SIMPLE PROOF OF A THEOREM ON PERMANENTS

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Let $A = (a_{ij})$ be an $n \times n$ complex matrix. The permanent of this matrix is

$$\operatorname{per} A = \sum_{\rho} \prod_{i=1}^{n} a_{i,\rho(i)},$$

where the sum is taken over all permutations ρ of the set $\{1, \ldots, n\}$.

In a recent paper [1] E. H. Lieb proved an interesting theorem (see below) which he applied to verify some conjectures of M. Marcus and M. Newman. The purpose of this note is to give a simple proof of Lieb's theorem.‡

THEOREM. *Let $A = (a_{ij})$ be an $n \times n$ hermitian positive semidefinite* (h.p.s.d.) *matrix partitioned as follows*

$$A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix},$$

where B is a $p \times p$ matrix. In addition suppose that A does not have a zero row. Let $A(\lambda)$ be the $n \times n$ matrix obtained from A by replacing B by the matrix λB, λ being a complex number. Then all the coefficients of the pth *degree polynomial $P(\lambda) = \operatorname{per} A(\lambda)$ are real and non-negative. Furthermore, if B and D are positive definite* (p.d.) *then the coefficient c_t of λ^t in $P(\lambda)$ is zero if and only if all subpermanents of C of order $p-t$ vanish.*

Proof. Let

$$\alpha = (\alpha_1, \ldots, \alpha_p), \qquad \alpha' = (\alpha_1, \ldots, \alpha_t), \qquad \alpha'' = (\alpha_{t+1}, \ldots, \alpha_p),$$

$$\sigma = (\sigma_1, \ldots, \sigma_{n-p}), \qquad \sigma' = (\sigma_1, \ldots, \sigma_{p-t}), \qquad \sigma'' = (\sigma_{p-t+1}, \ldots, \sigma_{n-p}),$$

where α and σ are permutations of the sets $\{1, \ldots, p\}$ and $\{p+1, \ldots, n\}$, respectively. When we need more such sequences we shall use the letters β and τ instead of α and σ, respectively.

The coefficient c_t is the sum of all permutation products

$$\prod_{i=1}^{n} a_{i,\rho(i)}$$

which contain t elements of B, $p-t$ elements of C, $p-t$ elements of C^* and $n-2p+t$ elements of D. We clearly have

$$Mc_t = \sum_{\alpha,\beta,\sigma,\tau} a_{\alpha'\beta'}\, a_{\alpha''\tau'}\, a_{\sigma'\beta''}\, a_{\sigma''\tau''},$$

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‡ We mention that the theorem in [1] contains an inaccuracy concerning the conditions under which $c_t = 0$. We present a corrected formulation. The proof given in [1] concerning the conditions under which $c_{p-1} > 0$ is still valid and so is the addendum on p. 129.

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where $M = t!(p-t)!^2(n-2p+t)!$ and, for instance,

$$a_{\alpha'\beta'} = \prod_{i=1}^{t} a_{\alpha_i,\beta_i}.$$

Since A is h.p.s.d. there exist n vectors f^i $(i = 1,\dots,n)$ such that

$$a_{ij} = (f^i, f^j) = \sum_{k=1}^{n} \overline{f^i_k} f^j_k,$$

where $f^i_k (k = 1,\dots,n)$ are coordinates of f^i. The bar denotes the complex conjugate. Using this representation of a_{ij} we get

$$Mc_t = \sum_{\alpha,\beta,\sigma,\tau} \sum_{I,J,K,R} \bar{F}^{\alpha'}_I F^{\beta'}_I \bar{F}^{\alpha''}_J F^{\tau'}_J \bar{F}^{\sigma'}_K F^{\beta''}_K \bar{F}^{\sigma''}_R F^{\tau''}_R,$$

where, for instance,

$$F^{\alpha'}_I = \prod_{s=1}^{t} f^{\alpha_s}_{i_s}.$$

The letters I, J, K, R denote the sequences of indices $I = (i_1,\dots,i_t)$, $J = (j_1,\dots,j_{p-t})$, $K = (k_1,\dots,k_{p-t})$, $R = (r_1,\dots,r_{n-2p+t})$. The sum over I, for instance, means the sum over all indices $i_1,\dots,i_t$. Each index runs through the values $1,\dots,n$.

Changing the order of summation, we get

$$Mc_t = \sum_{I,R}\left[\sum_J \Big(\sum_\alpha \bar{F}^{\alpha'}_I \bar{F}^{\alpha''}_J\Big)\Big(\sum_\tau F^{\tau'}_J F^{\tau''}_R\Big)\right]\left[\sum_K \Big(\sum_\beta F^{\beta'}_I F^{\beta''}_K\Big)\Big(\sum_\sigma \bar{F}^{\sigma'}_K \bar{F}^{\sigma''}_R\Big)\right]$$

$$= \sum_{I,R}\left|\sum_J \Big(\sum_\alpha \bar{F}^{\alpha'}_I \bar{F}^{\alpha''}_J\Big)\Big(\sum_\tau F^{\tau'}_J F^{\tau''}_R\Big)\right|^2.$$

Hence $c_t \geqq 0$. We have $c_t = 0$ if and only if

$$\sum_J \Big(\sum_\alpha \bar{F}^{\alpha'}_I \bar{F}^{\alpha''}_J\Big)\Big(\sum_\tau F^{\tau'}_J F^{\tau''}_R\Big) = 0,$$

i.e.,

$$\sum_{\alpha,\tau} \bar{F}^{\alpha'}_I F^{\tau''}_R a_{\alpha''\tau'} = 0$$

for all I and R. After summation over α'' and τ' this condition becomes

$$\sum_{\alpha',\tau''} \bar{F}^{\alpha'}_I F^{\tau''}_R \operatorname{per} C(\alpha', \tau'') = 0. \tag{1}$$

Here $C(\alpha', \tau'')$ denotes the submatrix of C which remains after deleting the rows α' and the columns τ'' of A.

It is obvious that (1) is satisfied if all subpermanents of C of order $p-t$ vanish. Conversely, if B and D are p.d. we shall prove that (1) implies that all these subpermanents vanish. Let β and σ be arbitrary. Since B is p.d. the vectors f^k $(k = 1,\dots,p)$ are linearly independent. Let g^k $(k = 1,\dots,p)$ be their reciprocal system of vectors. Similarly, let the system of vectors g^k $(k = p+1,\dots,n)$ be reciprocal to the system f^k $(k = p+1,\dots,n)$. Then we have

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$$\sum_I G_I^{\beta'} \bar{F}_I^{\alpha'} = \begin{cases} 1 & \text{if} \quad \beta' = \alpha', \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_R \bar{G}_R^{\sigma''} F_R^{\tau''} = \begin{cases} 1 & \text{if} \quad \sigma'' = \tau'', \\ 0 & \text{otherwise.} \end{cases}$$

Multiplying (1) by $G_I^{\beta'} \bar{G}_R^{\sigma''}$ and summing over I and R, we get per $C(\beta', \sigma'') = 0$. This proves the theorem.

I am grateful to Professor E. H. Lieb for the remark that it is sufficient to suppose that *B* and *D* are p.d. (*A* may be merely h.p.s.d.).

REFERENCE

1. E. H. Lieb, Proofs of some conjectures on permanents, *J. Mech. Math.* **16** (1966), 127–134.

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