ldots + if + (i+1)\} \\ \qquad\qquad (i = 1, 2, \ldots, n-2)\ , \\ f_2^{(n-1)} = ax^r(f^{n-2} + f^{n-3} + \ldots + f + 1)\ , \\ f_2^{(n)} = f^{n-1}\ . \end{cases}$$

After a cumbersome proof by induction we get, for $k = 2, 3, \ldots, n-1$,

$$(53)\quad \begin{cases} f_k^{(i)} = a^kx^{kr}\left\{f^i + \binom{k}{k-1}f^{i-1} + \binom{k+1}{k-1}f^{i-2} + \ldots \right. \\ \qquad \left. + \binom{k+i-2}{k-1}f + \binom{k+i-1}{k-1}\right\} \quad (i = 1, 2, \ldots, n-k) \\ f_k^{(n-j)} = a^jx^{jr}\left\{f^{n-j-1} + \binom{j}{j-1}f^{n-j-2} + \binom{j+1}{j-1}f^{n-j-3} + \ldots \right. \\ \qquad \left. + \binom{n-3}{j-1}f + \binom{n-2}{j-1}\right\} \quad (j = k-1, k-2, \ldots, 1) \\ f_k^{(n)} = f^{n-1}\ . \end{cases}$$

Now (53) leads to

$$(54)\quad \begin{cases} f_n^{(1)} = a^{n-1}x^{(n-1)r}\ , \\ f_n^{(n-j)} = a^jx^{jr}\left\{f^{n-j-1} + \binom{j}{j-1}f^{n-j-2} + \binom{j+1}{j-1}f^{n-j-3} + \ldots \right. \\ \qquad \left. + \binom{n-3}{j-1}f + \binom{n-2}{j-1}\right\} \quad (j = n-2, n-3, \ldots, 1) \\ f_n^{(n)} = f^{n-1} \end{cases}$$

after which the appearance of an interruption of order 1 is clear: $f_n^{(1)}$ is a monomial.

Application of the algorithm to (54) gives

$$(55)\quad \begin{cases} f_{n+1}^{(n-j)} = a^j x^{jr}\left\{f^{n-j-1} + \binom{j}{j-1}f^{n-j-2} + \binom{j+1}{j-1}f^{n-j-3} + \dots \right. \\ \qquad\qquad \left. + \binom{n-3}{j-1}f + \binom{n-2}{j-1}\right\} \quad (j = n-2,\ n-3,\ \dots,\ 1)\ , \\ f_{n+1}^{(n)} = f^{n-1}\ , \end{cases}$$

and further applications do not change the $(n-1)$-tuple

$$(56)\qquad f_\nu^{(i)} = f_{n+1}^{(i)} \quad (i = 2,\ 3,\ \dots,\ n;\ \nu = n+2,\ n+3,\ \dots)\ .$$

The form of the C-n-fraction, (49), follows from (50), (51), and from the first non-zero terms of the formal power series in (53), (54), (55), and (56). □

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