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ON A RECENT GENERALIZATION OF SEMIPERFECT RINGS

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Abstract

In a recent paper by Wang and Ding, it was stated that any ring which is generalized supplemented as a left module over itself is semiperfect. The purpose of this note is to show that Wang and Ding's claim is not true and that the class of generalized supplemented rings lies properly between the classes of semilocal and semiperfect rings. Moreover, we propose a corrected version of the theorem by introducing a wider notion of 'local' for submodules.

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1. Introduction

In [3], Bass characterized the rings R whose left R-modules have projective covers and termed them *left perfect rings*; these were characterized as semilocal rings which have a left t-nilpotent Jacobson radical $\mathrm{Jac}(R)$. Bass's *semiperfect rings* are those whose finitely generated left (or right) R-modules have projective covers; they can be characterized as semilocal rings which have the property that idempotents lift modulo $\mathrm{Jac}(R)$.

In [9], Kasch and Mares transferred the notions of perfect and semiperfect rings to modules, and characterized semiperfect modules by a lattice-theoretical condition as follows. A module M is called *supplemented* if for any submodule N of M, there exists a submodule L of M which is minimal and such that $M = N + L$. It is then shown that the left perfect rings are exactly those rings whose left R-modules are supplemented, while the semiperfect rings are rings whose finitely generated left R-modules are supplemented. Equivalently, for a ring R to be semiperfect it is enough that the left (or right) R-module be supplemented. Recall that a submodule N of

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a module M is said to be *small*, denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M. Weakening the 'supplemented' condition, one calls a module *weakly supplemented* if for every submodule N of M, there exists a submodule L of M with $N + L = M$ and $N \cap L \ll M$. The semilocal rings R are precisely those rings whose finitely generated left (or right) R-modules are weakly supplemented. Again, it is enough that R is weakly supplemented as a left (or right) R-module. There are many semilocal rings which are not semiperfect: for example, the localization of the integers at two distinct primes.

Recently, another notion of 'supplement' for submodules has emerged, called the *Rad-supplement*. A submodule N of a module M has a *generalized supplement* or *Rad-supplement* L in M if $N + L = M$ and $N \cap L \subseteq \operatorname{Rad}(L)$ (see [11]). Here $\operatorname{Rad}(L)$ denotes the *radical* of L, namely the intersection of all maximal submodules of L or, if L has no such submodules, simply L itself. If every submodule of M has a Rad-supplement, then M is called *Rad-supplemented* or *generalized supplemented*. Note that Rad-supplements L of M are also called *coneat submodules* in the literature, and can be characterized by the fact that any module with zero radical is injective with respect to the inclusion $L \subseteq M$ (see [5, 10.14] or [2]). Recall that a submodule N of a module M is called *cofinite* if M/N is finitely generated. In our terminology, Wang and Ding claimed that the following result holds.

THEOREM (Wang–Ding [11, Theorem 2.11]). *For any left R-module M, the following conditions are equivalent:*

(a) *every submodule of M is contained in a maximal submodule, and every cofinite submodule of M has a Rad-supplement in M;*
(b) *M has a small radical and can be written as an irredundant sum of local submodules.*

Here a module L is called local if $L/\operatorname{Rad}(L)$ is simple and $\operatorname{Rad}(L)$ is small in L.

Since it is well-known that finite sums of supplemented modules are supplemented (see [13, 41.2]), Wang and Ding's claim implies that a finitely generated module M is supplemented if and only if it is Rad-supplemented; in particular, in the case of $M = R$, R is semiperfect if and only if R is Rad-supplemented as a left (or right) R-module. The purpose of this note is to show that Wang and Ding's claim is not true and that the class of Rad-supplemented rings lies properly between the class of semilocal and the class of semiperfect rings. Moreover, we present a corrected version of the above theorem by introducing a different 'local' condition for modules.

Throughout the paper, R will be an associative ring with identity, and all modules will be unital left R-modules.

2. Examples

It is clear from the introduction that one has the following implications between conditions on submodules of a module with small radical:

$$\textit{supplements} \Rightarrow \textit{Rad-supplements} \Rightarrow \textit{weak supplements}.$$

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This further implies that any semiperfect ring is Rad-supplemented and any Rad-supplemented ring is semilocal.

2.1. Semilocal rings which are not Rad-supplemented Since any finitely generated module has a small radical, it is clear that finitely generated Rad-supplements are supplements. Thus, any left noetherian, left Rad-supplemented ring is semiperfect, and any left noetherian, semilocal, non-semiperfect ring would give an example of a semilocal ring which is not Rad-supplemented. It is well-known that the localization of the integers by two distinct primes p and q, namely

$$R = \left\{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ is neither divisible by } p \text{ nor by } q \right\},$$

is a semilocal noetherian ring which is not semiperfect; hence it provides such an example.

2.2. Rad-supplemented rings which are not semiperfect While a noetherian ring is Rad-supplemented if and only it is semiperfect, we now show that a ring with idempotent Jacobson radical is semilocal if and only if it is Rad-supplemented.

PROPOSITION 2.1. *Let R be a ring with idempotent Jacobson radical J. Then R is semilocal if and only if R is Rad-supplemented as a left or right R-module.*

PROOF. Let R be semilocal with idempotent Jacobson radical. Let I be a left ideal of R. If $I \subseteq J$, then I is small and R is a Rad-supplement of I. Therefore, suppose that $I \nsubseteq J$. In this case $(I + J)/J$ is a direct summand in the semisimple ring R/J, and there exists a left ideal K of R containing J such that $I + K = R$ and $I \cap K \subseteq J$. Since $J \subseteq K$ and J is idempotent, we have $J = J^2 \subseteq JK = \text{Rad}(K)$. Thus K is a Rad-supplement of I in R.

The converse is clear. □

Hence, any indecomposable nonlocal semilocal ring with idempotent Jacobson radical would be an example of a Rad-supplemented ring which is not semiperfect. To construct such a ring, we shall look at the endomorphism rings of uniserial modules. Recall that a unital ring R is called a *nearly simple uniserial domain* if it is uniserial as a left and right R-module, that is, its lattices of left, respectively right, ideals are linearly ordered such that $\text{Jac}(R)$ is the unique nonzero two-sided ideal of R. Let R be any nearly simple uniserial domain, let $r \in \text{Jac}(R)$ and let $S = \text{End}(R/rR)$ be the endomorphism ring of the uniserial cyclic right R-module R/rR. Then S is semilocal by a result of Herbera and Shamsuddin (see [7, 4.16]). Moreover, Puninski showed in [10, 6.7] that S has exactly three nonzero proper two-sided ideals, namely the two maximal ideals

$$I = \{f \in S \mid f \text{ is not injective}\} \quad \text{and} \quad K = \{g \in S \mid g \text{ is not surjective}\},$$

and the Jacobson radical $\text{Jac}(S) = I \cap K$, which is idempotent. Note that since R/rR is indecomposable, S also has no nontrivial idempotents. Furthermore, S is

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a prime ring. Hence S is an indecomposable, nonlocal semilocal ring with idempotent Jacobson radical.

Our example depends on the existence of nearly simple uniserial domains. Such rings were first constructed by Dubrovin in 1980 (see [6]), as follows. Let

$$G = \{f : \mathbb{Q} \to \mathbb{Q} \mid f(t) = at + b \text{ for } a, b \in \mathbb{Q} \text{ and } a > 0\}$$

be the group of affine linear functions on the field of rational numbers $\mathbb{Q}$. Choose any irrational number $\varepsilon \in \mathbb{R}$, and set

$$P = \{f \in G \mid \varepsilon \leq f(\varepsilon)\} \quad \text{and} \quad P^+ = \{f \in G \mid \varepsilon < f(\varepsilon)\}.$$

Note that P, and also P^+, defines a left order on G. Take an arbitrary field F and consider the semigroup group ring $F[P]$ in which the right ideal $M = \sum_{g \in P^+} gF[P]$ is maximal. The set $F[P] \setminus M$ is a left and right Ore set, and the corresponding localization R is a nearly simple uniserial domain (see [4, 6.5]). Thus, taking any nonzero element $r \in R$, $S = \mathrm{End}(R/rR)$ is a Rad-supplemented ring which is not semiperfect.

3. Cofinitely Rad-supplemented modules

We say that the module M is *cofinitely Rad-supplemented* if any cofinite submodule of M has a Rad-supplement. Note that every submodule of a finitely generated module is cofinite, hence a finitely generated module is Rad-supplemented if and only if it is cofinitely Rad-supplemented. Wang and Ding's theorem tried to describe cofinitely Rad-supplemented modules with small radical as sums of local modules. In order to correct their theorem, we introduce a different module-theoretic local condition.

3.1. w-local modules We say that a module is w-*local* if it has a unique maximal submodule. It is clear that a module is w-local if and only if its radical is maximal. The question of whether any projective w-local module must be local was first studied by Ware in [12, p. 250]; he proved that if R is commutative or left noetherian, or if idempotents lift modulo $\mathrm{Jac}(R)$, then any projective w-local module must be local (see [12, 4.9–11]). The first example of a w-local nonlocal projective module was given by Gerasimov and Sakhaev in [8].

In general, unless the module has small radical, a w-local module need not be local, as we now show by considering the abelian group $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$ for any prime p. Clearly, $J = \mathbb{Q} \oplus 0$ is maximal in M. We claim that $J = \mathrm{Rad}(M)$. Indeed, if N is another maximal submodule of M, then $M = N + J$ and thus $J/(N \cap J) \simeq M/N$ is simple; but this is impossible since $\mathbb{Q}$ has no simple factors. Therefore $J = \mathbb{Q} \oplus 0$ is the unique maximal submodule of (the nonlocal module) M.

The same construction is possible for any ring R with a nonzero left R-module Q such that $\mathrm{Rad}(Q) = Q$. Then, by taking any simple left R-module E, one concludes that $Q \oplus E$ is a w-local module which is not local. In other words, if every w-local module over a ring R is local, then $\mathrm{Rad}(Q) \neq Q$ for all nonzero left R-modules Q,

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that is, R is a *left max ring*. Conversely, any left module over a left max ring has a small radical, and hence any w-local module is local. Thus, we have just proved the following result.

LEMMA 3.1. *A ring R is a left max ring if and only if every w-local left R-module is local.*

3.2. Rad-supplements of maximal submodules While local (or hollow) modules are supplemented, w-local modules might not be.

LEMMA 3.2. *Every w-local module M is cofinitely Rad-supplemented.*

PROOF. Let U be a cofinite submodule of M. Since M/U is finitely generated, U is contained in a maximal submodule of M, and so $U \subseteq \operatorname{Rad} M$. Now M is a Rad-supplement of U in M, because $U + M = M$ and $U \cap M = U \subseteq \operatorname{Rad} M$. □

The gap in Wang and Ding's theorem arises from the fact that Rad-supplements of maximal submodules need not be local. They are, however, always w-local, as we shall prove next.

LEMMA 3.3. *Any Rad-supplement of a maximal submodule is w-local.*

PROOF. Let K be a maximal submodule of a module M, and let L be a Rad-supplement of K in M. Then $K + L = M$ and $K \cap L \subseteq \operatorname{Rad} L$. Since $M/K \cong L/K \cap L$ is simple, $\operatorname{Rad} L$ is a maximal submodule of L. Thus L is a w-local module. □

Let R be again a nearly simple uniserial domain, let $0 \neq r \in \operatorname{Jac}(R)$ and let $S = \operatorname{End}(R/rR)$. Then S has two maximal two-sided ideals I and K such that $I \cap K = J = J^2$, where $J = \operatorname{Jac}(S)$. Since $J \subset K$, we have $J = J^2 \subseteq JK \subseteq J$. Thus $I \cap K = J = JK = \operatorname{Rad}(K)$. Analogously, one can show that $I \cap K = \operatorname{Rad}(I)$. Hence I and K are mutual Rad-supplements. By [10, p. 239], K is a cyclic uniserial left ideal, while the left ideal I is not finitely generated. Hence K is a supplement of I. Moreover, K is a supplement of any maximal left ideal M of S that contains I, because $K + M \supseteq K + I = S$ and $K \cap M \subseteq K$ is small in K since K is uniserial. Therefore K is a local submodule. On the other hand, I is not a supplement of K, because otherwise K and I would be mutual supplements and hence direct summands, which is impossible.

3.3. Closure properties of cofinitely Rad-supplemented modules We establish some general closure properties of cofinitely Rad-supplemented modules. To begin with, we prove the following lemma.

LEMMA 3.4. *Let $N \subseteq M$ be modules such that N is cofinitely Rad-supplemented. If U is a cofinite submodule of M such that $U + N$ has a Rad-supplement in M, then U also has a Rad-supplement in M.*

PROOF. Let K be a Rad-supplement of $U + N$ in M. Since $N/[N \cap (U + K)] \cong M/(U + K)$ is finitely generated, $N \cap (U + K)$ is a cofinite submodule of N. Let

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L be a Rad-supplement of $N \cap (U + K)$ in N, that is, $L + (N \cap (U + K)) = N$ and $L \cap (U + K) \subseteq \text{Rad } L$. Then we have

$$M = U + N + K = U + K + L + (N \cap (U + K)) = U + K + L$$

and

$$U \cap (K + L) \subseteq (K \cap (U + L)) + (L \cap (U + K)) \subseteq \text{Rad } K + \text{Rad } L \subseteq \text{Rad}(K + L).$$

Hence $K + L$ is a Rad-supplement of U in M. □

With the preceding lemma in hand, we can now prove the following result.

THEOREM 3.5. *The class of cofinitely Rad-supplemented modules is closed under arbitrary direct sums and homomorphic images.*

PROOF. Let $\mathcal{C}$ denote the class of cofinitely Rad-supplemented left R-modules over a fixed ring R. We shall first show that $\mathcal{C}$ is closed under factor modules. First of all, it is clear that $\mathcal{C}$ is closed under isomorphisms, since being Rad-supplemented is a lattice-theoretical notion. Take any $M \in \mathcal{C}$ and $f : M \to X$, where X is a left R-module. We may assume that $f(M)$ is of the form M/N for some submodule N of M. Let K/N be a cofinite submodule of M/N. Then K is a cofinite submodule of M, so that K has a Rad-supplement L in M. In other words, $K + L = M$ and $K \cap L \subseteq \text{Rad}(L)$. We then have $K/N + (L + N)/N = M/N$ and

$$K \cap (L + N)/N = (K \cap L + N)/N \subseteq (\text{Rad } L + N)/N \subseteq \text{Rad}((L + N)/N).$$

Therefore $(L + N)/N$ is a Rad-supplement of K/N in M/N. Hence $M/N \in \mathcal{C}$.

Let $M = \bigoplus_{i \in I} M_i$, with $M_i \in \mathcal{C}$ and U a cofinite submodule of M. Then there is a finite subset $F = \{i_1, i_2, \ldots, i_k\} \subseteq I$ such that $M = U + \bigoplus_{n=1}^{k} M_{i_n}$. Since $U + \bigoplus_{n=2}^{k} M_{i_n}$ is cofinite and $M = M_{i_1} + (U + \bigoplus_{n=2}^{k} M_{i_n})$ trivially has a Rad-supplement in M, it follows from Lemma 3.4 that $U + \bigoplus_{n=2}^{k} M_{i_n}$ has a Rad-supplement in M. By using Lemma 3.4 repeatedly for $k - 1$ times, we get a Rad-supplement for U in M. Hence M is cofinitely Rad-supplemented. □

It follows from Theorem 3.5 that a ring R is left Rad-supplemented if and only if every left R-module is cofinitely Rad-supplemented.

Now we give an example showing that there are cofinitely Rad-supplemented modules which are not Rad-supplemented. Let $N \subseteq M$ be left R-modules. If M/N has no maximal submodule, then any proper Rad-supplement L of N in M also has no maximal submodule, that is, $L = \text{Rad}(L)$; this is because if L were a Rad-supplement of N in M and K a maximal submodule of L, then

$$M/(N + K) \simeq L/((N \cap L) + K) = L/K \neq 0$$

would be a nonzero simple module, which is a contradiction.

Next, let R be a discrete valuation ring with quotient field K. Then $F/T \cong K = \text{Rad}(K)$ as R-modules, for a free R-module F and $T \subseteq F$. Since R is hereditary, any submodule of F is projective and has a proper radical; thus T cannot have a Rad-supplement in F by the preceding remark, that is, F is not Rad-supplemented. On the other hand, since R is local and hence Rad-supplemented, F is cofinitely Rad-supplemented by Theorem 3.5.

We call a module M *radical-full* if $M = \text{Rad}(M)$. Obviously, radical-full modules are Rad-supplemented.

LEMMA 3.6. *Any extension of a radical-full module by a cofinitely Rad-supplemented module is cofinitely Rad-supplemented.*

PROOF. Let N be a cofinitely Rad-supplemented submodule of M such that M/N is radical-full. For any cofinite $U \subseteq M$, $(U + N)/N$ is a cofinite submodule of M/N, and hence $U + N = M$. Therefore, by Lemma 3.4, U has a Rad-supplement in M. □

3.4. Characterization of Rad-supplemented modules with small radical

Recall that Wang and Ding's theorem expresses a cofinitely Rad-supplemented module with small radical as an irredundant sum of local modules, which, as shown above, is not possible in general. Here we show that any cofinitely Rad-supplemented module can be written as the sum of w-local modules instead. The following theorem is a cofinitely Rad-supplemented analogue of a result in [1] by Alizade *et al.* for cofinitely supplemented modules.

THEOREM 3.7. *The following statements are equivalent for a module M.*

(a) *M is cofinitely Rad-supplemented.*
(b) *Every maximal submodule of M has a Rad-supplement in M.*
(c) *$M/w\text{Loc}(M)$ has no maximal submodules, where $w\text{Loc}(M)$ is the sum of all w-local submodules of M.*
(d) *$M/cgs(M)$ has no maximal submodules, where $cgs(M)$ is the sum of all cofinitely Rad-supplemented submodules of M.*

PROOF. (a) implies (b) is clear.

(b) implies (c). Suppose $w\text{Loc}(M) \subseteq K$ for some maximal submodule K of M. Let N be a Rad-supplement of K in M, that is, $K + N = M$ and $K \cap N \subseteq \text{Rad}\, N$. Then, by Lemma 3.3, N is a w-local submodule of M and so $N \subseteq w\text{Loc}(M) \subseteq K$, which is a contradiction. Thus $M/w\text{Loc}(M)$ has no maximal submodules.

(c) implies (d). Suppose K is a maximal submodule of M with $cgs(M) \subseteq K$. Since $M/w\text{Loc}(M)$ has no maximal submodules, we have $K + w\text{Loc}(M) = M$. So $K + L = M$ for some w-local submodule L of M. By Lemma 3.2, L is cofinitely Rad-supplemented and so $L \subseteq cgs(M) \subseteq K$, which is a contradiction. Hence $M/cgs(M)$ has no maximal submodules.

(d) implies (a). By Theorem 3.5, $cgs(M)$ is cofinitely Rad-supplemented and so, by Lemma 3.6, M is cofinitely Rad-supplemented. □

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For finitely generated modules M, for example $M = R$, we can rephrase the theorem as follows.

COROLLARY 3.8. *The following statements are equivalent for a finitely generated module M.*

(a) *M is Rad-supplemented.*
(b) *Every maximal submodule of M has a Rad-supplement in M.*
(c) *M is the sum of finitely many w-local submodules.*

4. Concluding remarks

Whereas a ring is semiperfect (that is, supplemented) if and only if it is a sum of local submodules, we have shown that a ring is left Rad-supplemented if and only if it is a sum of w-local submodules. Moreover, the class of Rad-supplemented rings lies strictly between the semilocal and the semiperfect rings. In particular, any example of a left Rad-supplemented ring which is not semiperfect must contain a w-local left ideal which is not local (cyclic).

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