Compositio Mathematica **128:** 299–322, 2001.

A Filtration on the Chow Groups of a Complex Projective Variety

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(Received: 15 September 1999; accepted in final form: 19 September 2000)

Abstract. Let $X/_{\mathbf{C}}$ be a projective algebraic manifold, and further let $CH^k(X)_{\mathbf{Q}}$ be the Chow group of codimension k algebraic cycles on X, modulo rational equivalence. By considering $\mathbf{Q}$-spreads of cycles on X and the corresponding cycle map into absolute Hodge cohomology, we construct a filtration $\{F^\ell\}_{\ell \geqslant 0}$ on $CH^k(X)_{\mathbf{Q}}$ of 'Bloch-Beilinson' type. In the event that a certain conjecture of Jannsen holds (related to the Bloch-Beilinson conjecture on the injectivity, modulo torsion, of the Abel–Jacobi map for smooth proper varieties over $\mathbf{Q}$), this filtration truncates. In particular, his conjecture implies that $F^{k+1} = 0$.

Mathematics Subject Classifications (2000). 14C30, 19E15.

Key words. Chow group, Abel–Jacobi map, regulator, Deligne cohomology, Hodge structure.

1. Introduction

Let $X/_{\mathbf{C}}$ be a smooth, projective algebraic manifold, and further let $z^k(X)$ be the free abelian group generated by codimension k irreducible subvarieties in X. The Chow group of codimension k cycles on X is given by:

$$CH^k(X) := \text{cokernel}\left(\bigoplus_{\text{codim}_X V = k-1} \mathbf{C}(V)^\times \xrightarrow{\text{div}} z^k(X) \right),$$

where div is the divisor map.

There are two well-known constructions of the cycle class map. First, there is the fundamental class map

$$\text{cl}_k : CH^k(X) \to H^{2k}_{\text{sing}}(X, \mathbf{Z}(k)),$$

*Supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

https://doi.org/10.1023/A:1011882030468 Published online by Cambridge University Press

and second, there is the Abel–Jacobi map

$$\Phi_k : CH^k_{\text{hom}}(X) \to J^k(X) := \frac{H^{2k-1}(X, \mathbf{C})}{F^0 H^{2k-1}(X, \mathbf{C}) + H^{2k-1}(X, \mathbf{Z}(k))}$$
$$\simeq \frac{H^{2k-1}(X, \mathbf{C})}{F^k H^{2k-1}(X, \mathbf{C}) + H^{2k-1}(X, \mathbf{Z})},$$

where $CH^k_{\text{hom}}(X) = \ker \text{cl}_k$, and where in the former description of $J^k(X)$, the Hodge structure given by $H^{2k-1}(X, \mathbf{C}) \simeq H^{2k-1}(X, \mathbf{Q}(k)) \otimes \mathbf{C}$ has weight -1.

The maps cl_k and Φ_k are in general neither injective nor surjective, and their kernels and images can be very complicated, as can be seen by the seminal works of Mumford [Mu] and Griffiths [Gr]. In fact, even for X defined over a field L of transcendence degree 1 over $\mathbf{Q}$, the kernel of $\Phi_{k,\mathbf{Q}} : CH^k(X_L)_{\mathbf{Q}} \to J^k(X(\mathbf{C}))_{\mathbf{Q}}$ can be nontrivial [Sch], where for any abelian group M, $M_{\mathbf{Q}} = M \otimes_{\mathbf{Z}} \mathbf{Q}$. The transcendence degree of the underlying field playing an essential role in the nontriviality of $\ker \Phi_{k,\mathbf{Q}}$ suggests that the opposite situation should occur if X is defined over a number field. In this direction, the following was conjectured by Bloch [Bl1] (and independently by Beilinson):

CONJECTURE 1.0. For smooth and proper Z defined over $\overline{\mathbf{Q}}$, the complex Abel–Jacobi map

$$\Phi_{k,\mathbf{Q}} : CH^k_{\text{hom}}(Z/_{\overline{\mathbf{Q}}})_{\mathbf{Q}} \hookrightarrow J^k(Z(\mathbf{C}))_{\mathbf{Q}}$$

is injective, where $\overline{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$.

Remarks 1.0.1. Bloch and Beilinson (*op. cit.*) originally formulate Conjecture (1.0) for smooth and proper Z defined over $\mathbf{Q}$. This is equivalent to the corresponding statement for smooth and proper Z defined over a number field [J2; p. 158], and hence for Z defined over $\overline{\mathbf{Q}}$. We also wish to point out that it is the conjectured injectivity of $\Phi_{k,\mathbf{Q}}$ in (1.0) that motivates us to restrict ourselves to the Chow groups tensored with $\mathbf{Q}$, viz. $CH^k(\ ) \otimes \mathbf{Q}$, rather than deal with the possibility of torsion cycles in the kernel of the corresponding $\Phi_k : CH^k_{\text{hom}}(Z/_{\mathbf{Q}}) \to J^k(Z(\mathbf{C}))$. Furthermore, the Künneth decomposition of the diagonal class in Theorem (1.2)(v) below, into algebraic classes, can only be expected to hold over $\mathbf{Q}$.

Since Φ_k is in general not injective for $X/_{\mathbf{C}}$, one anticipates that the kernel of Φ_k can be 'explained' by kernels of successive higher regulator maps, defining a filtration below.

$$CH^k(X/_{\mathbf{C}})_{\mathbf{Q}} = F^0 \supset F^1 \supset F^2 \supset \cdots \supset F^k \supset \{0\}, \tag{1.1}$$

where $F^1 = \ker \text{cl}_{k,\mathbf{Q}}$, and $F^2 = \ker \Phi_{k,\mathbf{Q}}$. This is fortified by Beilinson's conjectural formula:

$$Gr_F^{\ell} CH^k(X)_{\mathbf{Q}} = \mathrm{Ext}^{\ell}_{\mathcal{MM}}(\mathbf{1}, h^{2k-\ell}(X)(k)),$$

where $\mathcal{MM}$ is the conjectural category of mixed motives, and **1** is the trivial motive. There are a number of works in the literature which support the idea of a filtration. The reader can consult, for example, the works of [As], [A-S], [Gn], [Ra], [J2; p. 178], [Sa], and the references cited there.

Both cycle maps cl_k, Φ_k can be combined in the diagram below:

$$\begin{array}{ccccccccc} 0 & \to & CH^k_{\mathrm{hom}}(X) & \to & CH^k(X) & \to & CH^k(X)/CH^k_{\mathrm{hom}}(X) & \to & 0 \\ & & \Phi_k \downarrow & & \downarrow \Psi_k & & \downarrow \mathrm{cl}_k & & \\ 0 & \to & J^k(X) & \to & H^{2k}_{\mathcal{D}}(X, \mathbf{Z}(k)) & \to & \mathrm{Hg}^k(X) & \to & 0 \end{array}$$

where the Hodge group

$$\mathrm{Hg}^k(X) \subset H^{2k}(X, \mathbf{Z}(k))$$

can be identified with $\mathrm{hom}_{\mathrm{MH}}(\mathbf{Z}(0), H^{2k}(X, \mathbf{Z}(k)))$, and [Ca]

$$J^k(X) \simeq \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{Z}(0), H^{2k-1}(X, \mathbf{Z}(k))),$$

and where $H^i_{\mathcal{D}}(X, \mathbf{Z}(j)) := \mathbf{H}^i(\mathbf{Z}(j) \hookrightarrow \Omega_X^{\bullet < j})$ is Deligne cohomology. Here Ψ_k is the corresponding cycle map (see [E-V]). If again X is defined over $\overline{\mathbf{Q}}$, then by conjecture (1.0) above, Ψ_k is injective modulo torsion. Assuming this, one can then read off a 2 step filtration on $CH^k(X/_{\overline{\mathbf{Q}}})_{\mathbf{Q}}$. More specifically, from the short exact sequence

$$0 \to J^k(X(\mathbf{C}))_{\mathbf{Q}} \to H^{2k}_{\mathcal{D}}(X(\mathbf{C}), \mathbf{Q}(k)) \to H^{k,k}(X(\mathbf{C}), \mathbf{Q}(k)) \to 0,$$

there is the filtration:

$$\varphi_0 = H^{2k}_{\mathcal{D}}(X(\mathbf{C}), \mathbf{Q}(k)),$$
$$\varphi_1 = J^k(X(\mathbf{C}))_{\mathbf{Q}},$$
$$\varphi_2 = 0.$$

This correspondingly induces a filtration (via (1.0)) on Chow groups

$$F^0 CH^k(X/_{\overline{\mathbf{Q}}})_{\mathbf{Q}} = CH^k(X)_{\mathbf{Q}},$$
$$F^1 CH^k(X/_{\overline{\mathbf{Q}}})_{\mathbf{Q}} = CH^k_{\mathrm{hom}}(X)_{\mathbf{Q}},$$
$$F^2 CH^k(X/_{\overline{\mathbf{Q}}})_{\mathbf{Q}} = \{\ker : CH^k_{\mathrm{hom}}(X)_{\mathbf{Q}} \hookrightarrow J^k(X(\mathbf{C}))_{\mathbf{Q}}\} = \{0\}.$$

The idea of this paper is to conceive of a filtration on $CH^k(X/_{\mathbf{C}})_{\mathbf{Q}}$ based on reducing to this special situation. Roughly speaking then, a Bloch–Beilinson filtration (**B-B** filtration), is a filtration in (1.1) satisfying a number of desirable

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features. Our aim then is to show that if a certain generalization of conjecture (1.0) holds (see (2.1) below), then a B-B filtration exists. To be more precise, our main result is

THEOREM 1.2. *Assume given a smooth projective variety $X/_{\mathbf{C}}$. Then for all k, there is a filtration*

$$CH^k(X/_{\mathbf{C}})_{\mathbf{Q}} = F^0 \supset F^1 \supset \cdots \supset F^\ell \supset F^{\ell+1} \supset \cdots \supset F^k \supset F^{k+1} = F^{k+2} = \cdots,$$

which satisfies the following

(i) $F^1 = CH^k_{\text{hom}}(X/_{\mathbf{C}})_{\mathbf{Q}}$
(ii) $F^2 \subset \ker \Phi_{k,\mathbf{Q}} : CH^k_{\text{hom}}(X/_{\mathbf{C}})_{\mathbf{Q}} \to J^k(X(\mathbf{C}))_{\mathbf{Q}}$.
(iii) $F^\ell \bullet F^r \subset F^{\ell+r}$, *where $\bullet$ is the intersection product.*
(iv) F^ℓ *is preserved under push-forwards f_* and pull-backs f^*, where $f : X/_{\mathbf{C}} \to Y/_{\mathbf{C}}$ is a morphism of smooth projective varieties. [In short, F^ℓ is preserved under the action of correspondences between smooth projective varieties.]*
(v) $Gr^\ell_F := F^\ell/F^{\ell+1}$ *factors through the Grothendieck motive. More specifically, let us assume that the Künneth components of the diagonal class $[\Delta] = \oplus_{p+q=2n}[\Delta(p,q)] \in H^{2n}(X \times X, \mathbf{Q}(k))$ are algebraic. Then*

$$\Delta(2n-2k+r, 2k-r)_*|_{Gr^\ell_F CH^k(X/_{\mathbf{C}})_{\mathbf{Q}}} = \begin{cases} \text{Identity} & \text{if } r = \ell \\ 0 & \text{otherwise} \end{cases}.$$

(vi) *Let $D^k(X) := \cap_\ell F^\ell$. If conjecture (2.1) below holds, then $D^k(X) = 0$.*

Although we apply our results to $X/_{\mathbf{C}}$, one could easily modify our arguments so as to apply to X_L, where L is a subfield of $\mathbf{C}$. (On the other hand, by a standard norm argument, there is an injection $CH^k(X_L)_{\mathbf{Q}} \hookrightarrow CH^k(X/_{\mathbf{C}})_{\mathbf{Q}}$. Therefore a filtration on $CH^k(X/_{\mathbf{C}})_{\mathbf{Q}}$ induces a filtration on $CH^k(X_L)_{\mathbf{Q}}$.) In a few words, the basic idea of the proof is the following. Given $X/_{\mathbf{C}}$, we consider a $\overline{\mathbf{Q}}$-spread, namely we can view X as defined over K, viz. $X/_K$ where $K/\overline{\mathbf{Q}}$ is an extension of finite transcendence degree over $\overline{\mathbf{Q}}$. There exists projective schemes $X_S \xrightarrow{\rho} S$ over $\overline{\mathbf{Q}}$ for which the generic fiber X_η over the generic point $\eta \in S$ satisfies $X/_{\mathbf{C}} = X_{S,\eta} \times \mathbf{C}$ (via a suitable embedding $K \hookrightarrow \mathbf{C}$). Similarly, if $\xi \in CH^k(X/_{\mathbf{C}})_{\mathbf{Q}}$ is given, then by enlarging K (if necessary), ξ has a lifting $\tilde{\xi} \in CH^k(X_S)_{\mathbf{Q}}$. By restriction, we can consider $\tilde{\xi} \in CH^k(\rho^{-1}(U))_{\mathbf{Q}}$ for some Zariski open $U/_{\overline{\mathbf{Q}}} \subset S$, where $\rho : \rho^{-1}(U) \to U$ is smooth and proper. One then shows that under the assumption of conjecture (2.1), $CH^k(\rho^{-1}(U))_{\mathbf{Q}}$ embeds into absolute Hodge cohomology (Section 2). By constructing a Leray filtration on the 'lowest weight' part of absolute Hodge

https://doi.org/10.1023/A:1011882030468 Published online by Cambridge University Press

cohomology, one arrives at a corresponding filtration on $CH^k(\rho^{-1}(U))_{\mathbf{Q}}$. Taking direct limits, we arrive at

$$F^{\ell} CH^k(X_K)_{\mathbf{Q}} := \lim_{\substack{\rightarrow \\ U \subset S}} F^{\ell} CH^k(\rho^{-1}(U))_{\mathbf{Q}},$$

where $K = \overline{\mathbf{Q}}(S)$.

$$F^{\ell} CH^k(X/_{\mathbf{C}})_{\mathbf{Q}} := \lim_{\substack{\rightarrow \\ K \subset \mathbf{C}}} F^{\ell} CH^k(X_K)_{\mathbf{Q}}.$$

2. Relation of a Generalization of Conjecture 1.0 to a Conjecture of Jannsen

Let $\Gamma(-) := \hom_{\mathrm{MH}}(\mathbf{Q}(0), -)$, where hom is taken in the category of mixed Hodge structures, and where $\mathbf{1} := \mathbf{Q}(0)$ is the trivial Hodge structure of weight zero. Throughout this paper we will use either the notation $\Gamma(-)$ or $\hom_{\mathrm{MH}}(\mathbf{1}, -)$, and $\mathbf{1}$ or $\mathbf{Q}(0)$ depending on what appears notationally convenient at the given time, where for any subring $A \subset \mathbf{R}$, $A(k) = (2\pi\sqrt{-1})^k A \subset \mathbf{C}$. Note that $A(k)$ defines the trivial A-Hodge structure of weight $-2k$. Recall that $\overline{\mathbf{Q}} \subset \mathbf{C}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$.

CONJECTURE 2.0 (See [J2; 5.20]). *For a smooth complex quasi-projective variety V that can be defined over number field, the regulator map*

$$r : CH^j(V, 1)_{\mathbf{Q}} \to \Gamma(H^{2j-1}(V, \mathbf{Q}(j)))$$

is surjective, where $CH^j(-, 1)$ are the higher Chow groups introduced by Bloch [Bl2].

We now consider a generalization of (1.0) to smooth quasi-projective varieties.

CONJECTURE 2.1. *For any smooth quasi-projective variety $V/_{\overline{\mathbf{Q}}}$, the Abel–Jacobi map*

$$CH^k_{\mathrm{hom}}(V/_{\overline{\mathbf{Q}}})_{\mathbf{Q}} \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2k-1}(V, \mathbf{Q}(k))),$$

(as defined in [J2]) is injective.

Remarks 2.1.1

(i) If W is smooth and proper over $\mathbf{C}$, then $\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2k-1}(W, \mathbf{Q}(k))) \simeq J^k(W)_{\mathbf{Q}}$, and the corresponding map $CH^k_{\mathrm{hom}}(W)_{\mathbf{Q}} \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2k-1}(W, \mathbf{Q}(k)))$ is the classical Abel–Jacobi map. Thus indeed (2.1) generalizes (1.0).

(ii) The description of $\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2k-1}(V, \mathbf{Q}(k)))$ for quasi-projective V, in terms of a generalized jacobian, is given in (2.5.2) below.

(iii) In the case $k = 1$, conjecture (2.1) is true, even without the added assumption of V defined over a number field. We prove this in (2.5) below.

The purpose of this section is to prove the following.

THEOREM 2.2. *Conjecture (2.0) implies conjecture (2.1).*

Proof. The idea of proof essentially comes from [J2]. All varieties are thus defined over $\overline{\mathbf{Q}}$. (Thus for example, by Chow's lemma, conjecture (1.0) can be restricted to smooth projective varieties over $\overline{\mathbf{Q}}$). Now let $i : Z \hookrightarrow V$ be a [pure] codimension j algebraic subset of V. Then by purity, $H_Z^{2j-1}(V, \mathbf{Q}(j)) = 0$, hence the short exact sequence (where $U = V \backslash Z$)

$$0 \to H^{2j-1}(V, \mathbf{Q}(j)) \to H^{2j-1}(U, \mathbf{Q}(j)) \to H_Z^{2j}(V, \mathbf{Q}(j))^\circ \to 0,$$

where

$$H_Z^{2j}(V, \mathbf{Q}(j))^\circ := \ker : H_Z^{2j}(V, \mathbf{Q}(j)) \to H^{2j}(V, \mathbf{Q}(j)).$$

There is an exact sequence

$$\cdots \to \Gamma(H^{2j-1}(U, \mathbf{Q}(j))) \xrightarrow{\alpha} \Gamma(H_Z^{2j}(V, \mathbf{Q}(j))^\circ) \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2j-1}(V, \mathbf{Q}(j))) \to \cdots.$$

Note that if V is complete, then $H^{2j-1}(V, \mathbf{Q}(j))$ has pure weight $= -1$, thus

$$\Gamma(H^{2j-1}(V, \mathbf{Q}(j))) = 0,$$

and hence α is injective, although this is in general not the case. Now set $CH^j(V)^\circ_{\mathbf{Q}} = \ker : CH^j(V)_{\mathbf{Q}} \to H^{2j}(V, \mathbf{Q}(j))$. It seems natural to introduce the notation $CH^j_Z(V) := CH_{\dim V - j}(Z)$. Note that the cycle class map $CH^j_Z(V)_{\mathbf{Q}} \xrightarrow{\sim} H^{2j}_Z(V, \mathbf{Q}(j))$ is an isomorphism. This is because $CH^j_Z(V)_{\mathbf{Q}} = CH_{\dim Z}(Z)_{\mathbf{Q}}$ is freely generated by the irreducible components of Z, and that likewise $H^{2j}_Z(V, \mathbf{Q}(j))$ is freely generated by the fundamental classes of these components. Now set $CH^j_Z(V)^\circ_{\mathbf{Q}} = i^* CH^j(V)^\circ_{\mathbf{Q}}$, (recall the inclusion $i : Z \hookrightarrow V$). There is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\cdot \to & CH^j(U,1)_{\mathbf{Q}} & \to & CH^j_Z(V)^\circ_{\mathbf{Q}} & \xrightarrow{\beta} & CH^j(V)^\circ_{\mathbf{Q}} & \to \cdot \\
& \downarrow r & & \downarrow \wr & & \downarrow \Phi_j & \\
\cdot \to & \Gamma(H^{2j-1}(U, \mathbf{Q}(j))) & \xrightarrow{\alpha} & \Gamma(H^{2j}_Z(V, \mathbf{Q}(j))^\circ) & \to & \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2j-1}(V, \mathbf{Q}(j))) & \to \cdot,
\end{array} \tag{2.3}$$

where Φ_j is the Abel–Jacobi map, and β is induced by inclusion. It follows easily that

(a) r onto $\Rightarrow \Phi_j|_{\mathrm{Im}(\beta)}$ is injective.

(b) $\Phi_j|_{\mathrm{Im}(\beta)}$ and α injective $\Rightarrow r$ onto.

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Now suppose conjecture (2.0) holds and that V is quasi-projective over $\overline{\mathbf{Q}}$,with corresponding codimension k subvariety $Z/_{\overline{\mathbf{Q}}} \subset V$. Then viewing U as a complex variety, r is always onto, and hence $\Phi_{j_{|\mathrm{Im}(\beta)}}$ is injective by (a) above. Thus (2.0) $\Rightarrow$ (2.1), and we are done. □

Remark (2.4). If $V/_{\mathbf{C}}$ is a projective algebraic manifold for which Φ_j in (2.3) is not injective, then by (a) and (b) above, one arrives at a corresponding $U/_{\mathbf{C}}$ for which r in (2.3) is not onto. This was first observed by Jannsen, in providing a counterexample to a certain conjecture of Beilinson [J1; 3.12.c].

We now fulfill a promise made earlier.

PROPOSITION 2.5. *Let $U/_{\mathbf{C}}$ be a smooth quasi-projective variety. Then the Abel–Jacobi map*

$$CH^1_{\mathrm{hom}}(U)_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^1(U, \mathbf{Q}(1)))$$

is an isomorphism.

Proof. Let V be smooth, projective, and $U = V \backslash Z$, for some Zariski closed proper subset $Z \subset V$. Then from the commutative diagram of exact sequences

$$\begin{array}{ccccccc} CH^1_Z(V)_{\mathbf{Q}} & \to & CH^1(V)_{\mathbf{Q}} & \to & CH^1(U)_{\mathbf{Q}} & \to 0 \\ \downarrow \wr & & \mathrm{cl}_V \downarrow & & \downarrow \mathrm{cl}_U & \\ H^2_Z(V, \mathbf{Q}(1)) & \to & H^2(V, \mathbf{Q}(1)) & \to & W_0H^2(U, \mathbf{Q}(1)) & \to 0 \end{array}$$

we deduce that the restriction map

$$CH^1(V)^0_{\mathbf{Q}} \to CH^1(U)^0_{\mathbf{Q}}$$

is surjective, where $CH^1(V)^0_{\mathbf{Q}} := \ker \mathrm{cl}_V$ and $CH^1(U)^0_{\mathbf{Q}} := \ker \mathrm{cl}_U$. Now referring to the notation of (2.3), we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} CH^1_Z(V)^0_{\mathbf{Q}} & \to & CH^1(V)^0_{\mathbf{Q}} & \to & CH^1(U)^0_{\mathbf{Q}} & \to 0 \\ \downarrow \wr & & \Phi_1 \downarrow \wr & & \downarrow & \\ \Gamma(H^2_Z(V, \mathbf{Q}(1))^0) & \to & \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^1(V, \mathbf{Q}(1))) & \to & \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^1(U, \mathbf{Q}(1))) & \to 0, \end{array}$$

where we use the fact that $\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^2_Z(V, \mathbf{Q}(1))^0) = 0$. This is because $H^2_Z(V, \mathbf{Q}(1))^0$ is a pure Hodge structure of weight 0, and for a MHS H [J2; Lemma 9.2],

$$\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H) \simeq \frac{W_0H \otimes_{\mathbf{Z}} \mathbf{C}}{W_0H \; + \; F^0W_0H \otimes_{\mathbf{Z}} \mathbf{C}}. \tag{2.5.1}$$

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Thus it follows from the five lemma that the Abel–Jacobi map

$$CH^1(U)^0_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^1(U, \mathbf{Q}(1)))$$

is an isomorphism. Finally, by definition, $CH^1(U)^0_{\mathbf{Q}} = CH^1_{\mathrm{hom}}(U)_{\mathbf{Q}}$, and so we are done. □

As for final comments for this section, we deduce from (2.5.1) that

$$\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2j-1}(V, \mathbf{Q}(j))) \simeq \frac{W_0 H^{2j-1}(V, \mathbf{C})}{F^0 W_0 H^{2j-1}(V, \mathbf{C}) + W_0 H^{2j-1}(V, \mathbf{Q}(j))}. \tag{2.5.2}$$

Furthermore, there is an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2j-1}(V, \mathbf{Q}(j))) \to H^{2j}_{\mathcal{H}}(V, \mathbf{Q}(j)) \to \Gamma(H^{2j}(V, \mathbf{Q}(j))) \to 0, \tag{2.6}$$

where $H^*_{\mathcal{H}}(V, \mathbf{Q}(*))$ is absolute Hodge cohomology (see Section 3). Thus by conjecture (2.1), there is an injection

$$CH^j(V)_{\mathbf{Q}} \hookrightarrow H^{2j}_{\mathcal{H}}(V, \mathbf{Q}(j)),$$

for smooth quasi-projective V over $\overline{\mathbf{Q}}$. Now for $V = \rho^{-1}(U)$ (see the paragraph preceeding Section 2), one constructs a Leray filtration on the lowest weight part of $H^{2k}_{\mathcal{H}}(V, \mathbf{Q}(k))$. We work this out in the next section.

3. Absolute Hodge Cohomology and the Leray Filtration

For our narrow interests, we need to make use of a variant of Deligne cohomology that takes into account the weight filtration. This leads us to absolute Hodge cohomology. Since we are only interested in some formal properties of absolute Hodge cohomology, specifically the exact sequence in (2.6) above, we present here only a brief definition and refer the reader to the literature for more details. The definition of a mixed Hodge complex can be found for example in [Be], [J1], and [B-Z]. We adopt the notation in [J1]. Let $A \subset \mathbf{R}$ be a subring such that $A \otimes \mathbf{Q}$ is a field.

DEFINITION 3.0 (See [B-Z; def. 1.8]), or [J1; def. 2.1]). A mixed A-Hodge complex consists of the following.

(a) A complex $K^\bullet_A$ of A-modules (in the derived category), that is bounded below, such that $H^p(K_A)$ is an A-module of finite type for all p. (The reader who is not familiar with derived categories may think of an object in the derived categories of complexes as a complex defined up to *quasi-isomorphism*).

(b) A filtered complex $(K^\bullet_{A\otimes\mathbf{Q}}, W)$ of $A \otimes \mathbf{Q}$-vector spaces that is bounded below, and an isomorphism $K^\bullet_{A\otimes\mathbf{Q}} \xrightarrow{\sim} K^\bullet_A \otimes \mathbf{Q}$ in the derived category.

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(c) A bifiltered complex $(K^{\bullet}_{\mathbf{C}}, W, F)$ of $\mathbf{C}$-vector spaces, and a filtered isomorphism $\alpha : (K^{\bullet}_{\mathbf{C}}, W) \xrightarrow{\sim} (K^{\bullet}_{A\otimes\mathbf{Q}}, W) \otimes \mathbf{C}$ in the filtered derived category.
Further,

(d) For every $m \in \mathbf{Z}$,

$$Gr^m_W K^{\bullet}_{A\otimes\mathbf{Q}} \to (Gr^m_W K^{\bullet}_{\mathbf{C}}, F)$$

is a (polarizable) $A \otimes \mathbf{Q}$-Hodge complex of weight m, i.e. the differentials of $Gr^m_W K^{\bullet}_{\mathbf{C}}$ are strictly compatible with the induced filtration F, and F induces a pure (polarizable) $A \otimes \mathbf{Q}$-Hodge structure of weight $m+r$ on $H^r(Gr^m_W K^{\bullet}_{A\otimes\mathbf{Q}})$ for $r \in \mathbf{Z}$.

By definition of morphisms in the derived category, a mixed A-Hodge complex gives rise to a diagram

$$\begin{array}{ccccccc} & & {}'K^{\bullet}_{A\otimes\mathbf{Q}} & & & ({}'K^{\bullet}_{\mathbf{C}}, W) & \\ & {}^{\alpha_1}\nearrow & & \nwarrow^{\alpha_2} \quad {}^{\beta_1}\nearrow & & & \nwarrow^{\beta_2} \\ K^{\bullet}_A & & & (K^{\bullet}_{A\otimes\mathbf{Q}}, W) & & & (K^{\bullet}_{\mathbf{C}}, W, F), \end{array} \tag{3.1}$$

where $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ are actual morphisms of complexes, α_2 is a quasi-isomorphism, β_1 is a filtered morphism, and β_2 is a filtered quasi-isomorphism.

According to the work of Deligne (and Beilinson, see [J1; Theorems 2.2 and 2.3]), the construction of mixed A-Hodge complexes is equivalent to the construction of mixed A-Hodge structures. For the next definition, we need the following. If $\mu : M^{\bullet} \to N^{\bullet}$ is a morphism of complexes, then the cone of μ is the complex

$$\mathrm{Cone}\left(M^{\bullet} \xrightarrow{\mu} N^{\bullet}\right) = C^{\bullet}_{\mu} := M^{\bullet}[1] \oplus N^{\bullet},$$

with differential

$$M^{q+1} \oplus N^q \xrightarrow{\delta} M^{q+2} \oplus N^{q+1},$$
$$(a, b) \overset{\delta}{\mapsto} (-d(a), \mu(a) + d(b)).$$

The absolute Hodge cohomology is given by

$$H^{\bullet}\left(\mathrm{Cone}\left\{K^{\bullet}_A \bigoplus \hat{W}_0 K^{\bullet}_{A\otimes\mathbf{Q}} \bigoplus \hat{W}_0 \bigcap F^0 K^{\bullet}_{\mathbf{C}} \xrightarrow{(\alpha,\beta)} {}'K^{\bullet}_{A\otimes\mathbf{Q}} \bigoplus \hat{W}_0 {}'K^{\bullet}_{\mathbf{C}}\right\}[-1]\right),$$

where $\hat{W}_{\bullet} = (\mathrm{Dec}\ W)_{\bullet}$ is the filtration decalée [D2; II], and

$$(\alpha, \beta)(\xi_A, \xi_{\mathbf{Q}}, \xi_{\mathbf{C}}) = (\alpha_1\xi_A - \alpha_2\xi_{\mathbf{Q}}, \beta_1\xi_{\mathbf{Q}} - \beta_2\xi_{\mathbf{C}}).$$

A *cohomological* mixed A-Hodge complex on a space $\mathcal{W}$ is essentially a sheafified version of the definition of a mixed A-Hodge complex. The precise definition for example can be found in [B-Z; Definition 1.8]). A cohomological mixed A-Hodge

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complex naturally gives rise to a mixed Hodge complex by applying the functor $R\Gamma(\mathcal{W}, -)$ to the cohomological mixed Hodge complex (i.e. the result of applying $\Gamma(\mathcal{W}, -)$ to a corresponding acyclic resolution of a given complex of sheaves on $\mathcal{W}$). From now on we are now going to work in the setting where X is a projective algebraic manifold of dimension n, Y is a normal crossing divisor (NCD), and $j : X\backslash Y \hookrightarrow X$ is the inclusion. The cohomological mixed Hodge complex of interest is

$$(Rj_*\mathbf{Q}, (Rj_*\mathbf{Q}, W), (\Omega^\bullet_X\langle Y\rangle, W, F)); \tag{3.2}$$

the corresponding mixed Hodge complex will now be denoted by

$$(K^\bullet_A,\ (K^\bullet_{A\otimes\mathbf{Q}}, W),\ (K^\bullet_{\mathbf{C}}, W, F)),\quad \text{where } A = \mathbf{Q}.$$

Set

$$\mathcal{M}^\bullet := \mathrm{Cone}\Big\{K^\bullet_A \oplus \hat{W}_0 K^\bullet_{A\otimes\mathbf{Q}} \oplus \hat{W}_0 \bigcap F^0 K^\bullet_{\mathbf{C}} \xrightarrow{(\alpha,\beta)} {}'K^\bullet_{A\otimes\mathbf{Q}} \oplus \hat{W}_0{}' K^\bullet_{\mathbf{C}}\Big\}[-1],$$

with the given prescription in (3.2).

The corresponding absolute Hodge cohomology associated to (3.2) is then given by

$$H^\bullet_{\mathcal{H}}(X\backslash Y, \mathbf{Q}(k)) := H^\bullet(\mathcal{M}^\bullet). \tag{3.3}$$

There is a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2k-1}(X\backslash Y, \mathbf{Q}(k))) &\to H^{2k}_{\mathcal{H}}(X\backslash Y, \mathbf{Q}(k)) \\ \to \hom_{\mathrm{MH}}(\mathbf{1}, H^{2k}(X\backslash Y, \mathbf{Q}(k))) &\to 0. \end{aligned}$$

Now set

$$\underline{H}^{2k}_{\mathcal{H}}(X\backslash Y, \mathbf{Q}(k)) := \Phi(H^{2k}_{\mathcal{D}}(X, \mathbf{Q}(k))),$$

where we note that $H^{2k}_{\mathcal{D}}(X, \mathbf{Q}(k)) = H^{2k}_{\mathcal{H}}(X, \mathbf{Q}(k))$, and where Φ is given by restriction.

We are now going to view $X = X_S$ as fibered over a smooth projective variety S, and more particularly, we will refer to the setting below.

$$\begin{array}{ccccc} X\backslash Y & \overset{j}{\hookrightarrow} & X & \hookleftarrow & Y \\ \downarrow & & {}_\rho\downarrow & & \downarrow \\ S\backslash\Sigma & \hookrightarrow & S & \hookleftarrow & \Sigma \end{array}$$

where S is a projective algebraic manifold of dimension s, ρ is smooth over $S\backslash\Sigma$, and Σ, $Y := \rho^{-1}(\Sigma)$ are NCD's. This 'good situation' can always be arranged by the work of Hironaka. The main result of this section is:

PROPOSITION 3.4. *There is a 'Leray' filtration on* $\underline{H}^{2k}_{\mathcal{H}}(X\backslash Y, \mathbf{Q}(k))$ *with corresponding* ℓ^{th}*-graded piece* $E^{\ell,2k-\ell}_{\infty}$ *which satisfies the following. There is a short exact sequence*

$$0 \to \underline{E}^{\ell,2k-\ell}_{\infty} \to E^{\ell,2k-\ell}_{\infty} \to \underline{\underline{E}}^{\ell,2k-\ell}_{\infty} \to 0,$$

where

$$\underline{E}^{\ell,2k-\ell}_{\infty} \simeq \frac{\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1}H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_*\mathbf{Q}(k)))}{\hom_{\mathrm{MH}}(\mathbf{1}, Gr^0_W H^{2k-1}(X\backslash Y, \mathbf{Q}(k)))},$$

and

$$\underline{\underline{E}}^{\ell,2k-\ell}_{\infty} \simeq \hom_{\mathrm{MH}}(\mathbf{1}, H^{\ell}(S\backslash\Sigma, R^{2k-\ell}\rho_*\mathbf{Q}(k))).$$

Remarks 3.5. (i) We clarify the 'denominator' of $\underline{E}^{\ell,2k-\ell}_{\infty}$. First of all, there is a short exact sequence of MHS's

$$0 \to W_{-1}H^{2k-1}(X\backslash Y, \mathbf{Q}(k)) \to W_0 H^{2k-1}(X\backslash Y, \mathbf{Q}(k)) \to Gr^0_W H^{2k-1}(X\backslash Y, \mathbf{Q}(k)) \to 0.$$

Taking the corresponding $\mathrm{Ext}^{\bullet}$ long exact sequence, we arrive at the edge map

$$\hom_{\mathrm{MH}}(\mathbf{1}, Gr^0_W H^{2k-1}(X\backslash Y, \mathbf{Q}(k))) \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1}H^{2k-1}(X\backslash Y, \mathbf{Q}(k))).$$

Consider the Leray spectral sequence associated to the map $\rho : X\backslash Y \to S\backslash\Sigma$, and the associated Leray filtration $H^{2k-1}(X\backslash Y, \mathbf{Q}(k)) = \varphi_0 \supset \varphi_1 \supset \cdots$. By degeneration of this spectral sequence at E_2 [D1] and semi-simplicity considerations, we have $\varphi_{\ell-1}/\varphi_{\ell} \simeq H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_*\mathbf{Q}(k))$ and $W_{-1}\varphi_{\ell}$ is a direct summand of $W_{-1}H^{2k-1}(X\backslash Y, \mathbf{Q}(k))$ (as Hodge structures); moreover

$$W_{-1}H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_*\mathbf{Q}(k)) \simeq W_{-1}\varphi_{\ell-1}/W_{-1}\varphi_{\ell},$$

since the weight functor $W_{\bullet}(\;)$ is exact (see [J2] (6.3)). The denominator of $\underline{E}^{\ell,2k-\ell}_{\infty}$ means the corresponding 'image'

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1}H^{2k-1}(X\backslash Y, \mathbf{Q}(k))) & \hookleftarrow & \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1}\varphi_{\ell-1}) \\
\uparrow & & \\
\hom_{\mathrm{MH}}(\mathbf{1}, Gr^0_W H^{2k-1}(X\backslash Y, \mathbf{Q}(k))) & & \downarrow \\
& {\scriptstyle(\text{Image})} \searrow & \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1}H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_*\mathbf{Q}(k)))
\end{array}$$

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(ii) Let $t \in S\backslash\Sigma$. Then the composite

$$H^0(S\backslash\Sigma, R^\bullet\rho_*\mathbf{Q}) \underset{\simeq}{\overset{j_t^*}{\hookrightarrow}} H^\bullet(X_t, \mathbf{Q})^{\pi_1(S\backslash\Sigma)} \subset H^\bullet(X_t, \mathbf{Q})$$

is a morphism of mixed Hodge structures (see [Z; 8.4]). Hence $H^0(S\backslash\Sigma, R^\bullet\rho_*\mathbf{Q})$ is a pure Hodge structure. Therefore one deduces that

$$\underline{E}_\infty^{1,2k-1} \simeq \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^0(S\backslash\Sigma, R^{2k-1}\rho_*\mathbf{Q}(k))),$$

this being a consequence of the commutative diagram below.

$$\begin{array}{ccc}
\hom_{\mathrm{MH}}(\mathbf{1}, \mathrm{Gr}^0_W H^{2k-1}(X\backslash Y, \mathbf{Q}(k))) & \to & \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1}H^{2k-1}(X\backslash Y, \mathbf{Q}(k))) \\
\downarrow & & \downarrow \\
\hom_{\mathrm{MH}}(\mathbf{1}, \mathrm{Gr}^0_W H^0(S\backslash\Sigma, R^{2k-1}\rho_*\mathbf{Q}(k))) & \to & \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1}H^0(S\backslash\Sigma, R^{2k-1}\rho_*\mathbf{Q}(k))) \\
\| & & \\
0 & &
\end{array}$$

(iii) Similarly, if Σ, $Y = \emptyset$, then

$$\underline{E}_\infty^{\ell,2k-\ell} \simeq \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{\ell-1}(S, R^{2k-\ell}\rho_*\mathbf{Q}(k))),$$

the essential point here is that we are dealing with pure Hodge structures.

As part motivation for Proposition (3.4), we note that the exact sequence in (3.4) says something interesting even in the case $\ell = 1$. By a result of Deligne [D2] and Leray degeneration, the image of the composite

$$H^\bullet(X, \mathbf{Q}) \to H^\bullet(X\backslash Y, \mathbf{Q}) \to H^0(S\backslash\Sigma, R^\bullet\rho_*\mathbf{Q}),$$

is surjective. Thus by Proposition (3.4), taking $\ell = 1$, we arrive at the exact sequence

$$J^k(X)_{\mathbf{Q}} \to E_\infty^{1,2k-1} \to \left\{H^1(S\backslash\Sigma, R^{2k-1}\rho_*\mathbf{Q})\right\}^{(k,k)} \to 0,$$

where $\left\{H^1(S\backslash\Sigma, R^{2k-1}\rho_*\mathbf{Q})\right\}^{(k,k)} := \hom_{\mathrm{MH}}(\mathbf{Q}(-k), H^1(S\backslash\Sigma, R^{2k-1}\rho_*\mathbf{Q}))$, (shifting by twists), are the classes of Hodge type (k,k). Now bearing in mind the work on normal functions in [Z], one should be able to interpret $E_\infty^{1,2k-1}$ as corresponding to the normal functions associated to the fibering $\rho : X\backslash Y \to S\backslash\Sigma$, and that the map $E_\infty^{1,2k-1} \to \left\{H^1(S\backslash\Sigma, R^{2k-1}\rho_*\mathbf{Q})\right\}^{(k,k)}$ gives the cohomology class ('topological invariant') of a given normal function.

4. Proof of (3.4)

We make use of the fact that the Leray filtration and hence its corresponding graded pieces on $H^\bullet(X\backslash Y)$ are themselves mixed Hodge structures–the mixed Hodge

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structures being induced from $H^\bullet(X\backslash Y)$. (See [B-Z; Proposition 3.14] and [Z; (15.5)].) Recall the setting we consider is given by this diagram

$$\begin{array}{ccccc} X\backslash Y & \overset{j}{\hookrightarrow} & X & \hookleftarrow & Y \\ \downarrow & & {}_\rho\downarrow & & \downarrow \\ S\backslash\Sigma & \hookrightarrow & S & \hookleftarrow & \Sigma \end{array}$$

where S is a projective algebraic manifold of dimension s, ρ is smooth over $S\backslash\Sigma$, and both Σ and $Y := \rho^{-1}(\Sigma)$ are NCD's. Whenever the context is clear, we will also write $\rho : X\backslash Y \to S\backslash\Sigma$ for the obvious restriction of ρ to $X\backslash Y$. The fiber dimension of ρ will be denoted by m. Thus $n = s + m$, where $n = \dim X$. For a smooth quasi-projective $V/_{\mathbf{C}}$, we let $\Omega_V^\bullet$ (resp. $\Omega_{V^\infty}^\bullet$) be the sheaf of germs of holomorphic (resp. C^∞) forms on V, and for any sheaf $\mathcal{S}$ on V, $\mathcal{S}(V) := \Gamma(V, \mathcal{S})$, the global sections.

The proof of (3.4) will now proceed in three steps.

Step I. Construction of a filtration on cohomology.

We first introduce what we regard as an 'ideal' choice of filtration. We will later see that for technical reasons, our first choice of filtration will have to be modified slightly. Put

$$\begin{aligned}&\mathcal{L}^\ell\Omega^\bullet_{\{X\backslash Y\}^\infty}(X\backslash Y)\\ &\quad := \text{Image}\Big((\rho^*\Omega^{2s-\ell+1}_{\{S\backslash\Sigma\}^\infty} \otimes \Omega^{\bullet-2s+\ell-1}_{\{X\backslash Y\}^\infty})(X\backslash Y) \;\to\; \Omega^\bullet_{\{X\backslash Y\}^\infty}(X\backslash Y)\Big).\end{aligned}$$

The $\mathcal{C}^\infty$-forms compactly supported on $X\backslash Y$ are denoted by $\Omega^\bullet_{\{X\backslash Y\}^\infty_c}(X\backslash Y)$. Thus

$$\mathcal{L}^\ell\Omega^\bullet_{\{X\backslash Y\}^\infty_c}(X\backslash Y) := \mathcal{L}^\ell\Omega^\bullet_{\{X\backslash Y\}^\infty}(X\backslash Y) \;\cap\; \Omega^\bullet_{\{X\backslash Y\}^\infty_c}(X\backslash Y).$$

A 'Leray' filtration on $'\Omega^\bullet_{X^\infty}(X)$. Let $'\Omega^\bullet_{X^\infty}$ be the sheaf of distributions [J1] on $\Omega^{-\bullet}_{X^\infty}$. For a form $w \in \Omega^{-\bullet}_{X^\infty}$ and $D \in {}'\Omega^\bullet_{X^\infty}$, the differential $d : {}'\Omega^\bullet_{X^\infty} \to {}'\Omega^{\bullet+1}_{X^\infty}$ is given by the formula $dD(w) = (-1)^\bullet D(dw)$. The decomposition $\Omega^\bullet_{X^\infty} = \oplus_{p+q=\bullet}\Omega^{p,q}_{X^\infty}$ naturally induces a corresponding decomposition $'\Omega^\bullet_{X^\infty} = \oplus_{p+q=\bullet}{}'\Omega^{p,q}_{X^\infty}$, and Hodge filtration

$$F^{i\prime}\Omega^\bullet_{X^\infty} = \bigoplus_{\substack{p+q=\bullet\\ p\geqslant i}} {}'\Omega^{p,q}_{X^\infty}.$$

The natural embeddings of complexes

$$(\Omega^\bullet_X, F^i) \hookrightarrow (\Omega^\bullet_{X^\infty}, F^i) \hookrightarrow ({}'\Omega^\bullet_{X^\infty}[-2n], F^{i-n})$$

are filtered quasi-isomorphisms [J1; Lemma 1.2], where the latter embedding is given by the following. If w is a C^∞ (p, q)-form, then w determines a section of $'\Omega^{p-n,q-n}_{X^\infty}$ by

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the formula

$$\eta \mapsto \frac{1}{(2\pi\sqrt{-1})^n}\int_X \eta\wedge w.$$

We introduce the filtration

$$\mathcal{L}^{\ell\prime}\Omega^{\bullet}_{X^\infty}(X) := \{\xi \in {}'\Omega^{\bullet}_{X^\infty}(X) \mid \xi\cdot\mathcal{L}^{\ell}\Omega^{-\bullet}_{\{X\setminus Y\}^\infty_c}(X\setminus Y) = 0\}.$$

One has a corresponding spectral sequence with $E_1^{\ell,q} = H^{\ell+q}(Gr^{\ell}_{\mathcal{L}}{}'\Omega^{\bullet}_{X^\infty}(X))$, and corresponding filtration $\mathcal{L}^{\ell}H^{\bullet}(X)$. Now recall the perfect pairing (a duality of MHS's [F]):

$$H^{\bullet}(X\setminus Y,\mathbf{Q})\times H_c^{2n-\bullet}(X\setminus Y,\mathbf{Q})\to\mathbf{Q}, \tag{4.0}$$

Then in the composite pairing

$$H^{\bullet}(X,\mathbf{Q})\times H_c^{2n-\bullet}(X\setminus Y,\mathbf{Q})\to H^{\bullet}(X\setminus Y,\mathbf{Q})\times H_c^{2n-\bullet}(X\setminus Y,\mathbf{Q})\to\mathbf{Q},$$

$\mathcal{L}^{2s+1}H^{\bullet}(X)$ is the left kernel, i.e. the subspace of $H^{\bullet}(X,\mathbf{Q})$ that is annihilated by $H_c^{2n-\bullet}(X\setminus Y,\mathbf{Q})$. In other words

$$\mathcal{L}^{2s+1}H^{\bullet}(X) = \mathrm{Im}(H^{\bullet}_Y(X)\to H^{\bullet}(X)). \tag{4.0a}$$

The optimistic statement we would like to make is that

$$\mathrm{Im}(\mathcal{L}^{\ell}H^{\bullet}(X,\mathbf{Q}(k))\to H^{\bullet}(X\setminus Y,\mathbf{Q}(k))),$$

can be identified with

$$W_{\bullet-2k}\underline{\mathcal{L}}^{\ell}H^{\bullet}(X\setminus Y,\mathbf{Q}(k)),$$

where $\underline{\mathcal{L}}^{\ell}H^{\bullet}(X\setminus Y,\mathbf{Q}(k))$ is the Leray filtration of the fibering $\rho : X\setminus Y\to S\setminus\Sigma$ given in [G-H; pp. 462-465], which is made up of forms with local descriptions involving at least ℓ differentials from S on $H^{\bullet}(X\setminus Y,\mathbf{Q}(k))$, with graded piece $H^{\ell}(S\setminus\Sigma, R^{\bullet-\ell}\rho_*\mathbf{Q}(k))$, (and where the corresponding spectral sequence is known to degenerate at E_2). However, since we cannot prove this statement, we instead prove a slightly weaker result, which is sufficient for our purposes. This will involve a slight modication of our filtration, as we indicated earlier.

For a suitable metric on S, let Σ_ε be an ε neighbourhood of Σ in S, and let $Y_\varepsilon := \rho^{-1}(\Sigma_\varepsilon)$ (thus for example $X\setminus Y_\varepsilon$ is compact). We assume $\varepsilon > 0$ is sufficiently small. Note that Y is a deformation retract of $\overline{Y_\varepsilon}$, and hence $H_\bullet(Y)\simeq H_\bullet(\overline{Y_\varepsilon})$, and that if $H_\bullet(\cdots)$ is Borel-Moore homology, then $H_\bullet(X\setminus Y)\simeq H_\bullet(X\setminus\overline{Y_\varepsilon})$. In particular, it follows that the composite $H_\bullet(\overline{Y_\varepsilon})\to H_\bullet(X)\to H_\bullet(X\setminus Y)$ is zero. Let $\mathcal{L}^{\ell\prime}_\varepsilon\Omega^{\bullet}_{X^\infty}(X)$ be the result of replacing $\mathcal{L}^{\ell}\Omega^{-\bullet}_{\{X\setminus Y\},c}(X\setminus Y)$ in the definition of $\mathcal{L}^{\ell\prime}\Omega^{\bullet}_{X^\infty}(X)$ (preceeding (4.0)) by $\{\eta\in\mathcal{L}^{\ell}\Omega^{-\bullet}_{\{X\setminus Y\},c}(X\setminus Y)\mid \eta \text{ supported on } X\setminus Y_\varepsilon\}$. One has a corresponding $\mathcal{L}^{\ell}_\varepsilon H^{\bullet}(X)$. Let $j^* : \mathcal{L}^{\ell}_\varepsilon H^{\bullet}(X)\to H^{\bullet}(X\setminus Y)$ be the restriction map. We prove the following

LEMMA 4.1. $\mathrm{Im}(j^*) = \{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash Y)\} \cap H^\bullet(X)$, *where* $\underline{\mathcal{L}}^\ell H^\bullet(X\backslash Y)$ *is the Leray filtration as defined in [G-H]. (Henceforth, following the proof of this lemma, we will simply write* $\mathcal{L}^\ell H^\bullet(X\backslash Y)$ *instead of* $\underline{\mathcal{L}}^\ell H^\bullet(X\backslash Y)$*.)*

Proof. Let $\{w\} \in \{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash Y)\} \cap H^\bullet(X)$, where w is a differential form. Then on $X\backslash Y$, $(w + d\eta) \wedge \mathcal{L}^\ell \Omega^{2n-\bullet}_{\{X\backslash Y\}^\infty_c}(X\backslash Y) = 0$, for some form η on $X\backslash Y$. Now let φ be a C^∞ function on X satisfying

$$\varphi = \begin{cases} 1 & \text{on } X\backslash Y_\varepsilon \\ 0 & \text{on } Y_{\varepsilon/2} \end{cases}$$

Then $w + d(\varphi \cdot \eta) \in \mathcal{L}^{\ell}_{\varepsilon}{}'\Omega^{\bullet-2n}_{X^\infty}(X)$, hence $\mathrm{Im}(j^*) \supset \{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash Y)\} \cap H^\bullet(X)$. Going the other way, it is clear that $\{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash Y)\} \cap H^\bullet(X) \subset \{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash \overline{Y_\varepsilon}) \cap H^\bullet(X)$. But $H^\bullet(X\backslash \overline{Y_\varepsilon}) \simeq H^\bullet(X\backslash Y)$, and under this isomorphism,

$$\{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash \overline{Y_\varepsilon})\} \cap H^\bullet(X) = \{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash Y)\} \cap H^\bullet(X).$$

Moreover, it is easy to see that $\mathrm{Im}(j^*) \subset \{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash \overline{Y_\varepsilon})\} \cap H^\bullet(X)$. Hence,

$$\mathrm{Im}(j^*) \subset \{\underline{\mathcal{L}}^\ell H^\bullet(X\backslash Y)\} \cap H^\bullet(X),$$

and the lemma follows. □

Step II. Constructing the filtration on Deligne homology.

For the next part, we introduce the complex $\{\mathcal{C}_\bullet(X),\ \partial\}$ of integral C^∞–chains on X, with corresponding cohomological complex $\{{}'\mathcal{C}^\bullet(X) := \mathcal{C}_{-\bullet}(X), (-1)^\bullet\partial\}$. Note that ${}'\mathcal{C}^\bullet(X)$ can also be 'Leray' filtered via the morphism of complexes ${}'\mathcal{C}^\bullet(X) \to {}'\Omega^\bullet_{X^\infty}(X)$ given by integration of forms over chains, and the 'Leray' filtration on the latter term. Correspondingly, we consider the Deligne complex [J1]

$$\overline{\mathcal{M}}^\bullet_{\mathcal{D}} := \mathrm{Cone}\big\{{}'\mathcal{C}^\bullet(X; \mathbf{Q}(k-n)) \bigoplus F^{k-n}{}'\Omega^\bullet_{X^\infty}(X) \to {}'\Omega^\bullet_{X^\infty}(X)\big\}[-1], \tag{4.2}$$

where

$$H^\bullet(\overline{\mathcal{M}}^\bullet_{\mathcal{D}}) =: {}'H^\bullet_{\mathcal{D}}(X, \mathbf{Q}(k-n)) \simeq H^{\bullet+2n}_{\mathcal{D}}(X, \mathbf{Q}(k)),$$

and where the middle term is Deligne homology, and the latter isomorphism is given by Poincaré duality. Note that for example

$$\begin{aligned} &{}'H^{2k-2n}_{\mathcal{D}}(X, \mathbf{Q}(k-n) \\ &= \frac{\ker\left(D : \overline{\mathcal{M}}^{2k-2n}_{\mathcal{D}} := \left\{\begin{matrix} {}'\mathcal{C}^{2k-2n}(X; \mathbf{Q}(k-n)) \\ \bigoplus \quad F^{k-n}{}'\Omega^{2k-2n}_{X^\infty}(X) \\ \bigoplus \quad {}'\Omega^{2k-2n-1}_{X^\infty}(X) \end{matrix}\right\} \overset{D}{\to} \overline{\mathcal{M}}^{2k-2n+1}_{\mathcal{D}}\right) \quad (a,b,c) \overset{D}{\mapsto} (-da, -db, a-b+dc)}{D(\overline{\mathcal{M}}^{2k-2n-1}_{\mathcal{D}})} \end{aligned} \tag{4.3}$$

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Now for small $\varepsilon > 0$, set

$$\mathcal{L}^{\ell}\overline{\mathcal{M}}_{\mathcal{D}}^{2k-2n} := \mathcal{L}_{\varepsilon}^{\ell\prime}\mathcal{C}^{2k-2n}(X; \mathbf{Q}(k-n)) \oplus \mathcal{L}^{\ell}F^{k-n\prime}\Omega_{X^{\infty}}^{2k-2n}(X) \oplus \oplus \mathcal{L}_{\varepsilon}^{\ell-1\prime}\Omega_{X^{\infty}}^{2k-2n-1}(X). \tag{4.3a}$$

We consider the following filtration on ${}'H_{\mathcal{D}}^{\bullet}(X, \mathbf{Q}(k-n))$. Set

$$\mathcal{L}^{\ell\prime}H_{\mathcal{D}}^{2k-2n}(X, \mathbf{Q}(k-n)) = \frac{\{\ker D : \mathcal{L}^{\ell}\overline{\mathcal{M}}_{\mathcal{D}}^{2k-2n} \to \overline{\mathcal{M}}_{\mathcal{D}}^{2k-2n+1}\}}{D(\overline{\mathcal{M}}_{\mathcal{D}}^{2k-2n-1}) \cap \mathcal{L}^{\ell}\overline{\mathcal{M}}_{\mathcal{D}}^{2k-2n}}. \tag{4.3b}$$

Now let $\mathcal{L}^{\ell}K^{\bullet}$ be either $\mathcal{L}^{\ell\prime}\mathcal{C}^{\bullet}(X; \mathbf{Q}(k-n))$ or $\mathcal{L}^{\ell}F^{k-n\prime}\Omega_{X^{\infty}}^{\bullet}(X)$. Roughly, we want to show that if $\xi \in \mathcal{L}^{\ell}K^{2k-2n}$ has the property that ξ is a coboundary current, then $\xi = d\xi_0$ for some $\xi_0 \in \mathcal{L}^{\ell-1}K^{2k-2n-1}$. This can be verified on the one hand by harmonic theory principles, and on the other using triangulations. Before being more specific, we digress to consider the following situation. Let M, N be compact manifolds. By triangulizing both M and N, we consider the simplicial complex on the product $M \times N$ as the product of the simplicial complexes on M and N. Now working over $\mathbf{Q}$, and similar to the construction in [G-H, pp. 56-58]), we consider a basis of p-chains $\{\underline{\tau}_{\alpha}^{p}, \underline{\mu}_{\alpha}^{p}\}$ for M, where the $\{\underline{\tau}_{\alpha}^{p}\}$ are cycles. Similary, one has a basis of q-chains on N, $\{\underline{\underline{\tau}}_{\beta}^{q}, \underline{\underline{\mu}}_{\beta}^{q}\}$, where again the $\{\underline{\underline{\tau}}_{\beta}^{q}\}$ are cycles. Note that $\{\underline{\sigma}_{\alpha}^{p-1} := \partial\underline{\mu}_{\alpha}^{p}\}$ are independent (as no linear combination of the chains $\{\underline{\mu}_{\alpha}^{p}\}$ can be a cycle); and similarly for $\{\underline{\underline{\sigma}}_{\beta}^{q-1} := \partial\underline{\underline{\mu}}_{\beta}^{q}\}$. Now let

$$\xi = \sum_{\alpha,\beta,p,q} r_{\alpha,\beta,p,q}^{(1)}\underline{\tau}_{\alpha}^{p} \times \underline{\underline{\tau}}_{\beta}^{q} + \sum_{\alpha,\beta,p,q} r_{\alpha,\beta,p,q}^{(2)}\underline{\tau}_{\alpha}^{p} \times \underline{\underline{\mu}}_{\beta}^{q} + \sum_{\alpha,\beta,p,q} r_{\alpha,\beta,p,q}^{(3)}\underline{\mu}_{\alpha}^{p} \times \underline{\underline{\tau}}_{\beta}^{q} + + \sum_{\alpha,\beta,p,q} r_{\alpha,\beta,p,q}^{(4)}\underline{\mu}_{\alpha}^{p} \times \underline{\underline{\mu}}_{\beta}^{q},$$

where the r's are rational numbers. We will write $\xi \in L_{\ell}$ if $r_{\alpha,\beta,p,q}^{(i)} = 0$ for $p > \ell$ and $i = 1, 2, 3, 4$. Note that

$$\partial\xi = \sum_{\alpha,\beta,p,q} (-1)^{p} r_{\alpha,\beta,p,q}^{(2)}\underline{\tau}_{\alpha}^{p} \times \underline{\underline{\sigma}}_{\beta}^{q-1} + \sum_{\alpha,\beta,p,q} r_{\alpha,\beta,p,q}^{(3)}\underline{\sigma}_{\alpha}^{p-1} \times \underline{\underline{\tau}}_{\beta}^{q} + + \sum_{\alpha,\beta,p,q} r_{\alpha,\beta,p,q}^{(4)}[\underline{\sigma}_{\alpha}^{p-1} \times \underline{\underline{\mu}}_{\beta}^{q} + (-1)^{p}\underline{\mu}_{\alpha}^{p} \times \underline{\underline{\sigma}}_{\beta}^{q-1}].$$

Now suppose $\partial\xi \in L_{\ell}$. Then one can easily argue that $r_{\alpha,\beta,p,q}^{(i)} = 0$ for $p > \ell + 1$ and $i = 2, 3, 4$. Therefore it easily follows that $\partial\xi = \partial\xi_0$ for some $\xi_0 \in L_{\ell+1}$. It is not difficult to see that this is really a consequence of the degeneration of the Leray–Serre spectral sequence at E_2 associated to the map $\mathrm{Pr}_1 : M \times N \to M$. Next, getting back to the situation at hand, there is no harm in replacing $\mathcal{C}_{\bullet}(X)$ by the corresponding complex of *geometric chains* (see [Bo; p. 6]). That is, obtained from a direct limit of simplicial complexes on X under refinement of triangulations. This presumes a given piecewise-linear (PL-)structure on X, for which we acknowledge that an

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essentially canonical PL-structure with certain desireable properties exists (see [H; Theorem p. 170, Remark p. 178]). One can also consider the sheaf of (locally finite) geometric chains, which can be shown to be soft, albeit not necessarily flasque. It is mainly for this reason why we considered introducing ε in the definition of our filtration in (4.3b), the basic idea being that the restriction $\mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(X\backslash Y)$ need not be surjective, whereas the restriction $\mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(X\backslash Y_\varepsilon)$ is always surjective since $X\backslash Y_\varepsilon$ is closed in X. There is a corresponding $'\mathcal{C}^\bullet(X) := \mathcal{C}_{-\bullet}(X)$, where $\mathcal{C}_\bullet(X)$ now represents geometric chains, and the corresponding Deligne homology is independent of triangulations and PL-structure. Since $X\backslash Y$ is a smooth fibration over $S\backslash\Sigma$, hence a local product, a similar situation as in the product case occurs here. But here we have to be a little more careful. Again, for a suitable metric on X, we consider Y_ε, an ε neighbourhood of Y in X. Now if $\xi \in \mathcal{L}^{\ell\prime}\mathcal{C}^{2k-2n}(X; \mathbf{Q}(k-n))$ is a coboundary, then by a process of triangulation, we can assume that $\xi = d\gamma$, where ξ corresponds to a chain with $\dim_{\mathbf{R}}(|\rho(\xi \cap X\backslash Y_\varepsilon)|) \leqslant 2\dim_{\mathbf{C}} S - \ell$; moreover if we allow for the possibility of $\xi + \alpha = d\gamma$ for some coboundary chain α in X, which is supported on $\overline{Y_\varepsilon}$, then we can assume further that $\dim_{\mathbf{R}}(|\rho(\gamma \cap X\backslash Y_\varepsilon)|) \leqslant 2\dim_{\mathbf{C}} S - \ell + 1$. Again, this follows from the fact that over $S\backslash\Sigma$, X is a fibration, with degenerating Leray spectral sequence at E_2.

We now discuss the similar situation with currents. For the Hodge theoretic details, the reader may find it helpful to refer to [G-H; p. 84] for the statement of the Hodge theorem, as well as [G-H; pp. 111–116] for the standard Hodge–Kähler identities.

For a coboundary $\eta \in \mathcal{L}^\ell F^{k-n\prime}\Omega^{2k-2n}_{X^\infty}(X)$, it suffices for our purposes to prove the following weaker result. We can replace $F^{k-n\prime}\Omega^\bullet_{X^\infty}(X)$ in $\overline{\mathcal{M}^\bullet_{\mathcal{D}}}$ by $F^{k-n\prime}\Omega^\bullet_{X^\infty}(X)_{\partial-\text{closed}}$. Thus η being a coboundary (hence d-closed) implies that $\overline{\partial}\eta = 0$. If (from Hodge theory) $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$ and $G = G_{\overline{\partial}}$ is the Green's operator, then by the Hodge theorem, $\eta = \overline{\partial}\,\overline{\partial}^* G\eta$. Note that from Hodge theory, $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$ and that $[G, \partial] = 0$, hence if we set $v = \overline{\partial}^* G\eta$, then $\partial v = 0$ and $dv = \eta$. It is easy to show that $v \in \mathcal{L}^{\ell-1}$. In fact, it is trivial to check that $\overline{\partial}^*\mathcal{L}^\ell \subset \mathcal{L}^{\ell-1}$, this being a consequence of the explicit description of $\overline{\partial}^*$ that can for example be found in [G-H; p. 82]. Therefore it suffices to show that $G(\mathcal{L}^\ell) \subset \mathcal{L}^\ell$. But over a polydisk $D \subset S\backslash\Sigma$ where $\rho^{-1}(D) \approx D \times X_t$ trivializes (for some fixed $t \in D$), the Laplacian $\Delta = \Delta_d$ $(= 2\Delta_{\overline{\partial}} = 2\Delta_\partial)$ decomposes on 'decomposable forms' into $\Delta = \Delta_D + \Delta_{X_t}$ (see [G-H; p. 104]). Therefore it easily follows that $\Delta(\mathcal{L}^\ell) \subset \mathcal{L}^\ell$, and likewise one can argue that the same is true for G, due to the relationship of G with Δ given in the Hodge theorem. Thus v defines a current in $F^{k-n\prime}\Omega^{2k-2n-1}_{X^\infty}(X)$, as well as an element of $\mathcal{L}^{\ell-1}F^{k-n\prime}\Omega^{2k-2n-1}_{X^\infty}(X)$.

Remark. It is worthwhile mentioning that the property $\overline{\partial}^* G(\mathcal{L}^\ell) \subset \mathcal{L}^{\ell-1}$ should also be a consequence of a kernel description of $\overline{\partial}^* G$. For example, on $\mathbf{C}^n \times \mathbf{C}^n$, consider the related *Bochner–Martinelli kernel* $k(z, w)$ defined as follows. First, for $\zeta = (\zeta_1, \ldots, \zeta_n)$, set

$$\Phi(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n, \Phi_i(\zeta) = (-1)^{i-1}\zeta_i d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_i} \wedge \cdots \wedge d\zeta_n.$$

Define

$$k(z, w) = C_n \frac{\sum \overline{\Phi_i(z-w)} \wedge \Phi(w)}{||z-w||^{2n}},$$

for a suitable normalizing constant C_n. Then the operator

$$K\colon \Omega^{0,q}_{\{\mathbf{C}^n\}^\infty_c}(\mathbf{C}^n) \to \Omega^{0,q-1}_{\{\mathbf{C}^n\}^\infty}(\mathbf{C}^n)$$

given by

$$(K\varphi) = \int_{w \in \mathbf{C}^n} k(z, w) \wedge \varphi(w),$$

can be viewed as a first step towards a kernel description of $\overline{\partial}^* G$ on $\Omega^{0,q}_{\{\mathbf{C}^n\}^\infty_c}(\mathbf{C}^n)$. If we identify $S \leftrightarrow \mathbf{C}^s$, $X \leftrightarrow \mathbf{C}^n$, then in this case it is elementary to check that $K((\rho^*\Omega^{0,\ell}_{S^\infty_c}(S)) \wedge \Omega^{0,\bullet-\ell}_{X^\infty_c}(X)) \subset (\rho^*\Omega^{0,\ell-1}_{S^\infty}(S)) \wedge \Omega^{0,\bullet-\ell+1}_{X^\infty}(X)$. There are also analogues of $K\varphi$ for $\varphi \in \Omega^{p,q}_{\{\mathbf{C}^n\}^\infty_c}(\mathbf{C}^n)$.

Step III. Conclusion of the proof of (3.4).

Having defined a 'Leray' filtration on Deligne homology, it is now a consequence of the description of Deligne homology in (4.3) and the paragraphs following (4.3b), that the following is true. There is an exact sequence below, obtained by applying Poincaré duality and some Hodge theory.

$$\mathcal{L}^{\ell-1}H^{2k-1}(X; Y; \mathbf{C}) \to \mathcal{L}^{\ell}H^{2k}_{\mathcal{D}}(X, \mathbf{Q}(k)) \xrightarrow{\lambda} \mathcal{L}^{\ell}F^0H^{2k}(X, \mathbf{Q}(k)) \to 0, \tag{4.3c}$$

where $\mathcal{L}^{\ell}F^0H^{2k}(X, \mathbf{Q}(k))$ means $\mathcal{L}^{\ell}H^{2k}(X, \mathbf{Q}(k)) \cap F^0H^{2k}(X, \mathbf{Q}(k))$, $H^{2k-1}(X; Y)$ involves currents ξ for which $d\xi$ is supported on $\overline{Y_\varepsilon}$, and the following 'intersection' (given by restriction) makes sense and:

$$\mathcal{L}^{\ell-1}H^{2k-1}(X; Y; \mathbf{C}) \cap H^{2k-1}(X \backslash Y, \mathbf{C}) = \mathcal{L}^{\ell-1}W_{-1}H^{2k-1}(X \backslash Y, \mathbf{C})$$
$$\underline{\text{modulo}}\ W_{-1}H^{2k-1}(X \backslash Y, \mathbf{Q}(k)).$$

Recall the absolute Hodge cohomology $H^{2k}_{\mathcal{H}}(X \backslash Y, \mathbf{Q}(k))$ defined in (3.3) above and the short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2k-1}(X \backslash Y, \mathbf{Q}(k))) \to H^{2k}_{\mathcal{H}}(X \backslash Y, \mathbf{Q}(k))$$
$$\to \hom_{\mathrm{MH}}(\mathbf{1}, H^{2k}(X \backslash Y, \mathbf{Q}(k))) \to 0.$$

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There is a commutative diagram

$$\begin{array}{ccccc} & & CH^k(X)_{\mathbf{Q}} & \longrightarrow & CH^k(X\backslash Y)_{\mathbf{Q}} \\ & & \downarrow & & \downarrow \\ H^{2k}_{\mathcal{D}}(X,\mathbf{Q}(k)) & = & H^{2k}_{\mathcal{H}}(X,\mathbf{Q}(k)) & \xrightarrow{\Phi} & H^{2k}_{\mathcal{H}}(X\backslash Y,\mathbf{Q}(k)) \end{array} \tag{4.4}$$

where Φ is the obvious restriction map. By referring to the discussion following (3.3), we recall that $\underline{H}^{2k}_{\mathcal{H}}(X\backslash Y,\mathbf{Q}(k)) := \Phi(H^{2k}_{\mathcal{D}}(X,\mathbf{Q}(k)))$. Since the restriction $CH^k(X)\to CH^k(X\backslash Y)$ is surjective, it follows that the cycle map $CH^k(X\backslash Y)_{\mathbf{Q}} \to H^{2k}_{\mathcal{H}}(X\backslash Y,\mathbf{Q}(k))$ takes its values in $\underline{H}^{2k}_{\mathcal{H}}(X\backslash Y,\mathbf{Q}(k))$. Now set

$$\mathcal{L}^{\ell}\underline{H}^{2k}_{\mathcal{H}}(X\backslash Y,\mathbf{Q}(k)) := \Phi(\mathcal{L}^{\ell}H^{2k}_{\mathcal{D}}(X,\mathbf{Q}(k))).$$

Note that $W_{-1}H^{2k-1}(X\backslash Y,\mathbf{Q}(k))$ is a pure Hodge structure. It therefore follows that

$$\begin{aligned} &\{\mathcal{L}^{\ell}W_{-1}H^{2k-1}(X\backslash Y,\mathbf{C})\} \\ &\quad \bigcap \{W_{-1}H^{2k-1}(X\backslash Y,\mathbf{Q}(k)) + F^0W_{-1}H^{2k-1}(X\backslash Y,\mathbf{C})\} \\ &\quad = W_{-1}\mathcal{L}^{\ell}H^{2k-1}(X\backslash Y,\mathbf{Q}(k)) + F^0W_{-1}\mathcal{L}^{\ell}H^{2k-1}(X\backslash Y,\mathbf{C}). \end{aligned}$$

[*Proof.* Suppose, for some $\ell_0 < \ell$,

$$a \in \mathcal{L}^{\ell_0}W_{-1}H^{2k-1}(X\backslash Y,\mathbf{Q}(k)), \qquad b \in \mathcal{L}^{\ell_0}F^0W_{-1}H^{2k-1}(X\backslash Y,\mathbf{C})$$

are given such that $a+b \in \mathcal{L}^{\ell}W_{-1}H^{2k-1}(X\backslash Y,\mathbf{C})$. Then by a weight argument

$$[a] = [b] = [0] \ \in Gr^{\ell_0}_{\mathcal{L}}W_{-1}H^{2k-1}(X\backslash Y,\mathbf{C}) = W_{-1}H^{\ell_0}(S\backslash\Sigma, R^{2k-\ell_0-1}\rho_*\mathbf{C}).$$

Thus $a \in \mathcal{L}^{\ell_0+1}W_{-1}H^{2k-1}(X\backslash Y,\mathbf{Q}(k))$, $b \in \mathcal{L}^{\ell_0+1}F^0W_{-1}H^{2k-1}(X\backslash Y,\mathbf{C})$, and so on.]

Let $\tilde{Y}$ be a desingularization of Y. We consider the diagram below, where we note that the center column is not in general exact at the middle, whereas the right column is exact by the work of Deligne [D2; II(8.28)].

$$\begin{array}{ccccccc} & & H^{2k-2}_{\mathcal{D}}(\tilde{Y},\mathbf{Q}(k-1))\} & \to & [H^{2k-2}(\tilde{Y},\mathbf{Q}(k-1))]\cap F^0 & \to 0 \\ & & \downarrow & & \downarrow & \\ \mathcal{L}^{\ell-1}H^{2k-1}(X;Y;\mathbf{C}) & \to & \mathcal{L}^{\ell}H^{2k}_{\mathcal{D}}(X,\mathbf{Q}(k)) & \to & \mathcal{L}^{\ell}F^0H^{2k}(X,\mathbf{Q}(k)) & \to 0 \\ & & \downarrow & & \downarrow & \\ & & \mathcal{L}^{\ell}\underline{H}^{2k}_{\mathcal{H}}(X\backslash Y,\mathbf{Q}(k)) & \to & [\mathcal{L}^{\ell}W_0H^{2k}(X\backslash Y,\mathbf{Q}(k))]\cap F^0 & \to 0 \\ & & \downarrow & & \downarrow & \\ & & 0 & & 0 & \end{array}$$

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It follows by a diagram chase that there is a short exact sequence

$$0 \to \frac{\mathcal{L}^{\ell-1} W_{-1} H^{2k-1}(X\backslash Y, \mathbf{C})}{\left\{ \begin{array}{l} W_0 H^{2k-1}(X\backslash Y, \mathbf{Q}(k)) \\ \quad +F^0 W_0 H^{2k-1}(X\backslash Y, \mathbf{C}) \end{array} \right\} \cap \mathcal{L}^{\ell-1} W_{-1} H^{2k-1}(X\backslash Y, \mathbf{C})} \tag{4.5}$$
$$\to \mathcal{L}^{\ell} \underline{H}^{2k}_{\mathcal{H}}(X, \mathbf{Q}(k)) \to \hom_{\mathrm{MH}}(\mathbf{1}, \mathcal{L}^{\ell} H^{2k}(X\backslash Y, \mathbf{Q}(k))) \to 0.$$

Note that [Ca]

$$\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1} H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{Q}(k)))$$
$$\simeq \frac{W_{-1} H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{C})}{\left\{ \begin{array}{l} W_{-1} H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{Q}(k)) \\ \quad +F^0 W_{-1} H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{C}) \end{array} \right\}},$$

further, as we recall [J2],

$$\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, H^{2k-1}(X\backslash Y, \mathbf{Q}(k)))$$
$$\simeq \frac{W_0 H^{2k-1}(X\backslash Y, \mathbf{C})}{\{W_0 H^{2k-1}(X\backslash Y, \mathbf{Q}(k)) + F^0 W_0 H^{2k-1}(X\backslash Y, \mathbf{C})\}}.$$

Now set

$$\underline{E}^{\ell,2k-\ell}_{\infty} := \frac{W_{-1} H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{C})}{\left\{ \begin{array}{l} W_0 H^{2k-1}(X\backslash Y, \mathbf{Q}(k)) \\ \quad +F^0 W_0 H^{2k-1}(X\backslash Y, \mathbf{C}) \end{array} \right\} \cap W_{-1} H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{C})}$$
$$E^{\ell,2k-\ell}_{\infty} := Gr^{\ell}_{\mathcal{L}} \underline{H}^{2k}_{\mathcal{H}}(X\backslash Y, \mathbf{Q}(k))$$
$$\underline{\underline{E}}^{\ell,2k-\ell}_{\infty} := \hom_{\mathrm{MH}}(\mathbf{1}, H^{\ell}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{Q}(k)))$$

Taking $Gr^{\ell}_{\mathcal{L}}$ of (4.5) together with the serpent lemma, there is a short exact sequence

$$0 \to \underline{E}^{\ell,2k-\ell}_{\infty} \to E^{\ell,2k-\ell}_{\infty} \to \underline{\underline{E}}^{\ell,2k-\ell}_{\infty} \to 0.$$

It easily follows that the sequence below is exact

$$\hom_{\mathrm{MH}}(\mathbf{1}, Gr^0_W H^{2k-1}(X\backslash Y, \mathbf{Q}(k))) \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1} H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{Q}(k)))$$
$$\to \underline{E}^{\ell,2k-\ell}_{\infty} \to 0,$$

where the ‘map’

$$\hom_{\mathrm{MH}}(\mathbf{1}, Gr^0_W H^{2k-1}(X\backslash Y, \mathbf{Q}(k)) \to \mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1} H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_* \mathbf{Q}(k)))$$

is interpreted as in (3.5)(i). We deduce that

$$\underline{E}_\infty^{\ell,2k-\ell} \simeq \frac{\mathrm{Ext}^1_{\mathrm{MH}}(\mathbf{1}, W_{-1}H^{\ell-1}(S\backslash\Sigma, R^{2k-\ell}\rho_*\mathbf{Q}(k)))}{\mathrm{hom}_{\mathrm{MH}}(\mathbf{1}, Gr^0_W H^{2k-1}(X\backslash Y, \mathbf{Q}(k)))}.$$

This proves (3.4). □

Let L_X be the operation of taking the cup product with a hyperplane class of X. Then $L_X : H^\bullet(X\backslash Y, \mathbf{Q}(k)) \to H^{\bullet+2}(X\backslash Y, \mathbf{Q}(k+1))$ is a morphism of mixed Hodge structures; moreover since $m := \dim X_\eta$ (generic fiber), and over $S\backslash\Sigma$,

$$L_X^{m-i} : R^i\rho_*\mathbf{Q}(j) \xrightarrow{\sim} R^{2m-i}\rho_*\mathbf{Q}(m-i+j)$$

is an isomorphism (even if $m-i<0$). We deduce therefore that $L_X^{m-2k+\ell}$ induces isomorphisms

$$\begin{array}{ccccccccc} 0 & \to & \underline{E}_\infty^{\ell,2k-\ell} & \to & E_\infty^{\ell,2k-\ell} & \to & \underline{\underline{E}}_\infty^{\ell,2k-\ell} & \to & 0 \\ & & L_X^{m-2k+\ell}\downarrow\wr & & L_X^{m-2k+\ell}\downarrow\wr & & L_X^{m-2k+\ell}\downarrow\wr & & \\ 0 & \to & \underline{E}_\infty^{\ell,2(m-k+\ell)-\ell} & \to & E_\infty^{\ell,2(m-k+\ell)-\ell} & \to & \underline{\underline{E}}_\infty^{\ell,2(m-k+\ell)-\ell} & \to & 0 \end{array} \tag{4.6}$$

5. Proof of the Main Theorem (1.2)

Recall the final setting from Section 4, namely a short exact sequence

$$0 \to \underline{E}_\infty^{\ell,2k-\ell} \to E_\infty^{\ell,2k-\ell} \to \underline{\underline{E}}_\infty^{\ell,2k-\ell} \to 0,$$

and that L_X acts on the E_∞ terms; moreover since $m = \dim X_\eta$ (generic fiber) [$\Rightarrow n = m+s$], $L_X^{m-2k+\ell}$ determines an isomorphism on all three E_∞ terms as summarized in (4.6) above.

We now set $U = S\backslash\Sigma$ and consider the following prescription. We introduce a filtration F^ℓ on $CH^k(\rho^{-1}(U))_{\mathbf{Q}}$. Set

$$F^0CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} = CH^k(\rho^{-1}(U))_{\mathbf{Q}} = CH^k(X\backslash Y)_{\mathbf{Q}}.$$

Introduce

$$\psi_0 : CH^k(\rho^{-1}(U))_{\mathbf{Q}} \to E_\infty^{0,2k} = \underline{\underline{E}}_\infty^{0,2k} = \mathrm{hom}_{\mathrm{MH}}(\mathbf{Q}(0), H^0(S\backslash\Sigma, R^{2k}\rho_*\mathbf{Q}(k))),$$

and set

$$F^1CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} = \ker\psi_0.$$

It is clear that $F^1CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}}$ represents cycles that are relatively homologous to zero (i.e. homologous to zero fiberwise), thus in particular (1.2)(i) clearly holds.

One now has an induced map

$$\psi_1 : F^1 CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} \to E_\infty^{1,2k-1}.$$

Again, set

$$F^2 CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} = \ker \psi_1.$$

Correspondingly, we have an induced map $\psi_2 : F^2 CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} \to E_\infty^{2,2k-2}$, and so on. In general, we are looking at this setting.

$$\psi_\ell : F^\ell CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} \to E_\infty^{\ell,2k-\ell},$$
$$F^{\ell+1} CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} := \ker \psi_\ell.$$

PROPOSITION 5.0. *Recall the definition of $D^k(X)$ in (1.2)(vi). Then*

$$\varinjlim_{U\subset S} F^{k+1} CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} \subset D^k(X).$$

Proof. The idea of proof comes from [Ra]; however, as noted in [Ra], this idea is due to Beauville. Actually it is enough to show that

$$\varinjlim_{U\subset S} F^{k+1} CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} = \varinjlim_{U\subset S} F^{k+j} CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} \quad \text{for } j \geqslant 1.$$

Consider this commutative diagram

$$\begin{array}{ccc} F^{k+j} CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} & \xrightarrow{\psi_{k+j}} & E_\infty^{k+j,k-j} \\ L_X^{m-k+j} \downarrow & & \wr\downarrow L_X^{m-k+j} \\ F^{k+j} CH^{m+j}(\rho^{-1}(U)/U)_{\mathbf{Q}} & \longrightarrow & E_\infty^{k+j,2m-k+j} \end{array}$$

Since $F^{k+j+1} CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} = \ker \psi_{k+j}$ and that

$$\varinjlim_{U\subset S} CH^{m+j}(\rho^{-1}(U)/U)_{\mathbf{Q}} = CH^{m+j}(X_\eta)_{\mathbf{Q}} = 0 \text{ for } j \geqslant 1,$$

[where X_η = generic fiber of ρ] it follows that

$$\varinjlim_{U\subset S} F^{k+j+1} CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} = \varinjlim_{U\subset S} F^{k+j} CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}},$$

and we are done.

Note that (1.2)(vi) is clear. The remaining details (1.2)(ii)–(v) are worked out below.

Functoriality: Any correspondence between two complex projective algebraic manifolds X_1, X_2 can be 'spread out' ($\overline{\mathbf{Q}}$-spread) to a correspondence $\mathcal{Z}$ over

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$S\backslash\Sigma$, for some (S, Σ), in a diagram of the sort

$$\begin{array}{ccccc} \mathcal{Z} & \subset & (\mathcal{X}_1\backslash\mathcal{Y}_1) \times_{S\backslash\Sigma} (\mathcal{X}_2\backslash\mathcal{Y}_2) & \to & \mathcal{X}_2\backslash\mathcal{Y}_1 \\ & & \downarrow & & \\ & & \mathcal{X}_1\backslash\mathcal{Y}_1, & & \end{array}$$

where the $\mathcal{Y}_i$'s, being preimages of Σ, are NCD's, the arrows are smooth and proper morphisms, and where $X_i = \mathcal{X}_{i,\eta} \times \mathbf{C}$. It easily follows that correspondences act on $\underline{E}_\infty^{\ell,2k-\ell}$, $E_\infty^{\ell,2k-\ell}$ and $\underline{\underline{E}}_\infty^{\ell,2k-\ell}$, and Deligne cohomology, hence functoriality is clear. (1.2)(iv)&(v) easily follow.

Products: For products, this follows from the compatibility of the product $\cup$ on Deligne cohomology with the intersection product on Chow groups.

Abel–Jacobi equivalence: Let $\xi \in F^1CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}}$. By restriction to $X_t = \rho^{-1}(t) \overset{j_t}{\hookrightarrow} \rho^{-1}(U)$, $t \in U$, ξ determines a normal function $v_\xi : U \to \coprod_{t\in U} J^k(X_t)_{\mathbf{Q}}$. The normal function is described by the factorization

$$\begin{array}{ccccc} \xi \in F^1CH^k(\rho^{-1}(U)/U)_{\mathbf{Q}} & \to & E_\infty^{1,2k-1} & \overset{j_t^*}{\to} & J^k(X_t)_{\mathbf{Q}} \\ \downarrow & & & & \cap \\ H_{\mathcal{H}}^{2k}(\rho^{-1}(U), \mathbf{Q}(k)) & & \overset{j_t^*}{\to} & & H_{\mathcal{D}}^{2k}(X_t, \mathbf{Q}(k)). \end{array}$$

(1.2)(ii) easily follows from this.

Acknowledgements

The inspiration for writing this paper comes from [Ra]. From a different perspective, there are also the recent works of [As], [A-S] and [Gn]. We would like to thank the referee for his or her constructive criticisms; in particular the need to improve the presentation in Section 3.

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