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ON DEFICIENT-PERFECT NUMBERS

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Abstract

For a positive integer n, let $\sigma(n)$ denote the sum of the positive divisors of n. Let d be a proper divisor of n. We call n a deficient-perfect number if $\sigma(n) = 2n - d$. In this paper, we show that there are no odd deficient-perfect numbers with three distinct prime divisors.

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1. Introduction

For a positive integer n, let $\sigma(n)$ and $\omega(n)$ be the the sum of the positive divisors of n and the number of distinct prime divisors of n, respectively. Let d be a proper divisor of n. We call n a near-perfect number with redundant divisor d if $\sigma(n) = 2n + d$ and a deficient-perfect number with deficient divisor d if $\sigma(n) = 2n - d$. In particular, we call n an almost perfect number if $\sigma(n) = 2n - 1$. We know that if n is a power of 2, then n is an even almost perfect number. In 1978, Kishore [4] proved that if n is an odd almost perfect number, then $\omega(n) \geq 6$. In 2012, Pollack and Shevelev [5] presented an upper bound on the count of near-perfect numbers and constructed three types of near-perfect numbers. Recently, Ren and Chen [6] determined all near-perfect numbers with two distinct prime factors, and one sees from this classification that all such numbers are even. Following this, Tang *et al.* [8] proved that there is no odd near-perfect number with three distinct prime divisors and determined all deficient-perfect numbers with at most two distinct prime factors. For related problems, see [1–3, 7, 8].

In this paper, we obtain the following result.

THEOREM 1.1. *There are no odd deficient-perfect numbers with three distinct prime divisors.*

Throughout this paper, let m be a positive integer and a be any integer relatively prime to m. If h is the least positive integer such that $a^h \equiv 1 \pmod{m}$, then h is called the order of a modulo m, denoted by $\mathrm{ord}_m(a)$.

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2. Lemmas

LEMMA 2.1. *Let $n = \prod_{i=1}^{t} p_i^{\alpha_i}$ be the normal prime factorisation of n. If n is an odd deficient-perfect number, then the exponents α_i are even for all i.*

PROOF. Let d be a proper divisor of n. Then d is odd. Since $\sigma(n) = 2n - d$, $\sigma(n) \equiv 1 \pmod 2$. Since $\sigma(n)$ is a multiplicative function and the p_i are odd primes,

$$\sigma(n) = \prod_{i=1}^{t} \sigma(p_i^{\alpha_i}) = \prod_{i=1}^{t} (1 + p_i + \cdots + p_i^{\alpha_i}) \equiv \alpha_i + 1 \pmod 2.$$

Thus the exponents α_i are even for all i. □

LEMMA 2.2. *If $n = 3^{\alpha_1} 5^{\alpha_2} p^{\alpha_3}$ with $7 \le p \le 29$, then n is not an odd deficient-perfect number.*

PROOF. Assume that $n = 3^{\alpha_1} 5^{\alpha_2} p^{\alpha_3}$ is an odd deficient-perfect number with deficient divisor $d = 3^{\beta_1} 5^{\beta_2} p^{\beta_3}$, where $7 \le p \le 29$. Then

$$\sigma(3^{\alpha_1} 5^{\alpha_2} p^{\alpha_3}) = 2 \cdot 3^{\alpha_1} 5^{\alpha_2} p^{\alpha_3} - 3^{\beta_1} 5^{\beta_2} p^{\beta_3}, \tag{2.1}$$

where $\beta_i \le \alpha_i$, $1 \le i \le 3$ and $0 \le \beta_1 + \beta_2 + \beta_3 < \alpha_1 + \alpha_2 + \alpha_3$. Write

$$D_0 = 3^{\alpha_1 - \beta_1} 5^{\alpha_2 - \beta_2} p^{\alpha_3 - \beta_3}.$$

Then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} = \frac{\sigma(n)}{n} + \frac{1}{3^{\alpha_1 - \beta_1} 5^{\alpha_2 - \beta_2} p^{\alpha_3 - \beta_3}} = \frac{\sigma(n)}{n} + \frac{1}{D_0}. \tag{2.2}$$

By Lemma 2.1, $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2, 3$. Let

$$f(\alpha_1, \alpha_2, \alpha_3) = \Big(1 - \frac{1}{3^{\alpha_1 + 1}}\Big)\Big(1 - \frac{1}{5^{\alpha_2 + 1}}\Big)\Big(1 - \frac{1}{p^{\alpha_3 + 1}}\Big),$$

$$g(\alpha_1, \alpha_2, \alpha_3) = \frac{2^4 \cdot (p-1)}{3 \cdot 5 \cdot p} - \frac{2^3 \cdot (p-1)}{3^{\alpha_1 - \beta_1 + 1} \cdot 5^{\alpha_2 - \beta_2 + 1} \cdot p^{\alpha_3 - \beta_3 + 1}}.$$

Then, by (2.1),

$$f(\alpha_1, \alpha_2, \alpha_3) = g(\alpha_1, \alpha_2, \alpha_3). \tag{2.3}$$

Case 1. $p = 7$. Then

$$f(\alpha_1, \alpha_2, \alpha_3) \ge \Big(1 - \frac{1}{3^3}\Big)\Big(1 - \frac{1}{5^3}\Big)\Big(1 - \frac{1}{7^3}\Big) = 0.9524\ldots,$$

$$g(\alpha_1, \alpha_2, \alpha_3) < \frac{2^5}{5 \cdot 7} = 0.9142\ldots,$$

so (2.3) cannot hold.

Case 2. $p = 11$. Since $\mathrm{ord}_3(5) = \mathrm{ord}_3(11) = 2$ and $\alpha_i \equiv 0 \pmod 2$, $i = 2, 3$, we have $3 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3})$. Thus, by (2.1), $\beta_1 = 0$.

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If $\alpha_1 \geq 4$, then

$$f(\alpha_1, \alpha_2, \alpha_3) \geq \Big(1 - \frac{1}{3^5}\Big)\Big(1 - \frac{1}{5^3}\Big)\Big(1 - \frac{1}{11^3}\Big) = 0.9871\ldots,$$
$$g(\alpha_1, \alpha_2, \alpha_3) < \frac{2^5}{3 \cdot 11} = 0.9696\ldots,$$

so (2.3) cannot hold.

If $\alpha_1 = 2$ and $D_0 \geq 99$, then, by (2.2),

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2)}{3^2} \cdot \frac{5}{4} \cdot \frac{11}{10} + \frac{1}{99} < 2,$$

which is a contradiction.

If $\alpha_1 = 2$ and $D_0 = 45$, then $\alpha_2 - \beta_2 = 1$ and $\alpha_3 = \beta_3$. Thus, by (2.1),

$$13 \cdot \sigma(5^{\alpha_2} 11^{\alpha_3}) = \sigma(3^2 5^{\alpha_2} 11^{\alpha_3}) = 5^{\alpha_2 - 1} 11^{\alpha_3} \cdot 89,$$

which is impossible.

If $\alpha_1 = 2$ and $D_0 = 9$, then $\alpha_i = \beta_i$ for $i = 2, 3$. Thus, by (2.1),

$$13 \cdot \sigma(5^{\alpha_2} 11^{\alpha_3}) = \sigma(3^2 5^{\alpha_2} 11^{\alpha_3}) = 5^{\alpha_2} 11^{\alpha_3} \cdot 17,$$

which is impossible.

Case 3. $p = 13$. Since $\mathrm{ord}_5(3) = \mathrm{ord}_5(13) = 4$ and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 3$, we have $5 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} 13^{\alpha_3})$, so, by (2.1), $\beta_2 = 0$.

If $\alpha_1 \geq 4$, then

$$f(\alpha_1, \alpha_2, \alpha_3) \geq \Big(1 - \frac{1}{3^5}\Big)\Big(1 - \frac{1}{5^3}\Big)\Big(1 - \frac{1}{13^3}\Big) = 0.9874\ldots,$$
$$g(\alpha_1, \alpha_2, \alpha_3) < \frac{2^6}{5 \cdot 13} = 0.9846\ldots.$$

Thus (2.3) cannot hold.

If $\alpha_1 = 2$, then, by (2.2),

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2)}{3^2} \cdot \frac{5}{4} \cdot \frac{13}{12} + \frac{1}{5^2} < 2,$$

which is a contradiction.

Case 4. $p = 19$. Since $\mathrm{ord}_5(3) = 4, \mathrm{ord}_5(19) = 2$ and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 3$, we have $5 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} 19^{\alpha_3})$, so, by (2.1), $\beta_2 = 0$.

If $D_0 \geq 75$, then, by (2.2),

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{19}{18} + \frac{1}{75} < 2,$$

which is a contradiction.

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If $D_0 = 25$, then $\alpha_i = \beta_i$ for $i = 1, 3$ and $\alpha_2 = 2$. Thus, by (2.1),

$$31 \cdot \sigma(3^{\alpha_1} 19^{\alpha_3}) = \sigma(3^{\alpha_1} 5^2 19^{\alpha_3}) = 3^{\alpha_1} 19^{\alpha_3} \cdot 7^2,$$

which is a contradiction.

Case 5. $p = 17, 29$. Since $\mathrm{ord}_3(5), \mathrm{ord}_3(p), \mathrm{ord}_5(3), \mathrm{ord}_5(p), \mathrm{ord}_p(3), \mathrm{ord}_p(5)$ are even and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2, 3$, we have $3 \cdot 5 \cdot p \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} p^{\alpha_3})$, so, by (2.1), $\beta_1 = \beta_2 = \beta_3 = 0$. That is, n is an almost perfect number. By the result of Kishore [4], this is impossible.

Case 6. $p = 23$. Since $\mathrm{ord}_3(5) = \mathrm{ord}_3(23) = 2$, $\mathrm{ord}_5(3) = \mathrm{ord}_5(23) = 4$, and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2$, we have $3 \cdot 5 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} 23^{\alpha_3})$, so, by (2.1), $\beta_1 = \beta_2 = 0$. By (2.2),

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{23}{22} + \frac{1}{3^2 \cdot 5^2} < 2,$$

which is a contradiction. □

3. Proof of Theorem 1.1

Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ is an odd deficient-perfect number with deficient divisor $d = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3}$. Then

$$\sigma(p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}) = 2 \cdot p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} - p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3}, \tag{3.1}$$

where $\beta_i \le \alpha_i$, $1 \le i \le 3$, and $0 \le \beta_1 + \beta_2 + \beta_3 < \alpha_1 + \alpha_2 + \alpha_3$. By Lemma 2.1, $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2, 3$. Write

$$D = p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2 - \beta_2} p_3^{\alpha_3 - \beta_3}.$$

Then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} = \frac{\sigma(n)}{n} + \frac{1}{p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2 - \beta_2} p_3^{\alpha_3 - \beta_3}} = \frac{\sigma(n)}{n} + \frac{1}{D}. \tag{3.2}$$

If $p_1 \ge 5$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} + \frac{1}{5} < 2,$$

which is impossible. Thus $p_1 = 3$. If $p_2 \ge 19$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{19}{18} \cdot \frac{23}{22} + \frac{1}{3} < 2,$$

which is also impossible. Thus $p_2 \le 17$.

Case 1. $p_2 = 17$. If $p_3 \ge 23$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{17}{16} \cdot \frac{23}{22} + \frac{1}{3} < 2,$$

which is impossible. Thus $p_3 = 19$.

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Subcase 1.1. $D \geq 9$. Then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{17}{16} \cdot \frac{19}{18} + \frac{1}{9} < 2,$$

which is impossible.

Subcase 1.2. $D = 3$. Then $\alpha_1 - \beta_1 = 1$ and $\alpha_i = \beta_i$ for $i = 2, 3$. By (3.1),

$$\sigma(3^{\alpha_1} 17^{\alpha_2} 19^{\alpha_3}) = 5 \cdot 3^{\alpha_1 - 1} 17^{\alpha_2} 19^{\alpha_3}. \tag{3.3}$$

Noting that $\mathrm{ord}_5(3), \mathrm{ord}_5(17), \mathrm{ord}_5(19)$ are even and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2, 3$, we have $5 \nmid \sigma(3^{\alpha_1} 17^{\alpha_2} 19^{\alpha_3})$, so (3.3) cannot hold.

Case 2. $p_2 = 13$. If $p_3 \geq 41$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{13}{12} \cdot \frac{41}{40} + \frac{1}{3} < 2,$$

which is impossible. Thus $p_3 \leq 37$.

Subcase 2.1. $D \geq 9$. Then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{13}{12} \cdot \frac{17}{16} + \frac{1}{9} < 2,$$

which is impossible.

Subcase 2.2. $D = 3$. Then $\alpha_1 - \beta_1 = 1$ and $\alpha_i = \beta_i$ for $i = 2, 3$. By (3.1),

$$\sigma(3^{\alpha_1} 13^{\alpha_2} p_3^{\alpha_3}) = 5 \cdot 3^{\alpha_1 - 1} 13^{\alpha_2} p_3^{\alpha_3}. \tag{3.4}$$

If $p_3 = 17, 19, 23, 29$ or 37, then $\mathrm{ord}_5(p_3)$ is even. Moreover, $\mathrm{ord}_5(3) = \mathrm{ord}_5(13) = 4$ and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2, 3$, so $5 \nmid \sigma(3^{\alpha_1} 13^{\alpha_2} p_3^{\alpha_3})$. Thus (3.4) cannot hold.

If $p_3 = 31$, then since $\mathrm{ord}_{31}(3) = \mathrm{ord}_{31}(13) = 30$ and $\alpha_i \equiv 0 \pmod 2$ for $i = 1, 2$, $31 \nmid \sigma(3^{\alpha_1} 13^{\alpha_2} 31^{\alpha_3})$, so (3.4) cannot hold.

Case 3. $p_2 = 11$. If $p_3 \geq 101$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{11}{10} \cdot \frac{101}{100} + \frac{1}{3} < 2,$$

which is impossible. Thus $p_3 \leq 97$.

Subcase 3.1. $D \geq 9$. Then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} + \frac{1}{9} < 2,$$

which is impossible.

Subcase 3.2. $D = 3$. Then $\alpha_1 - \beta_1 = 1$ and $\alpha_i = \beta_i$ for $i = 2, 3$. By (3.1),

$$\sigma(3^{\alpha_1} 11^{\alpha_2} p_3^{\alpha_3}) = 5 \cdot 3^{\alpha_1 - 1} 11^{\alpha_2} p_3^{\alpha_3}. \tag{3.5}$$

Let

$$f_1(\alpha_1, \alpha_2, \alpha_3) = \Big(1 - \frac{1}{3^{\alpha_1+1}}\Big)\Big(1 - \frac{1}{11^{\alpha_2+1}}\Big)\Big(1 - \frac{1}{p_3^{\alpha_3+1}}\Big),$$

$$g_1(\alpha_1, \alpha_2, \alpha_3) = \frac{2^2 \cdot 5^2 \cdot (p_3 - 1)}{3^2 \cdot 11 \cdot p_3}.$$

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Then, by (3.5),

$$f_1(\alpha_1, \alpha_2, \alpha_3) = g_1(\alpha_1, \alpha_2, \alpha_3). \tag{3.6}$$

If $p_3 = 17, 23, 29, 41, 47, 53, 59, 71, 83$ or 89, then $\mathrm{ord}_3(p_3) = \mathrm{ord}_3(11) = 2$. Since $\alpha_i \equiv 0 \pmod 2$, $i = 2, 3$, we have $3 \nmid \sigma(3^{\alpha_1} 11^{\alpha_2} p_3^{\alpha_3})$, so (3.5) cannot hold.

If $p_3 = 31, 37, 61, 67, 73$ or 97, then $\mathrm{ord}_{p_3}(3)$ and $\mathrm{ord}_{p_3}(11)$ are even. Since $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2$, we have $p_3 \nmid \sigma(3^{\alpha_1} 11^{\alpha_2} p_3^{\alpha_3})$, so (3.5) cannot hold.

If $p_3 = 13$ or 19, then

$$f_1(\alpha_1, \alpha_2, \alpha_3) \geq \Big(1 - \frac{1}{3^3}\Big)\Big(1 - \frac{1}{11^3}\Big)\Big(1 - \frac{1}{13^3}\Big) = 0.9618\ldots,$$
$$g_1(\alpha_1, \alpha_2, \alpha_3) \leq \frac{2^2 \cdot 5^2 \cdot 18}{3^2 \cdot 11 \cdot 19} = 0.9569\ldots,$$

so (3.6) cannot hold.

If $p_3 = 43$ or 79 and $\alpha_1 = 2$, then $13 \mid \sigma(3^2 11^{\alpha_2} p_3^{\alpha_3})$, so (3.5) cannot hold.

If $p_3 = 43$ or 79 and $\alpha_1 = 4$, then, by (3.5),

$$\sigma(11^{\alpha_2} p_3^{\alpha_3}) = 5 \cdot 3^3 \cdot 11^{\alpha_2 - 2} p_3^{\alpha_3}. \tag{3.7}$$

If $\alpha_2 = 2$, then $7 \cdot 19 \cdot \sigma(p_3^{\alpha_3}) = 5 \cdot 3^3 \cdot p_3^{\alpha_3}$, which is impossible. Hence $\alpha_2 \geq 4$.

Noting that $\mathrm{ord}_{11}(43) = 2$, $\mathrm{ord}_{11}(79) = 10$ and $\alpha_3 \equiv 0 \pmod 2$, we have $11 \nmid \sigma(11^{\alpha_2} p_3^{\alpha_3})$, so (3.7) cannot hold.

If $p_3 = 43$ or 79 and $\alpha_1 \geq 6$, then

$$f_1(\alpha_1, \alpha_2, \alpha_3) \geq \Big(1 - \frac{1}{3^7}\Big)\Big(1 - \frac{1}{11^3}\Big)\Big(1 - \frac{1}{43^3}\Big) = 0.9987\ldots,$$
$$g_1(\alpha_1, \alpha_2, \alpha_3) \leq \frac{2^3 \cdot 5^2 \cdot 13}{3 \cdot 11 \cdot 79} = 0.9973\ldots,$$

so (3.6) cannot hold.

Case 4. $p_2 = 7$.

Subcase 4.1. $D \geq 21$. Then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10} + \frac{1}{21} < 2,$$

which is impossible.

Subcase 4.2. $D = p_3$. Then $\alpha_3 - \beta_3 = 1$ and $\alpha_i = \beta_i$ for $i = 1, 2$. If $p_3 \geq 13$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{13}{12} + \frac{1}{13} < 2,$$

which is impossible. Thus $p_3 = 11$ and, by (3.1),

$$\sigma(3^{\alpha_1} 7^{\alpha_2} 11^{\alpha_3}) = 3^{\alpha_1 + 1} 7^{\alpha_2 + 1} 11^{\alpha_3 - 1}. \tag{3.8}$$

Noting that $\mathrm{ord}_3(11)$ is even and $\alpha_3 \equiv 0 \pmod 2$, we know that $3 \nmid \sigma(11^{\alpha_1})$. By (3.8), $3 \mid \sigma(7^{\alpha_2})$, so $9 \mid 7^{\alpha_2+1} - 1$. Since $\mathrm{ord}_9(7) = 3$, we have $3 \mid \alpha_2 + 1$. Thus $\sigma(7^2) \mid \sigma(7^{\alpha_2})$, but $19 \mid \sigma(7^2)$, so (3.8) cannot hold.

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Subcase 4.3. $D = 9$. Then $\alpha_1 - \beta_1 = 2$ and $\alpha_i = \beta_i$ for $i = 2, 3$. If $p_3 \geq 17$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{17}{16} + \frac{1}{9} < 2,$$

which is impossible. Thus $p_3 = 11$ or 13. By (3.1),

$$\sigma(3^{\alpha_1} 7^{\alpha_2} p_3^{\alpha_3}) = 17 \cdot 3^{\alpha_1 - 2} 7^{\alpha_2} p_3^{\alpha_3}. \tag{3.9}$$

Noting that $\mathrm{ord}_{17}(3), \mathrm{ord}_{17}(7), \mathrm{ord}_{17}(p_3)$ are even and $\alpha_i \equiv 0 \pmod 2$ for $i = 1, 2, 3$, we have $17 \nmid \sigma(3^{\alpha_1} \cdot 7^{\alpha_2} \cdot p_3^{\alpha_3})$, and thus (3.9) cannot hold.

Subcase 4.4. $D = 7$. Then $\alpha_2 - \beta_2 = 1$ and $\alpha_i = \beta_i$ for $i = 1, 3$. By (3.1),

$$\sigma(3^{\alpha_1} 7^{\alpha_2} p_3^{\alpha_3}) = 13 \cdot 3^{\alpha_1} 7^{\alpha_2 - 1} p_3^{\alpha_3}. \tag{3.10}$$

If $p_3 \geq 19$, then, by (3.2),

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{19}{18} + \frac{1}{7} < 2,$$

which is impossible. Thus $p_3 = 11, 13$ or 17.

Let

$$f_2(\alpha_1, \alpha_2, \alpha_3) = \Big(1 - \frac{1}{3^{\alpha_1+1}}\Big)\Big(1 - \frac{1}{7^{\alpha_2+1}}\Big)\Big(1 - \frac{1}{p_3^{\alpha_3+1}}\Big),$$

$$g_2(\alpha_1, \alpha_2, \alpha_3) = \frac{2^2 \cdot 13 \cdot (p_3 - 1)}{7^2 \cdot p_3}.$$

Then, by (3.10),

$$f_2(\alpha_1, \alpha_2, \alpha_3) = g_2(\alpha_1, \alpha_2, \alpha_3). \tag{3.11}$$

If $p_3 = 11$ and $\alpha_1 \geq 4$, then

$$f_2(\alpha_1, \alpha_2, \alpha_3) \geq \Big(1 - \frac{1}{3^5}\Big)\Big(1 - \frac{1}{7^3}\Big)\Big(1 - \frac{1}{11^3}\Big) = 0.9922\ldots,$$

$$g_2(\alpha_1, \alpha_2, \alpha_3) = \frac{2^3 \cdot 5 \cdot 13}{7^2 \cdot 11} = 0.9647\ldots,$$

so (3.11) cannot hold.

If $p_3 = 11$ and $\alpha_1 = 2$, then, by (3.2),

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{\sigma(3^2)}{3^2} \cdot \frac{7}{6} \cdot \frac{11}{10} + \frac{1}{7} < 2,$$

which is impossible.

If $p_3 = 13$ or 17, then since $\mathrm{ord}_7(3), \mathrm{ord}_7(p_3)$ are even and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 3$, we have $7 \nmid \sigma(3^{\alpha_1} 7^{\alpha_2} p_3^{\alpha_3})$, so (3.10) cannot hold.

Subcase 4.5. $D = 3$. Then $\alpha_1 - \beta_1 = 1$ and $\alpha_i = \beta_i$ for $i = 2, 3$. By (3.1),

$$\sigma(3^{\alpha_1} 7^{\alpha_2} p_3^{\alpha_3}) = 5 \cdot 3^{\alpha_1 - 1} 7^{\alpha_2} p_3^{\alpha_3}. \tag{3.12}$$

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Moreover, $\mathrm{ord}_5(3) = \mathrm{ord}_5(7) = 4$ and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2$, so $5 \nmid \sigma(3^{\alpha_1} 7^{\alpha_2})$. Since $\mathrm{ord}_7(3) = 6$, we have $7 \nmid \sigma(3^{\alpha_1})$.

If $3 \mid \sigma(7^{\alpha_2})$, then $9 \mid 7^{\alpha_2+1} - 1$. Since $\mathrm{ord}_9(7) = 3$, we have $3 \mid \alpha_2 + 1$, so $\sigma(7^2) \mid \sigma(7^{\alpha_2})$ and hence $p_3 = 19$. Since $\mathrm{ord}_5(19) = 2$, we have $5 \nmid \sigma(19^{\alpha_3})$, so (3.12) cannot hold. Then $3 \nmid \sigma(7^{\alpha_2})$. By (3.12), $\sigma(3^{\alpha_1} 7^{\alpha_2}) = p_3^{\alpha_3}$ and $\sigma(p_3^{\alpha_3}) = 5 \cdot 3^{\alpha_1 - 1} 7^{\alpha_2}$. Then

$$p_3(3^{\alpha_1} 7^{\alpha_2} - 3^{\alpha_1+1} - 7^{\alpha_2+1} + 1) = -20 \cdot 3^{\alpha_1} 7^{\alpha_2} + 12. \tag{3.13}$$

Since $\alpha_1, \alpha_2 \geq 2$, we have $3^{\alpha_1} 7^{\alpha_2} > 3^{\alpha_1+1} + 7^{\alpha_2+1} - 1$, so (3.13) cannot hold.

Case 5. $p_2 = 5$. By Lemma 2.2, it is sufficient to consider $n = 3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3}$ with $p_3 \geq 31$.

Subcase 5.1. $D \geq 25$. Then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{31}{30} + \frac{1}{25} < 2,$$

which is impossible.

Subcase 5.2. $D = 15$. If $p_3 \geq 37$, then

$$2 = \frac{\sigma(n)}{n} + \frac{d}{n} < \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{37}{36} + \frac{1}{15} < 2,$$

which is impossible. Thus $p_3 = 31$ and, by (3.1),

$$\sigma(3^{\alpha_1} 5^{\alpha_2} 31^{\alpha_3}) = 29 \cdot 3^{\alpha_1 - 1} 5^{\alpha_2 - 1} 31^{\alpha_3}. \tag{3.14}$$

Noting that $\mathrm{ord}_{29}(3) = \mathrm{ord}_{29}(31) = 28$, $\mathrm{ord}_{29}(5) = 14$ and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2, 3$, we have $29 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2} 31^{\alpha_3})$, so (3.14) cannot hold.

Subcase 5.3. $D = 9$. Then $\alpha_1 - \beta_1 = 2$ and $\alpha_i = \beta_i$ for $i = 2, 3$. By (3.1),

$$\sigma(3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3}) = 17 \cdot 3^{\alpha_1 - 2} 5^{\alpha_2} p_3^{\alpha_3}. \tag{3.15}$$

If $\alpha_1 = 2$, then $13 \mid \sigma(3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3})$, so (3.15) cannot hold. Hence $\alpha_1 \geq 4$. Noting that $\mathrm{ord}_{17}(3)$, $\mathrm{ord}_{17}(5)$, $\mathrm{ord}_5(3)$ and $\mathrm{ord}_3(5)$ are even and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2$, we have $3 \cdot 5 \cdot 17 \nmid \sigma(3^{\alpha_1} 5^{\alpha_2})$. Thus $\sigma(3^{\alpha_1} 5^{\alpha_2}) = p_3^{\alpha_3}$ and $\sigma(p_3^{\alpha_3}) = 17 \cdot 3^{\alpha_1 - 2} 5^{\alpha_2}$. Then

$$p_3(3^{\alpha_1 - 2} 5^{\alpha_2} + 3^{\alpha_1+1} + 5^{\alpha_2+1} - 1) = 136 \cdot 3^{\alpha_1 - 2} 5^{\alpha_2} - 8.$$

Thus

$$(p_3 - 136) 3^{\alpha_1 - 2} 5^{\alpha_2} = -p_3(3^{\alpha_1+1} + 5^{\alpha_2+1} - 1) - 8. \tag{3.16}$$

We know that $3 \mid \sigma(p_3^{\alpha_3})$ and $5 \mid \sigma(p_3^{\alpha_3})$, since α_3 is even, so $p_3 \equiv 1 \pmod 3$ and $p_3 \equiv 1 \pmod 5$. Hence $p_3 \equiv 1 \pmod{15}$. Noting that there is no prime p_3 such that $p_3 \equiv 1 \pmod{15}$ and $31 \leq p_3 \leq 131$, it follows that (3.16) cannot hold.

Subcase 5.4. $D = 5$. Then $\alpha_2 - \beta_2 = 1$ and $\alpha_i = \beta_i$ for $i = 1, 3$. By (3.1),

$$\sigma(3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3}) = 3^{\alpha_1 + 2} 5^{\alpha_2 - 1} p_3^{\alpha_3}. \tag{3.17}$$

If $\alpha_1 = 2$, then $13 \mid \sigma(3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3})$, so (3.17) cannot hold. Hence $\alpha_1 \geq 4$.

Noting that $\mathrm{ord}_5(3) = 4, \mathrm{ord}_3(5) = 2$ and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2$, we have $3 \nmid \sigma(3^{\alpha_1}5^{\alpha_2})$, $5 \nmid \sigma(3^{\alpha_1}5^{\alpha_2})$. Thus, by (3.17), $\sigma(3^{\alpha_1}5^{\alpha_2}) = p_3^{\alpha_3}$ and $\sigma(p_3^{\alpha_3}) = 3^{\alpha_1+2} \cdot 5^{\alpha_2-1}$. Thus, by (3.17),

$$p_3(3^{\alpha_1+1}5^{\alpha_2-1} - 3^{\alpha_1+1} - 5^{\alpha_1+1} + 1) = -8 \cdot 3^{\alpha_1+2}5^{\alpha_2-1} + 8. \tag{3.18}$$

Noting that $\alpha_1 \geq 4$ and $\alpha_2 \geq 2$, we have $3^{\alpha_1+1}5^{\alpha_2-1} - 3^{\alpha_1+1} - 5^{\alpha_1+1} + 1 > 0$, so (3.18) cannot hold.

Subcase 5.5. $D = 3$. Then $\alpha_1 - \beta_1 = 1$ and $\alpha_i = \beta_i$ for $i = 2, 3$. By (3.1),

$$\sigma(3^{\alpha_1}5^{\alpha_2}p_3^{\alpha_3}) = 3^{\alpha_1-1}5^{\alpha_2+1}p_3^{\alpha_3}. \tag{3.19}$$

Since $\mathrm{ord}_5(3) = 4, \mathrm{ord}_3(5) = 2$ and $\alpha_i \equiv 0 \pmod 2$, $i = 1, 2$, we have $3 \nmid \sigma(3^{\alpha_1}5^{\alpha_2})$, $5 \nmid \sigma(3^{\alpha_1}5^{\alpha_2})$. Thus, by (3.19), $\sigma(3^{\alpha_1}5^{\alpha_2}) = p_3^{\alpha_3}$ and $\sigma(p_3^{\alpha_3}) = 3^{\alpha_1-1}5^{\alpha_2+1}$. Hence

$$p_3(3^{\alpha_1-1}5^{\alpha_2+1} - 3^{\alpha_1+1} - 5^{\alpha_2+1} + 1) = -8 \cdot 3^{\alpha_1-1}5^{\alpha_2+1} + 8. \tag{3.20}$$

Noting that $\alpha_1, \alpha_2 \geq 2$, we have $3^{\alpha_1-1}5^{\alpha_2+1} - 3^{\alpha_1+1} - 5^{\alpha_2+1} + 1 > 0$, so (3.20) cannot hold.

This completes the proof of Theorem 1.1. □

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