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COMPLEMENTED COPIES OF ℓ^1 AND PEŁCZYŃSKI'S PROPERTY (V^*) IN BOCHNER FUNCTION SPACES

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ABSTRACT. Let X be a Banach space and $(f_n)_n$ be a bounded sequence in $L^1(X)$. We prove a complemented version of the celebrated Talagrand's dichotomy, *i.e.*, we show that if $(e_n)_n$ denotes the unit vector basis of c_0, there exists a sequence $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$ such that for almost every ω, either the sequence $(g_n(\omega) \otimes e_n)$ is weakly Cauchy in $X\hat{\otimes}_\pi c_0$ or it is equivalent to the unit vector basis of ℓ^1. We then get a criterion for a bounded sequence to contain a subsequence equivalent to a complemented copy of ℓ^1 in $L^1(X)$. As an application, we show that for a Banach space X, the space $L^1(X)$ has Pełczyński's property (V^*) if and only if X does.

1. **Introduction.** Let X be a Banach space and $(\Omega, \Sigma, \lambda)$ be a finite measure space. If $1 \leq p < \infty$, we denote by $L^p(\lambda, X)$ the Banach space of all (class of) X-valued p-Bochner integrable functions with its usual norm. If E and F are Banach spaces, we denote by $E\hat{\otimes}_\pi F$ the projective tensor product of E and F. We will say that a sequence $(x_n)_n$ is equivalent to a complemented copy of ℓ^1 if $(x_n)_n$ is equivalent to the unit vector basis of ℓ^1 and its closed linear span is complemented in X.

One of the many important problems in the theory of Banach spaces is to recognize different structure of subspaces of a given space. In this paper, we will be mainly concerned with sequences in the Bochner space $L^1(\lambda, X)$ that are equivalent to a complemented copy of ℓ^1. Let us recall that in [18], Talagrand proved a fundamental theorem characterizing weakly Cauchy sequences and sequences that are equivalent to the unit vector basis of ℓ^1 in the Bochner space $L^1(\lambda, X)$, relating a given sequence $(f_n)_n$ to its values $\big(f_n(\omega)\big)_n$ in X. Our main goal is to provide a complemented version of Talagrand's result. One way one might tackle this problem is to consider for a given $(f_n)_n$ in $L^1(\lambda, X)$, the corresponding sequence $(f_n \otimes e_n)_n$ in $L^1(\lambda, X)\hat{\otimes}_\pi c_0$ (which can be viewed as the Bochner space $L^1(\lambda, X\hat{\otimes}_\pi c_0)$), where $(e_n)_n$ is the unit vector basis of c_0. The basic motivation behind this approach is the well known fact that a bounded sequence $(x_n)_n$ in a Banach space X contains a subsequence equivalent to a complemented copy of ℓ^1 if and only if the sequence $(x_n \otimes e_n)_n$ is not a weakly null sequence in $X\hat{\otimes}_\pi c_0$. Therefore the behavior of the sequence $(f_n \otimes e_n)_n$ will determine whether or not the sequence $(f_n)_n$ has a subsequence equivalent to a complemented copy of ℓ^1. As in [18], we try to relate the sequence $(f_n \otimes e_n)_n$ to its values $(f_n(\omega) \otimes e_n)_n$ in $X\hat{\otimes}_\pi c_0$ to see how a particular structure of the space X can be carried on to the Bochner space $L^1(\lambda, X)$. The main result of this

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paper is the following extension of Talagrand's theorem: Let $(f_n)_n$ be a bounded sequence in $L^1(\lambda, X)$, then there exists a sequence $g_n \in \mathrm{conv}(f_n, f_{n+1}, \ldots)$ such that $\big(g_n(\omega) \otimes e_n\big)_n$ is either weakly Cauchy or equivalent to the unit vector basis of ℓ^1. We follow a line of reasoning similar to that of Talagrand ([18]). Our main focus is to carry out all the steps in such a way that the convex combination is taken only on the sequence $(f_n)_n$ and not on the sequence $(f_n \otimes e_n)_n$.

In Section 3, we apply our main theorem for the study of property (V^*) introduced by Pełczyński in [13]. The most notable examples of Banach spaces that have property (V^*) are L^1-spaces and recently it was also proved that spaces that are L-summands in their biduals have property (V^*) (see [14]). It is then a natural question to ask for what Banach spaces X the space $L^1(\lambda, X)$ has property (V^*). The most one could hope for is that $L^1(\lambda, X)$ has property (V^*) if and only if X does. This question was studied by several authors. Partial results can be found in [1], [8], [16] and more recently in [11]. We present a complete positive answer to this question (see Theorem 2 below).

Our notation and terminology are standard and can be found in [5] and [6].

2. **Complemented version of Talagrand's theorem.** By way of motivation, let us begin with the following well known proposition that justifies our approach. The word operator will always mean linear bounded operator and $\mathcal{L}(X, Y)$ will stand for the Banach space of all operators from X to Y.

PROPOSITION 1. *Let X be a Banach space and $(x_n)_n$ be a bounded sequence in X that is equivalent to the ℓ^1 basis. Then the following statements are equivalent:*

(i) The sequence $(x_n)_n$ has a subsequence that is equivalent to a complemented copy of ℓ^1;

(ii) There exist an operator $T \in \mathcal{L}(X, \ell^1)$ and a subsequence $(y_n)_n$ of $(x_n)_n$ such that $\langle Ty_n, e_n\rangle \geq 1$ for all $n \in \mathbb{N}$.

Proposition 1 can be deduced from Theorem VII-5 of [5] (p. 72) and its proof.

Using the fact that the space $\mathcal{L}(X, \ell^1)$ is the dual of $X\hat{\otimes}_\pi c_0$, condition (ii) of Proposition 1 can be restated as:

$(*)$ There exist $T \in (X\hat{\otimes}_\pi c_0)^*$, $(y_n)_n$ a subsequence of $(x_n)_n$ such that $\langle T, y_n \otimes e_n\rangle \geq 1$ for all $n \in \mathbb{N}$.

Now $(*)$ implies that the sequence $(x_n \otimes e_n)_n$ is not a weakly null sequence. So the problem of whether or not $(x_n)_n$ has a subsequence that is equivalent to a complemented copy of ℓ^1 in X is reduced to the study of weak convergence of $(x_n \otimes e_n)_n$ in $X\hat{\otimes}_\pi c_0$.

The following result is our main criterion for determining if a given sequence in a Bochner space has a subsequence that is equivalent to a complemented copy of ℓ^1.

THEOREM 1. *Let X be a Banach space and $(\Omega, \Sigma, \lambda)$ be a probability space. Let $(f_n)_n$ be a bounded sequence in $L^1(\lambda, X)$. Then there exist a sequence $g_n \in \mathrm{conv}(f_n, f_{n+1}, \ldots)$ and two measurable subsets C and L of Ω with $\lambda(C \cup L) = 1$ such that:*

(a) for $\omega \in C$, the sequence $\big(g_n(\omega) \otimes e_n\big)_n$ is weakly Cauchy in the space $X\hat{\otimes}_\pi c_0$.

(b) for $\omega \in L$, the sequence $\big(g_n(\omega) \otimes e_n\big)_n$ is equivalent to the ℓ^1 basis in $X\hat{\otimes}_\pi c_0$.

The proof uses many (if not all) ideas from Talagrand's theorem so we recommend that the reader should get familiar with its proof first before reading our extension. However because of the complexity of the proof of Talagrand's theorem, we decided to present all critical details.

Using similar argument as in [7], we can assume without loss of generality that the sequence $(f_n)_n$ is such that $\sup_{n\in\mathbb{N}} \|f_n\|_\infty \leq 1$.

For convenience, we will use the following notation:

(i) For two sequences $(g_n)_n$ and $(f_n)_n$, we write $(g_n) \ll (f_n)$ if there exists $k \in \mathbb{N}$ so that $\forall n \geq k$, $g_n \in \mathrm{conv}(f_n, f_{n+1}, \ldots)$ and by passing to a subsequence (if necessary), we will always assume that there exist two sequences of integers (p_n) and $(q_n)_n$ such that $p_1 \leq q_1 < p_2 \leq q_2 \cdots$ and $g_n = \sum_{i=p_n}^{q_n} a_i f_i$;

(ii) For a Banach space Y, we denote by Y_1 the closed unit ball of Y.

CASE 1. The Banach space X is separable;

If the space X is separable, so is the space $X\hat{\otimes}_\pi c_0$ and therefore $\mathcal{L}(X,\ell^1)_1$ the unit ball of $\mathcal{L}(X,\ell^1) = (X\hat{\otimes}_\pi c_0)^*$ endowed with the weak*-topology is compact metrizable.

Let us now consider $(U_n)_n$ a countable basis for the weak*-topology on $\mathcal{L}(X,\ell^1)_1$. Set $U_0 = \mathcal{L}(X,\ell^1)_1$.

Following Talagrand [18], we denote by $\mathcal{K}$ the set of all (weak*) compact sets of $\mathcal{L}(X,\ell^1)_1$ and we say that a map $\omega \longrightarrow K(\omega)$ from Ω to $\mathcal{K}$ is measurable if for each $n \in \mathbb{N}$, the set $\{\omega \in \Omega\,;\, K(\omega) \cap U_n \neq \emptyset\}$ is measurable.

As in [18], we will make use of the following elementary lemma:

LEMMA 1. *If for each $k \in \mathbb{N}$, we have $(f_n^{k+1}) \ll (f_n^k)$, then there exists a sequence (k_n) such that if we set $g_n = f_n^{k_n}$ we have $(g_n) \ll (f_n^k)$ for all $k \in \mathbb{N}$.*

We are now ready to begin the proof of the Theorem.

Let $\omega \longrightarrow K(\omega)$ from Ω to $\mathcal{K}$ be a measurable map and V be a weak*-open subset of $\mathcal{L}(X,\ell^1)_1$ and let $g_n\colon \Omega \longrightarrow X_1$ be a bounded sequence in $L^\infty(\lambda, X)$ such that $(g_n) \ll (f_n)$. Let $g_n = \sum_{i=p_n}^{q_n} \lambda_i f_i$ be the representation of g_n as block convex combination of the f_n's.

We set:

$$\overline{g_n}(\omega) = \sup_{k\geq q_n} \sup\Big\{\Big\langle T\big(g_n(\omega)\big), e_k\Big\rangle, T \in V \cap K(\omega)\Big\} \tag{1}$$

$$\theta(g)(\omega) = \limsup_{n\to\infty} \overline{g_n}(\omega). \tag{2}$$

Notice that the definition of $\overline{g_n}$ depends on the representation of g_n as block convex combination of the f_n's. Note also that since V is fixed, it does not appear on the notation.

It is clear that $\|\overline{g_n}\|_\infty \leq 1$ and we claim that $\overline{g_n}$ is measurable. To see this notice that for each $k \in \mathbb{N}$, the map $\omega \longrightarrow \sup\{\langle T, g_n(\omega) \otimes e_k\rangle, T \in V \cap K(\omega)\}$ was already proved to be measurable by Talagrand: in fact for each $k \in \mathbb{N}$, the sequence $\big(g_n(\cdot) \otimes e_k\big)_n$ is bounded in $L^\infty(\lambda, X\hat{\otimes}_\pi c_0)$ and $K(\cdot)$ is a measurable map (see [18] p. 706). So by taking the supremum the claim follows.

LEMMA 2. *There exists $(g_n) \ll (f_n)$ such that if $(h_n) \ll (g_n)$ we have* $\lim_{n\to\infty} \|\theta(g) - \overline{h_n}\|_1 = 0$.

PROOF. The proof is done more or less the same as in [18]; let $g_n^1 = f_n$ and construct by induction sequences $g^p = (g_n^p)$ such that for $p \geq 1$, one has

$$\int \theta(g^p)(\omega)\, d\lambda(\omega) \leq 2^{-p} + \inf\Big\{ \int \theta(\phi)(\omega)\, d\lambda(\omega) \ ; \ \phi \ll g^{p-1} \Big\}.$$

By Lemma 1, there is a sequence (g_n) such that $g \ll g^p$ for each $p \in N$; in particular $g \ll u$.

Let $h \ll g$. We claim that $\theta(h) = \theta(g)$. To see the claim, write $h_n = \sum_{i=p_n}^{q_n} \alpha_i f_i$; $h_n = \sum_{j=a_n}^{b_n} \beta_j g_j$ and $g_n = \sum_{l=c_n}^{d_n} \gamma_l f_l$. We have

$$\begin{aligned} \overline{h_n}(\omega) &= \sup_{k \geq q_n} \sup\Big\{ \sum_{i=a_n}^{b_n} \beta_i \Big\langle T\big(g_i(\omega)\big), e_k \Big\rangle, T \in V \cap K(\omega) \Big\} \\ &\leq \sup_{k \geq q_n} \sup\Big\{ \sup_{i \in [a_n, b_n]} \Big\langle T\big(g_i(\omega)\big), e_k \Big\rangle, T \in V \cap K(\omega) \Big\} \end{aligned}$$

for each $n \in \mathbb{N}$ and $\omega \in \Omega$, so there exist $T \in V \cap K(\omega)$, $i_n \in [a_n, b_n]$ such that

$$\overline{h_n}(\omega) \leq \sup_{k \geq q_n} \Big\langle T\big(g_{i_n}(\omega)\big), e_k \Big\rangle + \frac{1}{2^n}$$

but $q_n \geq d_{b_n} \geq d_{j_n}$ so we get that

$$\begin{aligned} \overline{h_n}(\omega) &\leq \sup_{k \geq d_{j_n}} \Big\langle T\big(g_{i_n}(\omega)\big), e_k \Big\rangle + \frac{1}{2^n} \\ &\leq \overline{g_{i_n}}(\omega) + \frac{1}{2^n} \end{aligned}$$

and by taking the limsup, we get that $\theta(h) \leq \theta(g)$.

In the other hand we have for each $p \in \mathbb{N}$,

$$\begin{aligned} \inf\Big\{ \int \theta(\phi)(\omega)\, d\lambda(\omega), \phi \ll g^{p-1} \Big\} &\leq \int \theta(h)(\omega)\, d\lambda(\omega) \\ &\leq \int \theta(g)(\omega)\, d\lambda(\omega) \\ &\leq \inf\Big\{ \int \theta(\phi)(\omega)\, d\lambda(\omega), \phi \ll g^{p-1} \Big\} + 2^{-p} \end{aligned}$$

hence $\int \theta(h)(\omega)\, d\lambda(\omega) = \int \theta(g)(\omega)\, d\lambda(\omega)$.

We claim that $\theta(g) = \lim_{n\to\infty} \overline{h_n}$ for the weak*-topology in $L^\infty(\lambda)$: for that let ϕ be a cluster point of $(\overline{h_n})_n$. Since $\theta(h) = \limsup_{n\to\infty} \overline{h_n} \leq \theta(g)$, one has $\phi \leq \theta(g)$. Moreover if we choose $h_n{}' = \sum_{i=a_n}^{b_n} \alpha_i h_i$ such that $a_1 \leq b_1 < a_2 \leq b_2 < \cdots$ and $\| \sum_{i=a_n}^{b_n} \alpha_i \overline{h_i} - \phi \|_1 \leq 2^{-n}$, we have $\lim_{n\to\infty} \sum_{i=a_n}^{b_n} \alpha_i \overline{h_i}(\omega) = \phi(\omega)$ a.e. but for any $n \in \mathbb{N}$, the above estimate shows that: $\overline{h'_n}(\omega) \leq \overline{h_{i_n}}(\omega) + 2^{-n}$ and hence $\theta(g) = \theta(h') = \limsup_{n\to\infty} \overline{h'_n} \leq \phi \leq \theta(g)$ which shows that $\theta(g) = \phi$ a.e. and the claim is proved.

To conclude the proof of the lemma, notice that

$$\limsup_{n\to\infty} \overline{h_n} \leq \theta(g)$$

and $\lim_{n\to\infty} \int \overline{h_n}(\omega)\, d\lambda(\omega) = \int \theta(g)(\omega)\, d\lambda(\omega)$ so we get that $\lim_{n\to\infty} \|\overline{h_n} - \theta(g)\|_1 = 0$ and the lemma is proved. ∎

In a similar fashion we set for $g_n(\omega) = \sum_{i=p_n}^{q_n} \lambda_i f_i$ block convex combination of the f_n's

$$\text{(3)} \qquad \widetilde{g_n}(\omega) = \inf_{k\geq q_n} \inf\Big\{\Big\langle T\big(g_n(\omega)\big), e_k\Big\rangle, T \in V \cap K(\omega)\Big\}$$

$$\text{(4)} \qquad \varphi(g)(\omega) = \liminf_{n\to\infty} \widetilde{g_n}(\omega)$$

We have the corresponding lemma:

LEMMA 3. *There exists* $(g_n) \ll (f_n)$ *such that if* $(h_n) \ll (g_n)$ *we have* $\lim_{n\to\infty} \|\varphi(g) - \widetilde{h_n}\|_1 = 0$.

We are now ready to present the main construction of the proof. Let us fix $a < b$ and let τ be the first uncountable ordinal. Set $h_n^0 = f_n$ and $K_0(\omega) = \mathcal{L}(X, \ell^1)_1$. For $\alpha < \tau$, we will construct (as in [18]) sequences $h^\alpha = (h_n^\alpha)$, and measurable maps $K_\alpha \colon \Omega \longrightarrow \mathcal{K}$ with the following properties:

$$\text{(5)} \qquad \text{for } \beta < \alpha < \tau, \quad h^\alpha \ll h^\beta.$$

For $\alpha < \tau$ and $h \ll f$ (say $h_n = \sum_{i=a_n}^{b_n} \lambda_i f_i$ the representation of $(h_n)_n$ as block convex combination of $(f_n)_n$), if we define:

$$\text{(6)} \qquad \bar{h}_{n,k,\alpha}(\omega) = \sup_{m\geq b_n} \sup\Big\{\Big\langle T\big(h_n(\omega)\big), e_m\Big\rangle, T \in U_k \cap K_\alpha(\omega)\Big\}$$

$$\tilde{h}_{n,k,\alpha}(\omega) = \inf_{m\geq b_n} \inf\Big\{\Big\langle T\big(h_n(\omega)\big), e_m\Big\rangle, T \in U_k \cap K_\alpha(\omega)\Big\}$$

$$\theta_{k,\alpha}(h)(\omega) = \limsup_{n\to\infty} \bar{h}_{n,k,\alpha}(\omega)$$

$$\varphi_{k,\alpha}(h)(\omega) = \liminf_{n\to\infty} \tilde{h}_{n,k,\alpha}(\omega)$$

then for each α of the form $\beta + 1$, each $k \geq 1$ and each $h \ll h^\alpha$, we have $\lim_{n\to\infty} \|\theta_{k,\beta}(h^\alpha) - \bar{h}_{n,k,\beta}\|_1 = 0$; $\lim_{n\to\infty} \|\varphi_{k,\beta}(h^\alpha) - \tilde{h}_{n,k,\beta}\|_1 = 0$.

If α is limit, we set

$$\text{(7)} \qquad K_\alpha(\omega) = \bigcap_{\beta<\alpha} K_\beta(\omega)\,;$$

If $\alpha = \beta + 1$, we have

$$\text{(8)} \qquad K_\alpha(\omega) = \{T \in K_\beta(\omega), T \in U_k \Rightarrow \theta_{k,\beta}(h^\alpha) > b, \varphi_{k,\beta}(h^\alpha) < a\}.$$

The construction is done by induction. Suppose that the construction has been done for each ordinal $\beta < \alpha$. If α is limit, we set $K_\alpha(\omega) = \bigcap_{\beta<\alpha} K_\beta(\omega)$. Let β_n be an increasing sequence of ordinals with $\alpha = \sup \beta_n$. By Lemma 1, there exists (h^α) with $(h^\alpha) \ll (h^{\beta_n})$ for each $n \in \mathbb{N}$. Therefore for $\beta < \beta_n$, $(h^\alpha) \ll (h^{\beta_n}) \ll (h^\beta)$ and hence $(h^\alpha) \ll (h^\beta)$ so (5) is satisfied. The construction is done in the case of limit ordinal.

Suppose now that $\alpha = \beta + 1$. Using Lemma 2 and Lemma 3, one can construct a sequence (g^k) with $(g^1) = (h^\beta)$, $(g^{k+1}) \ll (g^k)$ and such that for $(h) \ll (g^k)$,

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$\lim_{n\to\infty} \|\theta_{k,\beta}(g^k) - \bar{h}_{n,k,\beta}\|_1 = 0$; $\lim_{n\to\infty} \|\varphi_{k,\beta}(g^k) - \tilde{h}_{n,k,\beta}\|_1 = 0$. Apply Lemma 1 to get a sequence (h^α) with $(h^\alpha) \ll (g^k)$ for each $k \geq 1$ and we claim that (h^α) satisfy (6). To see the claim let us fix $(h) \ll (h^\alpha)$. By the definition of (h^α), we have for each $k \geq 1$, $(h) \ll (g^k)$. It follows that $\lim_{n\to\infty} \|\theta_{k,\beta}(g^k) - \bar{h}_{n,k,\beta}\|_1 = 0$ and $\lim_{n\to\infty} \|\varphi_{k,\beta}(g^k) - \tilde{h}_{n,k,\beta}\|_1 = 0$. Since $(h^\alpha) \ll (g^k)$, we get that $\lim_{n\to\infty} \|\theta_{k,\beta}(g^k) - \bar{h}^\alpha_{n,k,\beta}\|_1 = 0$ and $\lim_{n\to\infty} \|\varphi_{k,\beta}(g^k) - \tilde{h}^\alpha_{n,k,\beta}\|_1 = 0$ which shows that $\theta_{k,\beta}(g^k) = \theta_{k,\beta}(h^\alpha)$ and $\varphi_{k,\beta}(g^k) = \varphi_{k,\beta}(h^\alpha)$ and the claim is proved.

Define now $K_\alpha(\omega)$ by (8). The set $K_\alpha(\omega)$ is easily seen to be closed and the measurability of $K_\alpha(\cdot)$ can be proved using similar argument as in [18]. The construction is complete.

CLAIM. *There exists $\alpha < \tau$ such that $K_\alpha(\omega) = K_{\alpha+1}(\omega)$ for a.e. $\omega \in \Omega$.*

In fact if we set for each $k \geq 1$, $\Omega^\alpha_k = \{\omega\,;\, U_k \cap K_\alpha(\omega) = \emptyset\}$ then for each $k \in \mathbb{N}$, the sequence $\big(\lambda(\Omega^\alpha_k)\big)_{\alpha<\tau}$ is increasing, hence eventually constant. Fix α such that for each $k \in \mathbb{N}$, we have $\lambda(\Omega^{\alpha+1}_k) = \lambda(\Omega^\alpha_k)$. It is clear that for $\omega \notin \bigcup_{k\geq 1}(\Omega^{\alpha+1}_k \setminus \Omega^\alpha_k)$, we have $K_\alpha(\omega) = K_{\alpha+1}(\omega)$. The claim is proved.

It should be noted that the approach used above is very similar to Debs' proof in [4].

We now set $(h) = (h^{\alpha+1})$, $C = \{\omega\,;\, K_\alpha(\omega) = \emptyset\}$ and $M = \{\omega\,;\, K_\alpha(\omega) = K_{\alpha+1}(\omega) \neq \emptyset\}$. Clearly $\lambda(C \cup M) = 1$ and for the rest of the proof we set $h_n = \sum_{i=p_n}^{q_n} \lambda_i f_i$ be the representation of $(h_n)_n$ as block convex combination of $(f_n)_n$. We have the following property of the measurable subset C:

LEMMA 4. *If $\omega \in C$ and $T \in \mathcal{L}(X, \ell^1)_1$ and $u \ll h$, then either*
(a) $\limsup_{n\to\infty}\big\langle T\big(u_n(\omega)\big), e_n\big\rangle \leq b$ *or*
(b) $\liminf_{n\to\infty}\big\langle T\big(u_n(\omega)\big), e_n\big\rangle \geq a$.

PROOF. Let $\omega \in C$ and $T \in \mathcal{L}(X, \ell^1)_1$. Fix $u \ll h\ (\ll f)$ say $u_n = \sum_{i=a_n}^{b_n} \alpha_i f_i$ for all $n \in \mathbb{N}$. Consider $S\colon c_0 \longrightarrow c_0$ defined as follows: $Se_{b_n} = e_n$ $\forall n \in \mathbb{N}$ and $Se_j = 0$ for $j \neq b_n$. The operator S is trivially linear and $\|S\| = 1$. Since $S^* \circ T \in K_0(\omega)$ and $S^* \circ T \notin K_\alpha(\omega)$, there is a least ordinal β for which $S^* \circ T \notin K_\beta(\omega)$. The ordinal β cannot be a limit so $\beta = \gamma + 1$ and $S^* \circ T \in K_\gamma(\omega)$. By the definition of $K_\beta(\cdot)$, there exists $k \in \mathbb{N}$ with $S^* \circ T \in U_k$ but either $\theta_{k,\gamma}(h^\beta)(\omega) \leq b$ or $\varphi_{k,\gamma}(h^\beta)(\omega) \geq a$. Now since $u \ll h^\beta$, we get that either

$$\limsup_{n\to\infty}\big\langle T\big(u_n(\omega)\big), e_n\big\rangle = \limsup_{n\to\infty}\langle S^* \circ T\big(u_n(\omega)\big), e_{b_n}\rangle \leq \theta_{k,\gamma}(u)(\omega) \leq \theta_{k,\gamma}(h^\beta)(\omega) \leq b$$

or

$$\liminf_{n\to\infty}\big\langle T\big(u_n(\omega)\big), e_n\big\rangle = \liminf_{n\to\infty}\big\langle S^* \circ T\big(u_n(\omega)\big), e_{b_n}\big\rangle \geq \varphi_{k,\gamma}(u)(\omega) \geq \varphi_{k,\gamma}(h^\beta) \geq a.$$

The lemma is proved. ∎

For the set M, we have the following lemma:

LEMMA 5. *There exists a subsequence $\big(n(i)\big)$ of the integers so that for almost every $\omega \in M$, there exists $k \in \mathbb{N}$ such that the sequence $(h_{n(i)}(\omega) \otimes e_i)_{i\geq k}$ is δ equivalent to the ℓ^1-basis in $X\hat{\otimes}_\pi c_0$, where $\delta = (b-a)/2$.*

PROOF. Again we adopt the methods in [18] to our situation. Let us denote by F the set of finite sequences of zeroes and ones. For $s \in F$, we will denote by $|s|$ the length of s. For $s = (s_1, \ldots, s_n)$ and $r = (r_1, \ldots, r_m)$ with $n \leq m$, we say that $s < r$ if $s_i = r_i$ for $i \leq n$. We will construct two sequences of integers $n(i)$, $m(i)$, measurable sets $B_i \subset M$ and measurable maps $Q(s, \cdot): M \longrightarrow \mathbb{N}$ such that the following conditions are satisfied:

$$q_{n(1)} < m(1) < q_{n(2)} < m(2) < \cdots < m(i) < q_{n(i+1)} < \cdots \tag{9}$$

$$\forall s \in F, \quad \sup\{Q(s,\omega)\,;\, \omega \in M\} < \infty\,; \tag{10}$$

$$\lambda(M \setminus B_i) \leq 2^{-i}\,; \tag{11}$$

$$\text{For } s, r \in F, s < r, \text{ and } \omega \in \bigcap_{|s| \leq i \leq |r|} B_i, \text{ one has } U_{Q(r,\omega)} \subset U_{Q(s,\omega)}\,; \tag{12}$$

$$\forall \omega \in M, s \in F, \quad K_\alpha(\omega) \cap U_{Q(s,\omega)} \neq \emptyset\,; \tag{13}$$

$$\forall p, \ \forall i \leq p, \quad \forall \omega \in \bigcap_{i \leq j \leq p} B_j, \tag{14}$$

$$s_i = 1 \Rightarrow \forall T \in U_{Q(s,\omega)}, \qquad \sup_{q_{n(i)} \leq k \leq m(i)} \Big\langle T\big(h_{n(i)}(\omega)\big), e_k \Big\rangle \geq b$$

$$s_i = 0 \Rightarrow \forall T \in U_{Q(s,\omega)}, \qquad \inf_{q_{n(i)} \leq k \leq m(i)} \Big\langle T\big(h_{n(i)}(\omega)\big), e_k \Big\rangle \leq a.$$

Again the construction is done by induction. Before doing so we need the following notation:

Let $n \in \mathbb{N}, j \in \mathbb{N}$ and $\alpha < \tau$. Fix $m \geq n$, the following notation will be used.

$$\bar{h}^{(m)}_{n,j,\alpha}(\omega) = \sup_{q_n \leq k \leq m} \sup\Big\{\big\langle T\big(h_n(\omega)\big), e_k\big\rangle, T \in U_j \cap K_\alpha(\omega)\Big\}$$

$$\tilde{h}^{(m)}_{n,j,\alpha}(\omega) = \inf_{q_n \leq k \leq m} \inf\Big\{\big\langle T\big(h_n(\omega)\big), e_k\big\rangle, T \in U_j \cap K_\alpha(\omega)\Big\}\,;$$

It is clear that $\bar{h}_{n,j,\alpha}(\omega) = \lim_{m\to\infty} \bar{h}^{(m)}_{n,j,\alpha}(\omega)$ a.e. and $\tilde{h}_{n,j,\alpha}(\omega) = \lim_{m\to\infty} \tilde{h}^{(m)}_{n,j,\alpha}(\omega)$ a.e.

For $i = 1$, recall that $U_0 = \mathcal{L}(X, \ell^1)_1$. Since $K_{\alpha+1}(\omega) \neq \emptyset$ for $\omega \in M$, one has $\theta_{0,\alpha}(h)(\omega) > b$ and $\varphi_{0,\alpha}(h)(\omega) < a$ but since

$$\lim_{n\to\infty} \|\theta_{0,\alpha}(h) - \bar{h}_{n,0,\alpha}\|_1 = \lim_{n\to\infty} \|\varphi_{0,\alpha}(h) - \tilde{h}_{n,0,\alpha}\|_1 = 0,$$

there exists an integer $n(1)$ such that if we set

$$B_1'' = \{\omega \in M, \bar{h}_{n(1),0,\alpha}(\omega) > b\,;\, \tilde{h}_{n(1),0,\alpha}(\omega) < a\}$$

we have $\lambda(M \setminus B_1'') \leq 2^{-3}$. Since $\bar{h}_{n(1),0,\alpha}(\omega) = \lim_{m\to\infty} \bar{h}^{(m)}_{n(1),0,\alpha}(\omega)$ a.e. and $\tilde{h}_{n(1),0,\alpha}(\omega) = \lim_{m\to\infty} \tilde{h}^{(m)}_{n(1),0,\alpha}(\omega)$ a.e., there exists an integer $m(1) > q_{n(1)}$ such that if we set

$$B_1' = \{\omega \in M, \bar{h}^{m(1)}_{n(1),0,\alpha}(\omega) > b\,;\, \tilde{h}^{m(1)}_{n(1),0,\alpha}(\omega) < a\},$$

we have $\lambda(M \setminus B_1') \leq 2^{-2}$.

Now for $\omega \in B'_1$, we have:

$$\sup_{q_{n(1)} \leq k \leq m(1)} \sup\left\{\left\langle T\big(h_{n(1)}(\omega)\big), e_k\right\rangle, T \in K_\alpha(\omega)\right\} > b$$

and

$$\inf_{q_{n(1)} \leq k \leq m(1)} \inf\left\{\left\langle T\big(h_{n(1)}(\omega)\big), e_k\right\rangle, T \in K_\alpha(\omega)\right\} < a.$$

For each $x \in X$, the maps $T \longrightarrow \sup_{q_{n(1)} \leq k \leq m(1)} \langle Tx, e_k\rangle$ and $T \longrightarrow \inf_{q_{n(1)} \leq k \leq m(1)} \langle Tx, e_k\rangle$ are continuous so the sets

$$\left\{T \in \mathcal{L}(X, \ell^1)_1, \quad \sup_{q_{n(1)} \leq k \leq m(1)} \langle Tx, e_k\rangle > b\right\}$$
$$\left\{T \in \mathcal{L}(X, \ell^1)_1, \quad \inf_{q_{n(1)} \leq k \leq m(1)} \langle Tx, e_k\rangle < a\right\}$$

are open subsets. By standard techniques one can choose measurable maps $Q_0(\cdot)$ and $Q_1(\cdot)$ from M to $\mathbb{N}$ such that

$$T \in U_{Q_1(\omega)} \Rightarrow \sup_{q_{n(1)} \leq k \leq m(1)} \left\langle T\big(h_{n(1)}(\omega)\big), e_k\right\rangle > b\,; \quad U_{Q_1(\omega)} \cap K_\alpha(\omega) \neq \emptyset$$
$$T \in U_{Q_0(\omega)} \Rightarrow \inf_{q_{n(1)} \leq k \leq m(1)} \left\langle T\big(h_{n(1)}(\omega)\big), e_k\right\rangle < a\,; \quad U_{Q_0(\omega)} \cap K_\alpha(\omega) \neq \emptyset.$$

There exists an integer l such that if $B_1 = \{\omega \in B'_1\ ;\ Q_0(\omega) < l, Q_1(\omega) < l\}$ we have $\lambda(M \setminus B_1) \leq 2^{-1}$. We define $Q\big((0), \omega\big) = Q_0(\omega)$ and $Q\big((1), \omega\big) = Q_1(\omega)$ for $\omega \in B_1$ and $Q\big((0), \omega\big) = Q\big((1), \omega\big) = 0$ for $\omega \in M \setminus B_1$. The required conditions (9)-(14) are satisfied.

Suppose now that the result has been proved for i. Let $l = \sup\{Q(s, \omega), |s| = i, \omega \in B_i\}$. Since $K_\alpha(\omega) = K_{\alpha+1}(\omega)$, for $\omega \in M$, condition (8) implies that for each $k \in \mathbb{N}$,

$$U_k \cap K_\alpha(\omega) \neq \emptyset \Rightarrow \theta_{k,\alpha}(h)(\omega) > b, \quad \varphi_{k,\alpha}(h)(\omega) < a.$$

We deduce as in the case $i = 1$ that there is an integer $n(i+1)$ such that $q_{n(i+1)} > m(i)$ and the set

$$B''_{i+1} = \{\omega \in M,\ \forall k \leq l, U_k \cap K_\alpha(\omega) \neq \emptyset \Rightarrow \bar{h}_{n(i+1),k,\alpha}(\omega) > b, \tilde{h}_{n(i+1),k,\alpha}(\omega) < a\}$$

satisfies $\lambda(M \setminus B''_{i+1}) \leq 2^{-i-3}$. Using similar argument as in the case $i = 1$, one can pick an integer $m(i+1) > q_{n(i+1)}$ so that the set

$$B'_{i+1} = \{\omega \in B''_{i+1},\ \forall k \leq l, U_k \cap K_\alpha(\omega) \neq \emptyset \Rightarrow \bar{h}^{(m(i+1))}_{n(i+1),k,\alpha}(\omega) > b, \tilde{h}^{(m(i+1))}_{n(i+1),k,\alpha}(\omega) < a\}$$

satisfies $\lambda(M \setminus B'_{i+1}) \leq 2^{-i-2}$.

For $\omega \in B'_{i+1}$, one has in particular for $s \in F$, $|s| = i$:

$$\bar{h}^{(m(i+1))}_{n(i+1),Q(s,\omega),\alpha}(\omega) > b\,; \quad \tilde{h}^{(m(i+1))}_{n(i+1),Q(s,\omega),\alpha}(\omega) < a.$$

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It follows that for $s \in F$, $|s| = i$, there exist measurable maps $Q_0(s,\cdot)$ and $Q_1(s,\cdot)$ from M to $\mathbb{N}$ such that

$$T \in U_{Q_0(s,\omega)} \Rightarrow \inf_{q_{n(i+1)} \le k \le m(i+1)} \left\langle T\big(h_{n(i+1)}(\omega)\big), e_k \right\rangle < a,$$
$$U_{Q_0(s,\omega)} \cap K_\alpha(\omega) \neq \emptyset \; ; \quad U_{Q_0(s,\omega)} \subset U_{Q(s,\omega)}$$

and

$$T \in U_{Q_1(s,\omega)} \Rightarrow \sup_{q_{n(i+1)} \le k \le m(i+1)} \left\langle T\big(h_{n(i+1)}(\omega)\big), e_k \right\rangle > b,$$
$$U_{Q_1(s,\omega)} \cap K_\alpha(\omega) \neq \emptyset \; ; U_{Q_1(s,\omega)} \subset U_{Q(s,\omega)}.$$

There exists an integer l' such that if we let

$$B_{i+1} = \{\omega \in B'_{i+1} \; ; \forall s \in F, |s| = i, Q_0(s,\omega), Q_1(s,\omega) \le l'\}$$

then $\lambda(M \setminus B_{i+1}) \le 2^{-i-1}$. The construction is done by setting

$$Q\big((s,0),\omega\big) = Q\big((s,1),\omega\big) = 0 \quad \text{if } \omega \in M \setminus B_{i+1}$$
$$Q\big((s,0),\omega\big) = Q_0(s,\omega) \; ; Q\big((s,1),\omega\big) = Q_1(s,\omega) \quad \text{if } \omega \in B_{i+1}.$$

Let $L = \bigcup_k \bigcap_{i \ge k} B_i$. It is clear that $\lambda(M \setminus L) = 0$ and we claim that if $\omega \in \bigcap_{i \ge k} B_i$, the sequence $\big(h_{n(i)}(\omega) \otimes e_i\big)_{i \ge k}$ is δ-equivalent to the unit vector basis of ℓ^1 in the Banach space $X \hat{\otimes}_\pi c_0$. To see the claim, let $p \ge k$ and consider a subset P of $[k,p]$. Let $s \in F$ be a sequence with $|s| = p$ and satisfies $s_i = 1$ if $i \in P$, $s_i = 0$ if $i \notin P$. From (14), there exists $T \in \mathcal{L}(X, \ell^1)_1$ with:

$$k \le i \le p, \quad i \in P \Rightarrow \sup_{q_{n(i)} \le m \le m(i)} \left\langle T\big(h_{n(i)}(\omega)\big), e_m \right\rangle \ge b$$
$$k \le i \le p, \quad i \notin P \Rightarrow \inf_{q_{n(i)} \le m \le m(i)} \left\langle T\big(h_{n(i)}(\omega)\big), e_m \right\rangle \le a.$$

Now for $i \in [k,p]$, choose $k(i) \in [q_{n(i)}, m(i)]$ so that

$$k \le i \le p, \quad i \in P \Rightarrow \sup_{q_{n(i)} \le m \le m(i)} \left\langle T\big(h_{n(i)}(\omega)\big), e_m \right\rangle = \left\langle T\big(h_{n(i)}(\omega)\big), e_{k(i)} \right\rangle$$
$$k \le i \le p, \quad i \notin P \Rightarrow \inf_{q_{n(i)} \le m \le m(i)} \left\langle T\big(h_{n(i)}(\omega)\big), e_m \right\rangle = \left\langle T\big(h_{n(i)}(\omega)\big), e_{k(i)} \right\rangle.$$

By (9), the sequence $k(i)$ is increasing so there exists an operator $S: c_0 \longrightarrow c_0$ of norm one such that $Se_i = e_{k(i)}$ and it is now clear that:

$$\text{(15)} \qquad k \le i \le p, \quad i \in P \Rightarrow \left\langle S^* \circ T\big(h_{n(i)}(\omega)\big), e_i \right\rangle \ge b$$
$$k \le i \le p, \quad i \notin P \Rightarrow \left\langle S^* \circ T\big(h_{n(i)}(\omega)\big), e_i \right\rangle \le a.$$

And the claim follows from Rosenthal's argument in [15] (see also [5] p. 205). The lemma is proved. ∎

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REMARK 1. Let $u \ll (h_{n(i)})_{i\in\mathbb{N}}$. Using the same argument as above, one can show that there exists a subsequence $(v_i)_i$ of $(u_i)_i$ such that $\big(v_i(\omega)\otimes e_i\big)_i$ is equivalent to the ℓ^1 basis in $X\hat{\otimes}c_0$ for a.e. $\omega \in L$.

To complete the proof of the theorem, let $\big(a(k), b(k)\big)$ be an enumeration of all pairs of rational numbers with $a < b$. By induction we construct sequences g^k and measurable sets C_k, L_k satisfying the following:

(i) $C_{k+1} \subset C_k$, $L_k \subset L_{k+1}$ and $\lambda(C_k \cup L_k) = 1$;

(ii) $\forall\omega \in C_k$, $\forall m \le k$, and $T \in \mathcal{L}(X, \ell^1)_1$ then either $\limsup_{n\to\infty}\big\langle T\big(g_n^m(\omega)\big), e_n\big\rangle \le b(k)$ or $\liminf_{n\to\infty}\big\langle T\big(g_n^m(\omega)\big), e_n\big\rangle \ge a(k)$;

(iii) $\forall\omega \in L_m \setminus L_{m-1}$ with $2 \le m \le k$, the sequence $\big(g_n^k(\omega)\otimes e_n\big)_n$ is $\big(b(m)-a(m)\big)/2$-equivalent to the unit vector basis of ℓ^1;

(iv) $g^{k+1} \ll g^k$.

Let $g^0 = f$, the steps above shows that one can find $g^1 \ll f$, measurable subsets C_1 and L_1 satisfying (i)–(iv). Suppose that g^k, C_k, L_k have been constructed. Again by the same reasoning for $a = a(k+1)$, $b = b(k+1)$ and $f_n = g_n^k$, there exist $g^{k+1} \ll g^k$ and measurable subsets C_{k+1}, L_{k+1} with $\lambda(C_{k+1} \cup L_{k+1}) = 1$ and by Lemma 4 and Lemma 5, conditions (i)–(iv) are satisfied.

We set $C = \bigcap_{k\ge1} C_k$, $L = \bigcup_{k\ge1} L_k$ and $g_n = g_n^n$. It is clear that $\lambda(C \cup L) = 1$ and $g_n \ll g_n^k$ for each $k \in \mathbb{N}$; in particular $g_n \ll f_n$. For $\omega \in C$, we have $\big(g_n(\omega)\otimes e_n\big)_n$ is weakly Cauchy. In fact since $g_n \ll g_n^k$, Lemma 4 asserts that for each $T \in \mathcal{L}(X, \ell^1)$ either $\limsup_{n\to\infty}\big\langle T\big(g_n(\omega)\big), e_n\big\rangle \le b(k)$ or $\liminf_{n\to\infty}\big\langle T\big(g_n(\omega)\big), e_n\big\rangle \ge a(k)$ for all $k \in \mathbb{N}$. Hence $\limsup_{n\to\infty}\big\langle T\big(g_n(\omega)\big), e_n\big\rangle = \liminf_{n\to\infty}\big\langle T\big(g_n(\omega)\big), e_n\big\rangle$. Now for $\omega \in L$, there exists k such that $\omega \in L_k$ and since $g_n \ll g_n^k$, by Lemma 5, $\big(g_n(\omega)\otimes e_n\big)_{n\ge m}$ is equivalent to the unit vector basis of ℓ^1 for some $m \in \mathbb{N}$. The proof of Theorem 1 is complete for the separable case.

CASE 2. General case;

One can reduce the general case to the separable one using the following result of Heinrich and Mankiewicz (see Proposition 3.4 of [10]):

LEMMA 6. *Let X be a Banach space and X_0 be a separable subspace of X. Then there exist a separable subspace Z of X that contains X_0 and an isometric embedding $J\colon Z^* \longrightarrow X^*$ such that $\langle z, Jz^*\rangle = \langle z, z^*\rangle$ for every $z \in Z$ and $z^* \in Z^*$. In particular $J(Z^*)$ is 1-complemented in X^*.*

Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1(\lambda, X)$. Since each f_n has (essentially) separable range, there exists a separable subspace X_0 of X such that for a.e. $\omega \in \Omega$, $f_n(\omega) \in X_0$. Let Z be a separable subspace as in the above lemma. The sequence $(f_n)_n$ is bounded in $L^1(\lambda, Z)$ so by Case 1, there exist $g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$, measurable subsets C and L of Ω with $\lambda(C\cup L) = 1$ such that for $\omega \in C$, the sequence $\big(g_n(\omega)\otimes e_n\big)_n$ is weakly Cauchy in $Z\hat{\otimes}_\pi c_0$ and for $\omega \in L$, the sequence $\big(g_n(\omega)\otimes e_n\big)_n$ is equivalent to the ℓ^1 basis in $Z\hat{\otimes}_\pi c_0$.

We claim that the same conclusion holds if we replace $Z\hat{\otimes}_\pi c_0$ by $X\hat{\otimes}_\pi c_0$. In fact if $\omega \in C$ and $T \in \mathcal{L}(X, \ell^1) = (X\hat{\otimes}_\pi c_0)^*$, the operator $T|_Z$ (the restriction of T on Z) belongs to $(Z\hat{\otimes}_\pi c_0)^*$ so we have

$$\begin{aligned}\lim_{n\to\infty}\langle T, g_n(\omega)\otimes e_n\rangle &= \lim_{n\to\infty}\left\langle T\big(g_n(\omega)\big), e_n\right\rangle\\ &= \lim_{n\to\infty}\langle T|_Z, g_n(\omega)\otimes e_n\rangle.\end{aligned}$$

Hence $\lim_{n\to\infty}\langle T, g_n(\omega)\otimes e_n\rangle$ exists so the sequence $(g_n(\omega)\otimes e_n)_n$ is weakly Cauchy in $X\hat{\otimes}_\pi c_0$.

Now for $\omega \in L$, let $(a_n)_n$ be a finite sequence of scalars. We have:

$$\begin{aligned}\Big\|\sum a_n g_n(\omega)\otimes e_n\Big\|_{X\hat{\otimes}_\pi c_0} &= \sup\Big\{\sum a_n\langle T, g_n(\omega)\otimes e_n\rangle\ ;\ T\in\mathcal{L}(X,\ell^1)_1\Big\}\\ &= \sup\Big\{\sum a_n\langle g_n(\omega), Se_n\rangle\ ;\ S\in\mathcal{L}(c_0, X^*)_1\Big\}\\ &\geq \sup\Big\{\sum a_n\langle g_n(\omega), J\circ Le_n\rangle\ ;\ L\in\mathcal{L}(c_0, Z^*)_1\Big\}\\ &= \sup\Big\{\sum a_n\langle g_n(\omega), Le_n\rangle\ ;\ L\in\mathcal{L}(c_0, Z^*)_1\Big\}\\ &= \Big\|\sum a_n g_n(\omega)\otimes e_n\Big\|_{Z\hat{\otimes}_\pi c_0} \geq \delta\sum|a_n|\end{aligned}$$

for some $\delta > 0$. So the sequence $\big(g_n(\omega)\otimes e_n\big)_n$ is equivalent to the ℓ^1 basis in $X\hat{\otimes}_\pi c_0$. The theorem is proved. ∎

REMARK. In [18], Talagrand extended his main theorem to the case of functions that are weak*-scalarly measurable. It is not clear to us if one can get a similar result as in Theorem 1 for weak*-scalarly measurable functions.

3. Applications: property (V^*) and (V^*)-sets for $L^1(\lambda, X)$.

DEFINITION 1. Let X be a Banach space. A series $\sum_{n=1}^\infty x_n$ in X is said to be *Weakly Unconditionally Cauchy* (W.U.C.) if for every $x^* \in X^*$, the series $\sum_{n=1}^\infty |x^*(x_n)|$ is convergent.

There are many criteria for a series to be a W.U.C. series (see for instance [5] or [19]).

DEFINITION 2. Assume that X and Y are Banach spaces. A bounded linear map $T: X \longrightarrow Y$ is said to be *Unconditionally* converging if T sends W.U.C. series in X to unconditionally convergent series in Y.

In his fundamental paper [13], Pełczyński proved the following proposition:

PROPOSITION 2. *For a Banach space X, the following assertions are equivalent:*

(i) A subset $H \subset X^*$ is relatively weakly compact whenever $\lim_{n\to\infty}\sup_{x^*\in H}|x^*(x_n)| = 0$ *for every* W.U.C. *series $\sum_{n=1}^\infty x_n$ in X;*

(ii) For any Banach space Y, every bounded operator $T: X \longrightarrow Y$ that is unconditionally converging is weakly compact.

DEFINITION 3. A Banach space X is said to have property (V) if it satisfies one of the equivalent conditions of Proposition 2.

As a dual property, we have the following definition:

DEFINITION 4. A Banach space X is said to have property (V^*) if a subset K of X is relatively weakly compact whenever $\lim_{n\to\infty} \sup_{x\in K} |x(x_n^*)| = 0$ for every W.U.C. series $\sum_{n=1}^{\infty} x_n^*$ in X^*.

DEFINITION 5. A subset K of a Banach space X is called a (V^*)-set if for every W.U.C. series $\sum_{n=1}^{\infty} x_n^*$ in X^*, the following holds: $\lim_{n\to\infty} \sup_{x\in K} |x(x_n^*)| = 0$.

Hence a Banach space X has property (V^*) if and only if every (V^*)-set in X is relatively weakly compact.

From a result of Emmanuele [8] (see also Godefroy and Saab [9]), one can deduce the following characterization of spaces that have property (V^*).

PROPOSITION 3. *A Banach space X has property (V^*) if and only if X is weakly sequentially complete and given any sequence $(x_n)_n$ in X that is equivalent to the unit vector basis of ℓ^1, there exists an operator $T: X \longrightarrow \ell^1$ such that $(Tx_n)_n$ is not relatively compact in ℓ^1.*

The above proposition shows in particular that a Banach space X has property (V^*) if and only if X is weakly sequentially complete and every sequence that is equivalent to the unit vector basis of ℓ^1 has a subsequence equivalent to a complemented copy of ℓ^1.

In this section we will concentrate on property (V^*) and we shall refer the reader to [2] and [13] for more on property (V).

In [16], Saab and Saab showed (see Proposition 3 of [16]) that a Banach space with the separable complementation property has property (V^*) if and only if each of its separable subspaces has property (V^*). On the next proposition, we will show that property (V^*) is in fact separably determined.

PROPOSITION 4. *A Banach space X has property (V^*) if and only if all of its separable subspace has property (V^*).*

PROOF. Since property (V^*) is easily seen to be stable by subspaces, one implication is immediate.

For the converse, we will use the result of Heinrich and Mankiewicz stated in Lemma 6 above.

Assume that every separable subspace of X has property (V^*). The space X is trivially weakly sequentially complete. Let K be a bounded subset of X that is not relatively weakly compact. There exists a sequence $(x_n)_n$ in K that is equivalent to the unit vector basis of ℓ^1 in X and let $X_0 = \overline{\text{span}}\{x_n\ ;\ n \in \mathbb{N}\}$. The space X_0 is separable and consider Z as in Lemma 6. Since Z is separable, by assumption it has property (V^*) and therefore there exists a W.U.C. series $\sum_{k=1}^{\infty} z_k^*$ in Z^* such that $\limsup_{k\to\infty} \sup_{n\in\mathbb{N}} \langle z_k^*, x_n\rangle > 0$. Let $x_k^* = J(z_k^*)$; the series $\sum_{k=1}^{\infty} x_k^*$ is a W.U.C. series in X^* and

$$\langle x_k^*, x_n\rangle = \langle J(z_k^*), x_n\rangle = \langle z_k^*, x_n\rangle.$$

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So $\limsup_{k\to\infty} \sup_{n\in\mathbb{N}} \langle x_k^*, x_n\rangle > 0$ which shows that K is not a (V^*)-set. ∎

We are now ready to present the main theorem of this section.

Let E be a Banach lattice with weak unit. By the classical representation (see [12]), there exists a probability space $(\Omega, \Sigma, \lambda)$ such that $L^\infty(\lambda) \subset E \subset L^1(\lambda)$ with E being an ideal and the inclusion being continuous.

Define $E(X)$ to be the space of (class of) measurable map $f\colon \Omega \longrightarrow X$ so that the measurable function $V(f)$ defined by $V(f)(\omega) = \|f(\omega)\|_X$ belongs to E. The space $E(X)$ endowed with the norm $\|f\| = \|V(f)\|_E$ is a Banach space.

We have the following stability result:

THEOREM 2. *Let X be a Banach space and E be a Banach lattice that does not contain any copy of c_0. The space X has property (V^*) if and only if $E(X)$ has property (V^*).*

PROOF. If $E(X)$ has property (V^*), then the space X has property (V^*) since property (V^*) is stable by subspace.

Conversely, assume that X has property (V^*). By Proposition 4, we can assume without loss of generalities that E and X are separable. By the classical representation, there exists a probability space $(\Omega, \Sigma, \lambda)$ such that $L^\infty(\lambda) \subset E \subset L^1(\lambda)$ and it is clear that $L^\infty(\lambda, X) \subset E(X) \subset L^1(\lambda, X)$. Since X is weakly sequentially complete and E does not contain any copy of c_0, the space $E(X)$ is weakly sequentially complete (see [18]).

Let $(f_n)_n$ be a bounded sequence in $E(X)$ that is equivalent to the unit vector basis of ℓ^1. We will show that $(f_n)_n$ is not a (V^*)-set. If $(f_n)_n$ is not uniformly integrable then $(f_n)_n$ cannot be a (V^*)-set (see Proposition 3.1 of [1]) so we will assume that $(f_n)_n$ is uniformly integrable.

By Talagrand's theorem, there exists a sequence $g_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots)$ and a measurable subset Ω' of Ω, with $\lambda(\Omega') > 0$ and such that for each $\omega \in \Omega'$, there exists $k \in \mathbb{N}$ so that $\big(g_n(\omega)\big)_{n\geq k}$ is equivalent to the unit vector basis of ℓ^1 in X. Define

$$\varphi_n = g_n \chi_{\Omega'}, \quad n \in \mathbb{N}.$$

Applying Theorem 1 to the sequence $(\varphi_n)_n$, there exist C and L measurable subsets of Ω with $\lambda(C \cup L) = 1$ and a sequence $\psi_n \in \operatorname{conv}(\varphi_n, \varphi_{n+1}, \ldots)$ so that

(1) for $\omega \in C$, $\big(\psi_n(\omega) \otimes e_n\big)_n$ is weakly Cauchy in $X\hat{\otimes}_\pi c_0$;
(2) for $\omega \in L$, there exists $k \in \mathbb{N}$ so that $\big(\psi_n(\omega) \otimes e_n\big)_{n\geq k}$ is equivalent to the unit vector basis of ℓ^1 in $X\hat{\otimes}_\pi c_0$.

CASE 1. Assume that $\lambda(L) > 0$:

It is clear (see for instance [18]) that the sequence $(\psi_n \otimes e_n)_n$ is equivalent to the ℓ^1 basis in $L^1(\lambda, X\hat{\otimes}_\pi c_0)$ and by identification, the sequence $(\psi_n \otimes e_n)_n$ is equivalent to the ℓ^1 basis in $L^1(\lambda, X)\hat{\otimes}_\pi c_0$ so it cannot be a weakly null sequence. Therefore the sequence $(\psi_n)_n$ contains a subsequence that is equivalent to a complemented copy of ℓ^1 in $L^1(\lambda, X)$. Now since the inclusion map from $E(X)$ into $L^1(\lambda, X)$ is continuous, the sequence $(\psi_n)_n$ contains a subsequence that is equivalent to a complemented copy of ℓ^1 in $E(X)$. As a consequence, the set $\{\psi_n \ ; \ n \leq 1\}$ (and hence $\{\varphi_n \ ; \ n \geq 1\}$) is not a (V^*)-set which implies of course that the set $\{f_n, n \geq 1\}$ is not a (V^*)-set.

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CASE 2. Assume that $\lambda(L) = 0$.

Since $\lambda(C \cup L) = 1$ we have $\lambda(C) = 1$. Note that for each $\omega \in \Omega'$, the sequence $\big(g_n(\omega)\big)_{n\geq k}$ is equivalent to the unit vector basis of the ℓ^1 for some $k \in \mathbb{N}$. Now since $\psi_n(\omega) \in \operatorname{conv}\big(g_n(\omega), g_{n+1}(\omega), \ldots\big)$, the sequence $\big(\psi_n(\omega)\big)_{n\geq k}$ is equivalent to the unit vector basis of ℓ^1 and since X has property (V^*), the sequence $\big(\psi_n(\omega)\big)_{n\geq k}$ contains a subsequence equivalent to a complemented copy of ℓ^1 and therefore the sequence $\big(\psi_n(\omega)\otimes e_n\big)_n$ cannot be a weakly null sequence in $X\hat{\otimes}_\pi c_0$. In the other hand $(\psi_n(\omega)\otimes e_n)_n$ is weakly Cauchy (by the definition of C) so for each $\omega \in \Omega'$ fixed, there exists an operator $T \in \mathcal{L}(X, \ell^1)_1$ so that $\lim_{n\to\infty}\big\langle T\big(\psi_n(\omega)\big), e_n\big\rangle > 0$. Now we shall choose the operator above measurably using the following proposition.

PROPOSITION 5. *There exists a map $T\colon \Omega \longrightarrow \mathcal{L}(X, \ell^1)_1$ with the following properties:*

(a) $T(\omega) = 0\ \omega \in \Omega \setminus \Omega'$;

(b) $\lim_{n\to\infty}\big\langle T(\omega)\big(\psi_n(\omega)\big), e_n\big\rangle > 0\ \omega \in \Omega'$;

(c) The map $\omega \longrightarrow T(\omega)x$ is norm-measurable for each $x \in X$.

We need few steps to prove the proposition.

Notice first that since X is separable so is the space $X\hat{\otimes}_\pi c_0$ and therefore the unit ball of its dual $\mathcal{L}(X, \ell^1)_1$ is compact metrizable for the weak*-topology (in particular it is a Polish space). The space $\mathcal{L}(X, \ell^1)_1 \times (X\hat{\otimes}_\pi c_0)^{\mathbb{N}}$ with the product topology is a Polish space and let $\mathcal{A}$ be a subset of $\mathcal{L}(X, \ell^1)_1 \times (X\hat{\otimes}_\pi c_0)^{\mathbb{N}}$ defined as follows:

$$\{T, (\xi_n)_n\} \in \mathcal{A} \Leftrightarrow \lim_{n\to\infty}\langle T, \xi_n\rangle > 0.$$

The set $\mathcal{A}$ is trivially a Borel subset of $\mathcal{L}(X, \ell^1)_1 \times (X\hat{\otimes}_\pi c_0)^{\mathbb{N}}$. Let $\Pi\colon \mathcal{L}(X, \ell^1)_1 \times (X\hat{\otimes}_\pi c_0)^{\mathbb{N}} \longrightarrow (X\hat{\otimes}_\pi c_0)^{\mathbb{N}}$ be the 2-nd projection; the operator Π is of course continuous and therefore $\Pi(\mathcal{A})$ is analytic. By Theorem 8.5.3 of [3], there is a universally measurable map $\Theta\colon \Pi(\mathcal{A}) \longrightarrow \mathcal{L}(X, \ell^1)_1$ such that the graph of Θ is a subset of $\mathcal{A}$. Notice also that for $\omega \in \Omega'$, we have by the above argument that the sequence $\big(\psi_n(\omega)\otimes e_n\big)_n$ belongs to $\Pi(\mathcal{A})$. Now we define T as follows:

$$T(\omega) = \begin{cases} \Theta\Big(\big(\psi_n(\omega)\otimes e_n\big)_n\Big) & \text{for } \omega \in \Omega' \\ 0 & \text{otherwise.} \end{cases}$$

The map T is the composition of a universally measurable map Θ and the λ- measurable map $\omega \longrightarrow \big(\psi_n(\omega)\otimes e_n\big)_n$ so it is measurable for the weak*-topology. Now for any $x \in X$, the map $\omega \longrightarrow T(\omega)x$ is a ℓ^1-valued map and is *weak**-scalarly measurable and since ℓ^1 is a separable dual, it is norm-measurable. Now for $\omega \in \Omega'$, we get that

$$\Big\langle T(\omega), \big(\psi_n(\omega)\otimes e_n\big)_n\Big\rangle \in \mathcal{A}$$

which by the definition of $\mathcal{A}$ is equivalent to: $\lim_{n\to\infty}\big\langle T(\omega)\big(\psi_n(\omega)\big), e_n\big\rangle > 0$ and the proposition is proved. ∎

To finish the proof of the theorem, let $\gamma(\omega) = \lim_{n\to\infty}\big\langle T(\omega)\big(\psi_n(\omega)\big), e_n\big\rangle$. The map $\omega \longrightarrow \gamma(\omega)$ is measurable and for each $\omega \in \Omega'$, $\gamma(\omega) > 0$. Now define $S\colon E(X) \longrightarrow \ell^1$ as follows:

$$S(f) = \text{Bochner} - \int_{\Omega'} T\big(\omega\big)(f(\omega)\big)\, d\lambda(\omega)$$

for each $f \in E(X)$. Since $\|T(\omega)\| \leq 1$ a.e. and $E(X) \subset L^1(\lambda, X)$, the integrand in the above integral is easily seen to be Bochner integrable.

The operator S is linear bounded with $\|S\| \leq 1$ and it is easy to verify that

$$\lim_{n\to\infty} \langle S(\psi_n), e_n\rangle = \int_{\Omega'} \gamma(\omega)\, d\lambda(\omega) = \gamma > 0.$$

So there exists $N \in \mathbb{N}$ so that for $n \geq N$, $\langle S(\psi_n), e_n\rangle > \gamma/2$ and by Proposition 1, $(\psi_n)_{n\geq N}$ has a subsequence that is equivalent to a complemented copy of ℓ^1 and therefore the set $\{f_n \,;\, n \geq 1\}$ is not a (V^*)-set. The proof is complete. ∎

Let us finish by asking the following question: Let (Ω, Σ) be a measure space and Y be a Banach space. We denote by $M(\Omega, Y)$ the space of Y-valued countably additive measures with bounded variation endowed with the variation norm.

QUESTION. Assume that $Y = X^*$ is a dual space. Does property (V^*) pass from Y to $M(\Omega, Y)$?

Note that for a non dual space, the answer is negative: Talagrand constructed in [17] a Banach lattice E that does not contain c_0 (so has property (V^*) by [16]) but $M(\Omega, E)$ contains c_0 (hence failing property (V^*)).

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