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# A NEW EXTENSION OF DARBO'S FIXED POINT THEOREM USING RELATIVELY MEIR–KEELER CONDENSING OPERATORS

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**Abstract**

We consider relatively Meir–Keeler condensing operators to study the existence of best proximity points (pairs) by using the notion of measure of noncompactness, and extend a result of Aghajani *et al.* ['Fixed point theorems for Meir–Keeler condensing operators via measure of noncompactness', *Acta Math. Sci. Ser. B* **35** (2015), 552–566]. As an application of our main result, we investigate the existence of an optimal solution for a system of integrodifferential equations.

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## 1. Introduction

Throughout this paper, $\mathcal{B}(x; r)$ denotes the closed ball at $x$ with radius $r > 0$ in a normed linear space $X$. One of the most important fixed point theorems due to Brouwer states that the Euclidean unit ball $\mathcal{B}(0; 1) \subseteq \mathbb{R}^n$ has the fixed point property for continuous functions, that is, every continuous function $T : \mathcal{B}(0; 1) \to \mathcal{B}(0; 1)$ has at least one fixed point.

Let $X$ be a topological space and $Y \subseteq X$. Then $Y$ is called a retract of $X$ if there exists a continuous mapping $f : X \to Y$ such that $f(x) = x$ for all $x \in Y$. In this case, $f$ is called a retraction of $X$ into $Y$. Brouwer's fixed point theorem is equivalent to the assertion that there is no indefinitely differentiable retraction $f : \mathcal{B}(0; 1)\,(\subseteq \mathbb{R}^n) \to \mathcal{S}(0; 1)$, where $\mathcal{S}(0; 1) = \{x \in \mathbb{R}^n : \|x\| = 1\}$ (see [3, Theorem 1.2]).

In 1930, Schauder [11] generalised Brouwer's fixed point theorem to Banach spaces as follows.

THEOREM 1.1. *Let $A$ be a nonempty, compact and convex subset of a Banach space $X$ and $T : A \to A$ be a continuous mapping. Then $T$ has a fixed point.*

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The compactness of the set $A$ in Theorem 1.1 plays an important role in the proof of Schauder's fixed point theorem. It is natural to ask if Theorem 1.1 holds whenever $A$ is a bounded, closed and convex subset of a Banach space $X$. This problem was answered by Kakutani in [10], where he presented an example of a continuous self-mapping defined on a unit ball of the Hilbert space $l^2(\mathbb{Z})$ which is fixed-point free.

Schauder's fixed point theorem is a very useful tool for proving the existence of solutions to many nonlinear problems, especially problems concerning ordinary and partial differential equations, and it has a number of extensions.

Definition 1.2. Let $X$ and $Y$ be normed linear spaces and $K$ be a subset of $X$. A mapping $T : K \to Y$ is said to be a compact operator if $T$ is continuous and maps bounded sets into relatively compact sets.

Here, we recall a well-known generalisation of Schauder's fixed point theorem.

Theorem 1.3. *Let $K$ be a nonempty, bounded, closed and convex subset of a Banach space $X$ and $T : K \to K$ be a compact operator. Then $T$ has a fixed point.*

An improved version of Theorem 1.3 was presented by Darbo [4] using the notion of *measure of noncompactness*.

Definition 1.4 (Kuratowski, 1930). Let $(X, d)$ be a metric space and $\Sigma$ be the family of all nonempty and bounded subsets of $X$. The function $\alpha : \Sigma \to [0, \infty)$ defined by

$$\alpha(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter } \leq \varepsilon\},$$

for all $B \in \Sigma$, is called the Kuratowski measure of noncompactness.

Similarly, the function $\chi : \Sigma \to [0, \infty)$ defined by

$$\chi(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many balls with radii } \leq \varepsilon\},$$

for all $B \in \Sigma$, is called the *Hausdorff measure of noncompactness*. It was introduced in [9] as an extension of the Kuratowski measure of noncompactness. We refer to [3] for more interesting information related to measures of noncompactness.

The essential properties of the Kuratowski and Hausdorff measures of noncompactness can be listed as follows.

Definition 1.5. Let $(X, d)$ be a complete metric space and $\Sigma$ be the family of bounded subsets of $X$. A function $\mu : \Sigma \to [0, \infty)$ is called a measure of noncompactness (MNC, for short) if it satisfies the following conditions:

(i) $\mu(A) = 0$ if and only if $A$ is relatively compact;
(ii) $\mu(A) = \mu(\overline{A})$ for all $A \in \Sigma$;
(iii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma$.

If $\mu$ is an MNC on $\Sigma$, then the following properties follow immediately (see [3]):

$(p_1)$ if $A \subseteq B$, then $\mu(A) \leq \mu(B)$;
$(p_2)$ $\mu(A \cap B) \leq \min\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma$;

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($p_3$) if $A$ is a finite set, then $\mu(A) = 0$;
($p_4$) if $\{A_n\}$ is a decreasing sequence of nonempty, bounded and closed subsets of $X$ such that $\lim_{n\to\infty} \mu(A_n) = 0$, then $A_\infty := \bigcap_{n\geq 1} A_n$ is nonempty and compact.

Further, if $X$ is a Banach space and $\overline{\text{con}}(A)$ denotes the *closed and convex hull* of a set $A$, then:

($p_5$) $\mu(\overline{\text{con}}(A)) = \mu(A)$ for all $A \in \Sigma$;
($p_6$) $\mu(tA) = |t|\mu(A)$ for any number $t$ and $A \in \Sigma$;
($p_7$) $\mu(A + B) \leq \mu(A) + \mu(B)$ for all $A, B \in \Sigma$.

We are now ready to state Darbo's fixed point theorem for mappings which may not be compact.

Theorem 1.6 [4]. *Let $A$ be a nonempty, bounded, closed and convex subset of a Banach space $X$ and $\mu$ be a measure of noncompactness on $X$. Suppose that $T : A \to A$ is a continuous mapping such that, for some $r \in [0, 1)$,*

$$\mu(T(K)) \leq r\mu(K)$$

*for all nonempty and bounded $K \subseteq A$. Then $T$ has a fixed point.*

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be a *Meir–Keeler contraction* provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x, y \in X, \quad \varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon.$$

For such mappings, we have the following generalisation of the Banach contraction principle.

Theorem 1.7. *Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a Meir–Keeler contraction mapping. Then $T$ has a unique fixed point and the Picard iteration sequence $\{T^n x_0\}$ converges to the fixed point of $T$ for any $x_0 \in X$.*

Very recently, an extension of Theorem 1.6 was proved using the Meir–Keeler contraction condition.

Definition 1.8 [1]. Let $A$ be a nonempty subset of a Banach space $X$ and $\mu$ be a measure of noncompactness on $X$. An operator $T : A \to X$ is said to be a Meir–Keeler condensing operator if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(K) < \varepsilon + \delta \Rightarrow \mu(T(K)) < \varepsilon$$

for all nonempty and bounded $K \subseteq A$.

Theorem 1.9 [1, Theorem 2.2]. *Let $A$ be a nonempty, bounded, closed and convex subset of a Banach space $X$ and $\mu$ be a measure of noncompactness on $X$. Suppose that $T : A \to A$ is a continuous Meir–Keeler condensing operator. Then $T$ has a fixed point.*

Theorem 1.9 was used to study the existence of a solution for a class of functional integral equations of Volterra type (see [1, Theorem 4.1]).

This paper is organised as follows: in Section 2, we recall some basic definitions and notions. In Section 3, we generalise Theorem 1.9 to a new class of mappings, called relatively Meir–Keeler condensing operators, in order to study best proximity pairs of points. Finally, in Section 4, we apply our results to the existence of an optimal solution for a system of integrodifferential equations, extending some of the recent results of [8].

## 2. Mathematical background

A Banach space $X$ is said to be *strictly convex* provided that the following implication holds for $x, y, p \in X$ and $R > 0$:

$$\begin{cases} \|x-p\| \le R, \\ \|y-p\| \le R, \\ x \neq y \end{cases} \Rightarrow \quad \left\| \frac{x+y}{2} - p \right\| < R.$$

Remark 2.1. Hilbert and $L^p$ spaces with $1 < p < \infty$ are strictly convex Banach spaces.

Let $A$ and $B$ be two nonempty subsets of a normed linear space $X$. We shall say that a pair $(A, B)$ of subsets of a Banach space $X$ satisfies a property if both $A$ and $B$ satisfy that property. For example, $(A, B)$ is convex if and only if both $A$ and $B$ are convex; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C,\ B \subseteq D$. From now on, $\mathcal{B}(x; r)$ will denote the closed ball in the Banach space $X$ centred at $x \in X$ with radius $r > 0$. Also, $\operatorname{diam}(A)$ stands for the diameter of the set $A$. We mention that if $A$ is a nonempty and compact subset of a Banach space $X$, then $\overline{\operatorname{con}}(A)$ is compact (see [5]).

Also, we shall adopt the following notation:

$$\operatorname{dist}(A, B) = \inf\{\|x-y\| : (x, y) \in A \times B\},$$
$$A_0 = \{x \in A : \exists\, y' \in B : \|x-y'\| = \operatorname{dist}(A, B)\},$$
$$B_0 = \{y \in B : \exists\, x' \in A : \|x'-y\| = \operatorname{dist}(A, B)\}.$$

Definition 2.2. A nonempty pair $(A, B)$ in a normed linear space $X$ is said to be proximinal if $A = A_0$ and $B = B_0$.

Note that if $(A, B)$ is a nonempty, weakly compact and convex pair in a Banach space $X$, then $(A_0, B_0)$ is also nonempty, weakly compact and convex.

A mapping $T : A \cup B \to A \cup B$ is called *cyclic* if $T(A) \subseteq B$ and $T(B) \subseteq A$ and it is called *noncyclic* if $T(A) \subseteq A$ and $T(B) \subseteq B$. In case $T$ is cyclic, we would like to find an element $x$ such that $x$ is in proximity to $Tx$ in some sense. This is the aim of *best proximity point theory*. In fact, we want to find sufficient conditions on the mapping $T$ and the pair $(A, B)$ to give the existence of a best proximity point $p \in A \cup B$ such that $\|p - Tp\| = \operatorname{dist}(A, B)$. If $T$ is noncyclic, the existence of a best proximity pair, that is, a point $(u, v) \in A \times B$ for which $u = Tu$, $v = Tv$ and $\|u - v\| = \operatorname{dist}(A, B)$, can be considered.

In [6], Eldred *et al.* established the existence of best proximity points (pairs) for cyclic (noncyclic) relatively nonexpansive mappings using a geometric notion of *proximal normal structure* defined on a nonempty and convex pair in a Banach space. We recall that the mapping $T : A \cup B \to A \cup B$ is called relatively nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } (x, y) \in A \times B.$$

The following existence results follow from the fact that every nonempty, compact and convex pair in a Banach space $X$ has the proximal normal structure (see [7, Theorem 3.5]).

Theorem 2.3 [8, Theorem 3.2]. *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a Banach space $X$ such that $A_0$ is nonempty. Assume that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If $T$ is compact, that is, $(\overline{T(A)}, \overline{T(B)})$ is a compact pair, then $T$ has a best proximity point.*

Theorem 2.4 [8, Theorem 4.1]. *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a strictly convex Banach space $X$ such that $A_0$ is nonempty. Assume that $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping. If $T$ is compact, then $T$ has a best proximity pair.*

Motivated by Definition 1.8, we introduce the following new classes of cyclic and noncyclic mappings.

Definition 2.5. Let $(A, B)$ be a nonempty and convex pair in a Banach space $X$ and $\mu$ be an MNC on $X$. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic (noncyclic) Meir–Keeler condensing operator if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any nonempty, bounded, closed, convex, proximinal and $T$-invariant pair $(K_1, K_2) \subseteq (A, B)$ with $\text{dist}(K_1, K_2) = \text{dist}(A, B)$,

$$\varepsilon \leq \mu(K_1 \cup K_2) < \varepsilon + \delta(\varepsilon) \Rightarrow \mu((T(K_1) \cup T(K_2))) < \varepsilon.$$

Example 2.6. To illustrate the significance of the definition, let $(A, B)$ be a nonempty and convex pair in a Banach space $X$ such that $B$ is relatively compact and let $\mu$ be an MNC on $X$. If $T : A \cup B \to A \cup B$ is a cyclic mapping for which $T|_B$ is Meir–Keeler condensing, then $T$ is a cyclic Meir–Keeler condensing operator.

Proof. Let $\varepsilon > 0$ be given. Since $T|_B$ is Meir–Keeler condensing, there exists $\delta(\varepsilon) > 0$ such that for any bounded subset $H$ of $B$,

$$\varepsilon \leq \mu(H) < \varepsilon + \delta(\varepsilon) \Rightarrow \mu(T(H)) < \varepsilon. \tag{2.1}$$

Now suppose that, for a nonempty, bounded, closed, convex and proximinal pair $(K_1, K_2) \subseteq (A, B)$ which is $T$-invariant and for which $\text{dist}(K_1, K_2) = \text{dist}(A, B)$, we have $\varepsilon \leq \mu(K_1 \cup K_2) < \varepsilon + \delta$. Since $B$ is relatively compact and $T$ is cyclic,

$$\mu(T(K_1)) \leq \mu(K_2) \leq \mu(B) = \mu(\overline{B}) = 0.$$

Thus, $\varepsilon \leq \mu(K_1) \leq \varepsilon + \delta(\varepsilon)$. Using (2.1),

$$\mu(T(K_1) \cup T(K_2)) = \max\{\mu(T(K_1)), \mu(T(K_2))\} = \mu(T(K_2)) < \varepsilon$$

and the result follows. □

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## 3. Main results

In this section, we establish best proximity point results for the new classes of cyclic and noncyclic mappings in Definition 2.5. First, we consider the cyclic case.

Theorem 3.1. *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a Banach space $X$ such that $A_0$ is nonempty and $\mu$ be an* MNC *on $X$. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping which is Meir–Keeler condensing. Then $T$ has a best proximity point.*

Proof. First, $(A_0, B_0)$ is nonempty by the hypothesis on $A_0$. It is easy to verify that $(A_0, B_0)$ is closed, convex and proximinal. If $x \in A_0$, then there exists $y \in B_0$ such that $\|x - y\| = \mathrm{dist}(A, B)$. Since $T$ is a relatively nonexpansive mapping,

$$\|Tx - Ty\| \le \|x - y\| = \mathrm{dist}(A, B),$$

which implies that $Tx \in B_0$, that is, $T(A_0) \subseteq B_0$. Similarly, $T(B_0) \subseteq A_0$ and so $T$ is cyclic on $A_0 \cup B_0$. Put $\mathcal{G}_0 := A_0$ and $\mathcal{H}_0 := B_0$ and, for all $n \in \mathbb{N}$, define

$$\mathcal{G}_n = \overline{\mathrm{con}}(T(\mathcal{G}_{n-1})), \quad \mathcal{H}_n = \overline{\mathrm{con}}(T(\mathcal{H}_{n-1})).$$

We now have

$$\mathcal{G}_1 = \overline{\mathrm{con}}(T(\mathcal{G}_0)) = \overline{\mathrm{con}}(T(A_0)) \subseteq B_0 = \mathcal{H}_0.$$

Thus, $T(\mathcal{G}_1) \subseteq T(\mathcal{H}_0)$ and so $\mathcal{G}_2 = \overline{\mathrm{con}}(T(\mathcal{G}_1)) \subseteq \overline{\mathrm{con}}(T(\mathcal{H}_0)) = \mathcal{H}_1$. Continuing this process, we conclude by induction that $\mathcal{G}_{n+1} \subseteq \mathcal{H}_n$. Similarly, we can see that $\mathcal{H}_n \subseteq \mathcal{G}_{n-1}$ for all $n \in \mathbb{N}$. Thus,

$$\mathcal{G}_{n+2} \subseteq \mathcal{H}_{n+1} \subseteq \mathcal{G}_n \subseteq \mathcal{H}_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Hence, $\{(\mathcal{G}_{2n}, \mathcal{H}_{2n})\}_{n\ge 0}$ is a decreasing sequence of nonempty, closed and convex pairs in $A_0 \times B_0$. Moreover,

$$T(\mathcal{H}_{2n}) \subseteq T(\mathcal{G}_{2n-1}) \subseteq \overline{\mathrm{con}}(T(\mathcal{G}_{2n-1})) = \mathcal{G}_{2n},$$
$$T(\mathcal{G}_{2n}) \subseteq T(\mathcal{H}_{2n-1}) \subseteq \overline{\mathrm{con}}(T(\mathcal{H}_{2n-1})) = \mathcal{H}_{2n}.$$

Therefore, for all $n \in \mathbb{N}$, the pair $(\mathcal{G}_{2n}, \mathcal{H}_{2n})$ is $T$-invariant. On the other hand, if $(p, q) \in A_0 \times B_0$ is a proximal pair, then

$$\mathrm{dist}(\mathcal{G}_{2n}, \mathcal{H}_{2n}) \le \|T^{2n}p - T^{2n}q\| \le \|p - q\| = \mathrm{dist}(A, B) \quad \text{for all } n \in \mathbb{N}.$$

We shall show by induction that the pair $(\mathcal{G}_n, \mathcal{H}_n)$ is proximinal for all $n \in \mathbb{N} \cup \{0\}$. This is obvious if $n = 0$. Suppose that $(\mathcal{G}_n, \mathcal{H}_n)$ is proximinal. Let $x \in \mathcal{G}_{n+1} = \overline{\mathrm{con}}(T(\mathcal{G}_n))$ be an arbitrary element. Then $x = \sum_{j=1}^{k} \lambda_j T(x_j)$ with $x_j \in \mathcal{G}_n$, $k \in \mathbb{N}$, $\lambda_j \ge 0$ and $\sum_{j=1}^{k} \lambda_j = 1$. Proximinality of the pair $(\mathcal{G}_n, \mathcal{H}_n)$ implies that there exists $y_j \in \mathcal{H}_n$ for $1 \le j \le k$ such that $\|x_j - y_j\| = \mathrm{dist}(\mathcal{G}_n, \mathcal{H}_n)$ $(= \mathrm{dist}(A, B))$. Put $y = \sum_{j=1}^{k} \lambda_j T(y_j)$. Then $y \in \overline{\mathrm{con}}(T(\mathcal{H}_n)) = \mathcal{H}_{n+1}$ and

$$\|x - y\| = \left\| \sum_{j=1}^{k} \lambda_j T(x_j) - \sum_{j=1}^{k} \lambda_j T(y_j) \right\| \le \sum_{j=1}^{k} \lambda_j \|x_j - y_j\| = \mathrm{dist}(A, B)$$

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and so the pair $(\mathcal{G}_{n+1}, \mathcal{H}_{n+1})$ is proximinal. We now consider the following possible cases.

*Case 1.* If $\max\{\mu(\mathcal{G}_{2k}), \mu(\mathcal{H}_{2k})\} = 0$ for some $k \in \mathbb{N}$, then

$$T : \mathcal{G}_{2k} \cup \mathcal{H}_{2k} \to \mathcal{G}_{2k} \cup \mathcal{H}_{2k}$$

is a compact and cyclic relatively nonexpansive mapping and the result follows from Theorem 2.3.

*Case 2.* Assume that $\max\{\mu(\mathcal{G}_{2n}), \mu(\mathcal{H}_{2n})\} > 0$ for all $n \in \mathbb{N}$. Put $\varepsilon_n := \mu(\mathcal{G}_{2n} \cup \mathcal{H}_{2n})$. Since $T$ is a cyclic Meir–Keeler condensing operator, there exists $\delta_n := \delta(\varepsilon_n)$ such that

$$\mu(T(\mathcal{G}_{2n}) \cup T(\mathcal{H}_{2n})) < \varepsilon_n \quad \text{for all } n \in \mathbb{N}.$$

Also, for all $n \in \mathbb{N}$,

$$\varepsilon_{n+1} = \mu(\mathcal{G}_{2n+2} \cup \mathcal{H}_{2n+2}) = \max\{\mu(\mathcal{G}_{2n+2}), \mu(\mathcal{H}_{2n+2})\} \le \max\{\mu(\mathcal{G}_{2n}), \mu(\mathcal{H}_{2n})\} = \varepsilon_n.$$

Thus, $\{\varepsilon_n\}$ is a decreasing sequence of positive real numbers and $\lim_{n\to\infty} \varepsilon_n = r \ge 0$. We claim that $r = 0$. Suppose on the contrary that $r > 0$. Then there exists $l \in \mathbb{N}$ such that $r \le \varepsilon_l < r + \delta(r)$. Again using the fact that $T$ is a cyclic Meir–Keeler condensing operator, we conclude that

$$\begin{aligned}\varepsilon_{l+1} &= \mu(\mathcal{G}_{2l+2} \cup \mathcal{H}_{2l+2}) = \max\{\mu(\mathcal{G}_{2l+2}), \mu(\mathcal{H}_{2l+2})\}\\ &\le \max\{\mu(\mathcal{H}_{2l+1}), \mu(\mathcal{G}_{2l+1})\} = \max\{\mu(\overline{\mathrm{con}}(T(\mathcal{H}_{2l}))), \mu(\overline{\mathrm{con}}(T(\mathcal{G}_{2l})))\}\\ &= \max\{\mu(T(\mathcal{H}_{2l})), \mu(T(\mathcal{G}_{2l}))\} = \mu(T(\mathcal{G}_{2l}) \cup T(\mathcal{H}_{2l})) < r,\end{aligned}$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} \mu(\mathcal{G}_{2n} \cup \mathcal{H}_{2n}) = \max\Big\{\lim_{n\to\infty} \mu(\mathcal{G}_{2n}), \lim_{n\to\infty} \mu(\mathcal{H}_{2n})\Big\} = 0.$$

Set

$$\mathcal{G}_\infty = \bigcap_{n=0}^{\infty} \mathcal{G}_{2n} \quad \text{and} \quad \mathcal{H}_\infty = \bigcap_{n=0}^{\infty} \mathcal{H}_{2n}.$$

By property $(p_4)$ of MNC (see Definition 1.5), the pair $(C_\infty, \mathcal{D}_\infty)$ is nonempty and compact. It is also convex and $T$-invariant with $\mathrm{dist}(A, B) = \mathrm{dist}(\mathcal{G}_\infty, \mathcal{H}_\infty)$. This ensures that $T$ has a best proximity point. □

Next we prove the analogous result for noncyclic Meir–Keeler condensing mappings.

THEOREM 3.2. *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a strictly convex Banach space $X$ such that $A_0$ is nonempty and $\mu$ be an* MNC *on $X$. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping which is Meir–Keeler condensing. Then $T$ has a best proximity pair.*

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PROOF. As in Theorem 3.1, define $\mathcal{G}_n = \overline{\text{con}}(T(\mathcal{G}_{n-1}))$ and $\mathcal{H}_n = \overline{\text{con}}(T(\mathcal{H}_{n-1}))$ for all $n \in \mathbb{N}$. Proceeding as in the proof of Theorem 3.1, we can see that $\{(\mathcal{G}_n, \mathcal{H}_n)\}$ is a decreasing sequence of nonempty, closed, convex, proximinal and $T$-invariant pairs such that $\text{dist}(\mathcal{G}_n, \mathcal{H}_n) = \text{dist}(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$. If $\max\{\mu(\mathcal{G}_k), \mu(\mathcal{H}_k)\} = 0$ for some $k \in \mathbb{N}$, then the result follows from Theorem 2.4. Otherwise, if we set

$$\mathcal{G}_\infty = \bigcap_{n=0}^{\infty} \mathcal{G}_n \quad \text{and} \quad \mathcal{H}_\infty = \bigcap_{n=0}^{\infty} \mathcal{H}_n,$$

then $(\mathcal{G}_\infty, \mathcal{H}_\infty)$ is a nonempty, compact and convex pair and $\text{dist}(\mathcal{G}_\infty, \mathcal{H}_\infty) = \text{dist}(A, B)$. Because

$$T : \mathcal{G}_\infty \cup \mathcal{H}_\infty \to \mathcal{G}_\infty \cup \mathcal{H}_\infty$$

is a noncyclic relatively nonexpansive mapping, an application of Theorem 2.4 shows that $T$ has a best proximity pair. □

As corollaries of Theorems 3.1 and 3.2, we give the following results.

COROLLARY 3.3 [8, Theorem 3.4]. *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a Banach space $X$ such that $A_0$ is nonempty and let $\mu$ be an* MNC *on $X$. Let $T : A \cup B \to A \cup B$ be a cyclic relatively nonexpansive mapping such that for any nonempty, closed, convex, proximinal and $T$-invariant pair $(K_1, K_2)$ with* $\text{dist}(K_1, K_2) = \text{dist}(A, B)$,

$$\mu(T(K_1) \cup T(K_2)) \le r\mu(K_1 \cup K_2)$$

*for some $r \in (0, 1)$. Then $T$ has a best proximity point.*

COROLLARY 3.4 [8, Theorem 4.3]. *Let $(A, B)$ be a nonempty, bounded, closed and convex pair in a strictly convex Banach space $X$ such that $A_0$ is nonempty and let $\mu$ be an* MNC *on $X$. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping such that for any nonempty, closed, convex, proximinal and $T$-invariant pair with* $\text{dist}(K_1, K_2) = \text{dist}(A, B)$,

$$\mu(T(K_1) \cup T(K_2)) \le r\mu(K_1 \cup K_2)$$

*for some $r \in (0, 1)$. Then $T$ has a best proximity pair.*

## 4. Application to integrodifferential equations

In this section, we prove a theorem establishing the existence of an optimal solution of certain systems of integrodifferential equations with local initial conditions.

Let $a, b, h$ be positive real numbers with $h < a$. For a given real number $t_0$ and a Banach space $X$, we consider the Banach space $C(I, X)$ consisting of continuous mappings from $I = [t_0 - a, t_0 + a]$ into $X$, endowed with the supremum norm. Also, let $V_1 = B(x_1; b)$ and $V_2 = B(x_2; b)$ be closed balls in $X$, where $x_1, x_2 \in X$. Assume that

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$k_i : I \times I \times V_i \to X$ and $f_i : I \times V_i \times V_i \to X$, with $i = 1, 2$, are continuous mappings and $k_i$ is $k_i$-invariant. Here we consider the problem:

$$\begin{cases} x'(t) = f_1\Big(t, x(t), \displaystyle\int_{t_0}^{t} k_1(t, s, x(s))\, ds\Big), & x(t_0) = x_1, \\ y'(t) = f_2\Big(t, y(t), \displaystyle\int_{t_0}^{t} k_2(t, s, y(s))\, ds\Big), & y(t_0) = x_2, \end{cases} \tag{4.1}$$

where the integral is the Bochner integral. Let $J = [t_0 - h, t_0 + h]$ and define $C(J, V_1) = \{x : J \to V_1$ such that $x \in C(J, X)$ and $x(t_0) = x_1\}$ and $C(J, V_2) = \{y : J \to V_2$ such that $y \in C(J, X)$ and $y(t_0) = x_2\}$. Clearly, $(C(J, V_1), C(J, V_2))$ is a bounded, closed and convex pair in $C(J, X)$. Also, for any $(x, y) \in C(J, V_1) \times C(J, V_2)$, we have $\|x_1 - x_2\| \le \sup_{t \in J} \|x(t) - y(t)\| = \|x - y\|$ and, so, $\mathrm{dist}(C(J, V_1), C(J, V_2)) = \|x_1 - x_2\|$.

Now, let $T : C(J, V_1) \cup C(J, V_2) \to C(J, X)$ be the operator defined by

$$Tx(t) = \begin{cases} x_2 + \displaystyle\int_{t_0}^{t} f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\, ds\Big)\, d\sigma, & x \in C(J, V_1), \\ x_1 + \displaystyle\int_{t_0}^{t} f_2\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_2(\sigma, s, x(s))\, ds\Big)\, d\sigma, & x \in C(J, V_2). \end{cases} \tag{4.2}$$

We show that $T$ is a cyclic operator. Indeed, for $x \in C(J, V_1)$,

$$\begin{aligned} \|Tx(t) - x_2\| &= \Big\| \int_{t_0}^{t} f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\, ds\Big)\, d\sigma \Big\| \\ &\le \Big| \int_{t_0}^{t} \Big\| f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\, ds\Big)\Big\|\, d\sigma \Big| \\ &\le M_1 h, \end{aligned}$$

where $M_i = \sup\{\|f_i(t, x(t), \int_{t_0}^{t} k_i(t, s, x(s))\, ds)\| : (t, x) \in I \times V_i\}$, $i = 1, 2$. Assume that $h < b/\max\{M_1, M_2\}$. Then $\|Tx(t) - x_2\| \le b$ for all $t \in J$ and so $Tx \in C(J, V_2)$. The same argument shows that $x \in C(J, V_2)$ implies that $Tx \in C(J, V_1)$.

With this in mind, for $0 < h < b/\max\{M_1, M_2\}$, the hypotheses are as follows:

($H_1$) let $\mu$ be an MNC on $C(J, X)$ such that for any $r > 0$ there exists $\delta(r) > 0$ such that $r \le \mu(W_1 \cup W_2) < r + \delta(r)$ for any bounded $(W_1, W_2) \subseteq (V_1, V_2)$ implies that $\mu(f_1(I \times W_1 \times W_1) \cup f_2(I \times W_2 \times W_2)) < r/h$;

($H_2$) for all $(x, y) \in C(J, V_1) \times C(J, V_2)$,

$$\Big\| f_1\Big(t, x(t), \int_{t_0}^{t} k_1(t, s, x(s))\, ds\Big) - f_2\Big(t, y(t), \int_{t_0}^{t} k_2(t, s, y(s))\, ds\Big)\Big\| \le \frac{1}{h}(\|x(t) - y(t)\| - \|x_1 - x_0\|).$$

We recall the following extension of the mean-value theorem, which we arrange according to our notation.

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THEOREM 4.1. *Let $I$, $J$, $X$, $V_i$, $k_i : I \times I \times V_i \to X$ and $f_i : I \times V_i \times V_i \to X$ with $i = 1, 2$ be given as above. Let $t_0, t \in J$ with $t_0 < t$. Then, for $(i, j) \in \{(1, 2), (2, 1)\}$,*

$$\begin{aligned} x_j &+ \int_{t_0}^{t} f_i\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_i(\sigma, s, x(s))\, ds\Big)\, d\sigma \\ &\in x_j + (t - t_0)\overline{\text{con}}\Big(\Big\{f_i\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_i(\sigma, s, x(s))\, ds\Big) : \sigma \in [t_0, t]\Big\}\Big). \end{aligned} \tag{4.3}$$

We say that $z \in C(J, V_1) \cup C(J, V_2)$ is an optimal solution for the system (4.1) provided that $\|z - Tz\| = \text{dist}(C(J, V_1), C(J, V_2))$, that is, $z$ is a best proximity point of the operator $T$ in (4.2). We prove the following result.

THEOREM 4.2. *Under hypotheses $(H_1)$ and $(H_2)$, if $h < b/\max\{M_1, M_2\}$, then the problem (4.1) has an optimal solution.*

PROOF. Since $T$ is a cyclic operator, it follows trivially that $T(C(J, V_1))$ is a bounded subset of $C(J, V_2)$. We show that $T(C(J, V_1))$ is also an equicontinuous subset of $C(J, V_2)$. Suppose that $t, t' \in J$ and $x \in C(J, V_1)$. Then

$$\begin{aligned} &\|Tx(t) - Tx(t')\| \\ &= \Big\|\int_{t_0}^{t} f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\, ds\Big)\, d\sigma - \int_{t_0}^{t'} f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\, ds\Big)\, d\sigma\Big\| \\ &\le \Big|\int_{t}^{t'} \Big\|f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\, ds\Big)\Big\|\, d\sigma\Big| \le M_1|t - t'|, \end{aligned}$$

that is, $T(C(J, V_1))$ is equicontinuous. The same argument applies to $T(C(J, V_2))$.

Next, we show that $T$ is a Meir–Keeler condensing operator. Suppose that the pair $(K_1, K_2) \subseteq (C(J, V_1), C(J, V_2))$ is nonempty, closed, convex, proximinal and $T$-invariant and that $\text{dist}(K_1, K_2) = \text{dist}(C(J, V_1), C(J, V_2)) (= \|x_1 - x_2\|)$. By a generalised version of the Arzelà–Ascoli theorem (see Ambrosetti [2]) and hypothesis $(H_1)$,

$$\begin{aligned} &\mu(T(K_1) \cup T(K_2)) = \max\{\mu(T(K_1)), \mu(T(K_2))\} \\ &= \max\Big\{\sup_{t\in J}\{\mu(\{Tx(t) : x \in K_1\})\}, \sup_{t\in J}\{\mu(\{Ty(t) : y \in K_2\})\}\Big\} \\ &= \max\Big\{\sup_{t\in J}\Big\{\mu\Big(\Big\{x_2 + \int_{t_0}^{t} f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\, ds\Big)\, d\sigma : x \in K_1\Big\}\Big)\Big\}, \\ &\qquad \sup_{t\in J}\Big\{\mu\Big(\Big\{x_1 + \int_{t_0}^{t} f_2\Big(\sigma, y(\sigma), \int_{t_0}^{\sigma} k_2(\sigma, s, y(s))\, ds\Big)\, d\sigma : y \in K_2\Big\}\Big)\Big\}\Big\}. \end{aligned}$$

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In view of (4.3),

$$
\begin{aligned}
&\mu(T(K_1) \cup T(K_2)) \\
&\leq \max\Big\{\sup_{t\in J}\Big\{\mu\Big(\Big\{x_2 + (t-t_0)\overline{\mathrm{con}}\Big(\Big\{f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\,ds\Big) : \sigma \in [t_0, t]\Big\}\Big)\Big\}\Big)\Big\}, \\
&\qquad \sup_{t\in J}\Big\{\mu\Big(\Big\{x_1 + (t-t_0)\overline{\mathrm{con}}\Big(\Big\{f_2\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_2(\sigma, s, x(s))\,ds\Big) : \sigma \in [t_0, t]\Big\}\Big)\Big\}\Big)\Big\}\Big\} \\
&\leq \max\Big\{\sup_{0\leq\lambda\leq h}\{\mu(\{x_2 + \lambda\overline{\mathrm{con}}(\{f_1(J\times K_1\times K_1)\})\})\}, \\
&\qquad \sup_{0\leq\lambda\leq h}\{\mu(\{x_1 + \lambda\overline{\mathrm{con}}(\{f_2(J\times K_2\times K_2)\})\})\}\Big\} \\
&= \max\{h\mu(f_1(J\times K_1\times K_1)), h\mu(f_2(J\times K_2\times K_2))\} \\
&= h\mu(\{f_1(J\times K_1\times K_1)\cup f_2(J\times K_2\times K_2)\}) < h\frac{r}{h} = r.
\end{aligned}
$$

We conclude that $T$ is a Meir–Keeler condensing operator.

The last step of the proof is to show that $T$ is cyclic relatively nonexpansive. Indeed, for any $(x, y) \in C(J, V_1) \times C(J, V_2)$,

$$
\begin{aligned}
\|Tx(t) - Ty(t)\| &= \Big\|\Big(x_2 + \int_{t_0}^{t} f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\,ds\Big)d\sigma\Big) \\
&\qquad - \Big(x_1 + \int_{t_0}^{t} f_2\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_2(\sigma, s, x(s))\,ds\Big)d\sigma\Big)\Big\| \\
&\leq \|x_1 - x_2\| + \Big|\int_{t_0}^{t}\Big\|f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_1(\sigma, s, x(s))\,ds\Big) \\
&\qquad - f_1\Big(\sigma, x(\sigma), \int_{t_0}^{\sigma} k_2(\sigma, s, y(s))\,ds\Big)\Big\|\,d\sigma\Big| \\
&\leq \|x_1 - x_2\| + \frac{1}{h}\Big|\int_{t_0}^{t}(\|x(s) - y(s)\| - \|x_1 - x_2\|)\,ds\Big| \\
&\qquad\qquad \text{(by hypothesis } (H_2)) \\
&\leq \|x_1 - x_2\| + (\|x - y\| - \|x_1 - x_2\|) = \|x - y\|.
\end{aligned}
$$

Thus, $\|Tx - Ty\| \leq \|x - y\|$. All the hypotheses of Theorem 3.1 hold and so the operator $T$ has a best proximity point $z \in C(J, V_1) \cup C(J, V_2)$ which is an optimal solution for the system (4.1). □

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