LIMIT COMPLEXITIES, MINIMAL DESCRIPTIONS, AND n-RANDOMNESS

RODNEY DOWNEY, LU LIU, KENG MENG NG, AND DANIEL TURETSKY

Abstract. Let K denote prefix-free Kolmogorov complexity, and let K^A denote it relative to an oracle A. We show that for any n, $K^{\emptyset^{(n)}}$ is definable purely in terms of the unrelativized notion K. It was already known that 2-randomness is definable in terms of K (and plain complexity C) as those reals which infinitely often have maximal complexity. We can use our characterization to show that n-randomness is definable purely in terms of K. To do this we extend a certain "limsup" formula from the literature, and apply Symmetry of Information. This extension entails a novel use of semilow sets, and a more precise analysis of the complexity of Δ^0_2 sets of minimal descriptions.

§1. Introduction. A cornerstone of algorithmic randomness is Schnorr's Theorem[1] that X is Martin-Löf (ML) random iff $K(X\upharpoonright n) \geq^+ n$ for all n (Chaitin [3]). Thus X being ML-random is equivalent to having all initial segments weakly K-random. Now this result relativizes and hence X is A-random iff $K^A(X\upharpoonright n) \geq^+ n$, so that, for example, X is 2-random, meaning X is ML-random relative to $\emptyset'$, iff $K^{\emptyset'}(X\upharpoonright n) \geq^+ n$, for all n.

It is slightly surprising that 2-randomness can also be characterized using K (and C) *without relativization.*

THEOREM 1.1.

- (*Miller* [10], *Nies, Stephan, and Terwijn* [14]) *X is 2-random iff $C(X\upharpoonright n) \geq^+ n$ for infinitely many n.*
- (*Miller* [11]) *X is 2-random iff $K(X\upharpoonright n) \geq^+ n + K(n)$ for infinitely many n.*

Theorem 1.1 suggests the following motivating question:

QUESTION 1.2. *Is it possible to define $(k+1)$-randomness using unrelativized initial segment complexity K, not the relativized notion $K^{(k)} =_{\text{def}} K^{\emptyset^{(k)}}$? Indeed, is it possible to define $K^{(k)}$ using K?*

On the face of it, why should there be a characterization of X being $\emptyset^{(900)}$-random, have a definition purely involving the behaviour $K(X\upharpoonright n)$ for $n \in \omega$?

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[1] We assume that the reader is familiar with the rudiments of algorithmic randomness and Kolmogorov complexity, such as initial segments of [5] or [13]. We will use C to denote plain complexity and K for prefix-free complexity.

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Indeed Solovay [18] (see [5, Section 10.2.3]) showed that $K^{\emptyset'}$ and K have apparently complex relationships such as $K^{\emptyset'}(n) \leq K(n) - \alpha(n) + O(\log \alpha(n))$, where $\alpha(n) = \min\{K(m) \mid m \geq n\}$, a fact we use in Section 6.

Nevertheless, we'll show that Question 1.2 has a positive answer. To prove this, we will give a new analysis of the behaviour of sets of minimal descriptions, where τ is a minimal description of σ if $U(\tau) = \sigma$ and $|\tau| = K(\sigma)$. Such τ's must be incompressible, let there be a shorter description of σ. For example, we will show that although every Δ^0_2 set of minimal descriptions computes $\emptyset'$, there are, for our purposes, sufficiently tame sets of minimal descriptions called "semilow" sets, a notion which grew from Soare's automorphism machinery [16].

1.1. $k \geq 2$**.** To prove that we can define $(k+1)$-randomness using only K, our starting point is the following attractive result.

THEOREM 1.3. (Bienvenu, Muchnik, Shen, and Vereshchagin [2])

$$K^{\emptyset'}(\sigma) =^+ \limsup_n K(\sigma \mid n).$$

The same result holds for C in place of K.

Here $K(\sigma \mid n)$ is the conditional complexity of σ given n. The reader might think that Theorem 1.3 does the job: We could relativize the result to get $K^{\emptyset''}(\sigma) = \limsup_n K^{\emptyset'}(\sigma \mid n)$ and then replace $K^{\emptyset'}(\sigma \mid n)$ by a $\limsup_m K((\sigma \mid n) \mid m)$.

The problem is that $K(\sigma \mid n)$ is not really an unrelativized notion, even though no oracles appear in a standard definition such as that in Li–Vitanyi [9]

$$K(\sigma \mid n) = \min\{|y| : U(\langle y, n\rangle) = \sigma\}.$$

However, since n is fixed, this definition hard-codes n as an oracle. More precisely, the definition in [9] is easily seen to be equivalent to the definition of conditional complexity used in [5]

$$K(\sigma \mid n) = K^{\overline{n}}(\sigma),$$

where $\overline{n}$, a self-delimited version of n, is provided as an oracle.

Perhaps we might be able to get rid of the finite oracles. You might imagine that finite oracles have little effect, but this is not true in general. Indeed, in Section 6, we will see that finite strings can have very strong compression power. There we prove the following:

THEOREM 1.4. *For all e there is a string ρ such that for almost all τ, $K(\tau \mid \rho) < K(\tau) - e$. That is, for $\xi = \overline{\rho}$, $K^{\xi}(\tau) < K(\tau) - e$.*

So we need a new plan.

1.2. The plan. Our plan is to leverage $K(\sigma \mid n)$, and we will do this using Symmetry of Information (Levin and Gács [8], Chaitin [3]) which says that

$$K(\sigma, n) =^+ K(n) + K(\sigma \mid n^*).$$

Here the reader should recall that $K(v, \rho)$ is the complexity of the pair $\langle v, \rho\rangle$, and that for any string τ, τ^* is the first minimal description to occur of length $K(\tau)$ with $U(\tau^*) = \tau$. (In the case of C we will write τ^*_C.)

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While τ^* is a particular minimal code for τ (the first to appear in a fixed effective search), it may not be the only code for τ of length $K(\tau)$. We will also be interested in all minimal codes, and so we adopt the following notation.

NOTATION 1.5. *For a universal machine U (prefix-free or otherwise), let $N_U = \{n^* : n \in \omega\}$, where n^* is defined based on U. Let $M_U = \{\rho : U(\rho)\downarrow \wedge |\rho| = K(U(\rho))\}$. When U is clear from context, we will omit the subscript.*

Thus N_U contains only the first minimal description for any given string, while M_U contains every minimal description. Observe that $N_U \subseteq M_U$.

REMARK 1.6. Note that τ^* is interchangeable with $(\tau, K(\tau))$, in that there are computable functions to uniformly pass from one to the other: in one direction, $\tau^* \mapsto (U(\tau^*), |\tau^*|)$; in the other direction, given $(\tau, K(\tau))$, we search for the first string ρ of length $K(\tau)$ with $U(\rho) = \tau$. We will use τ^*, but the reader who prefers may substitute $(\tau, K(\tau))$ anywhere this occurs in the paper.

Similarly, any minimal code from M_U may be replaced with the corresponding $(\tau, K(\tau))$. Again, given $\rho \in M_U$, it maps to $(U(\rho), |\rho|)$. In the other direction the map is not uniform, but there is a constant bound on the number of minimal codes a string may have. Thus if ρ is a minimal code for τ, ρ can be obtained from $(\tau, K(\tau))$ with only a constant amount of extra information: we begin enumerating all strings of length $K(\tau)$ which code τ, and we must know τ's position in this enumeration.

In Section 5 we will prove the following.

THEOREM 1.7. *For any universal prefix-free machine U,*

$$\limsup_n K(\sigma \mid n^*) =^+ \limsup_{\tau \in M_U} K(\sigma \mid \tau) =^+ K^{\emptyset'}(\sigma).$$

Notice that by rearranging Symmetry of Information, we obtain $K(\sigma \mid n^*) =^+ K(\sigma, n) - K(n)$. Hence $K^{\emptyset'}(\sigma) =^+ \limsup_n [K(\sigma, n) - K(n)]$, giving a definition of $K^{\emptyset'}$ purely in terms of K without relativization.

By relativizing Theorem 1.3, we obtain

$$K^{A'}(\sigma) =^+ \limsup_n K^A(\sigma \mid n),$$

for all A.[2] By appropriately iterating and relativizing Theorem 1.7, we obtain a definition of $K^{\emptyset^{(k)}}$ for all $k \in \omega$, e.g.,

$$\begin{aligned} K^{\emptyset^{(2)}}(\sigma) &=^+ \limsup_n [K^{\emptyset'}(\sigma, n) - K^{\emptyset'}(n)] \\ &=^+ \limsup_n \left(\limsup_m [K(\sigma, n, m) - K(m)] - \limsup_m [K(n, m) - K(m)] \right). \end{aligned}$$

From this follows a definition of k-randomness purely in terms of unrelativized K.

1.3. The complexity of the sets M_U and N_U. The question arises how should we prove the $\limsup_n K(\sigma \mid n^*)$ theorem? The answer really comes from understanding

[2] Indeed $=^+ \limsup_n K(\sigma \mid A\restriction n)$.

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the behaviour of the set of minimal descriptions. One hint came from the unpublished work of Hirschfeldt.

THEOREM 1.8 (Hirschfeldt, unpublished). $C^{\emptyset'}(\sigma) =^+ \limsup_n C(\sigma \mid n^*_C)$.

We will prove this result in Section 2. The method is to construct an infinite low subset of N_V, for V the machine generating C, and use some relativization tricks.

We had hoped to use this method for K, but unfortunately we were able to prove a result saying that this is impossible.

THEOREM 1.9. *Let U be a universal prefix-free machine, and let S be an infinite Δ^0_2 subset of M_U. Then $\emptyset' \leq_T S$.*

Note that M_U and N_U are of degree $\mathbf{0}'$ (e.g., [5]). For example, $N_U \geq_{wtt} \Omega$ by the Coding Theorem (see [5, Section 3.9]), and Ω is wtt-complete. In view of Theorem 1.9, it would seem reasonable to suggest that they are introreducible, which in this context would mean that every infinite subset computes $\emptyset'$. However, Joseph Miller proved that this is not the case. We include this also in Section 3.

We remark that the proof can also be adapted to show that an infinite Δ^0_2 hitting set for a Solovay function[3] must also be Turing complete. We prove these results, which are of independent interest, in Section 3.

In the end we found a way around these problems using an idea from studies in the automorphism group of the lattice of computably enumerable sets (Soare [16]) A set S is called *semi-low* iff

$$\{e \mid S \cap W_e \neq \emptyset\} \leq_T \emptyset',$$

i.e., a pointwise version of being low. This notion was introduced as a method towards characterizing when the lattice of supersets of a c.e. set was isomorphic to the lattice of all c.e. sets. In Section 5, we will show that, although there cannot be an infinite low subset of M_U or N_U, there *can* be a semi-low one (which we believe to be of independent interest, especially contrasted with the previous theorem). In Section 4 we will prove that this is enough for our main result. There have been other uses of semi-lowness outside of the lattice of c.e. sets, such as Downey and Melnikov [7] in the study of abelian groups, but these seem sporadic at best.

1.4. The machine existence theorem. As we will be using it several times, we state the machine existence theorem and fix our notation surrounding it.

DEFINITION 1.10. For a set $A \subseteq 2^{<\omega} \times \omega$, $\mathrm{wt}(A) = \sum_{(\sigma,s)\in A} 2^{-s}$.

THEOREM 1.11 (KC Theorem, Coding Theorem, or Machine Existence Theorem; see Downey and Hirschfeldt [5, Section 3.6]). *If $A \subseteq 2^{<\omega} \times \omega$ is c.e. and has $wt(A) \leq 1$, then there is a prefix-free machine V such that for every $(\sigma, s) \in A$, there is a ρ with $|\rho| = s$ and $V(\rho) = \sigma$. Further, an index for V can be effectively obtained from a c.e. index for A.*

COROLLARY 1.12. *If $A \subseteq 2^{<\omega} \times \omega$ is c.e. with $wt(A) < \infty$, then for all $(\sigma, s) \in A$, $K(\sigma) \leq^+ s$.*

[3]That is, a computable F such that $F(\sigma) \geq^+ K(\sigma)$ for all σ, and $F(\sigma) =^+ K(\sigma)$ for infinitely many σ.

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Sets A of this form are sometimes called *request sets*.

The following is more of a proof technique, but we will state it as a corollary.

COROLLARY 1.13. *As part of a uniform construction, we may effectively obtain an $\varepsilon > 0$ such that if we enumerate $A \subseteq 2^{<\omega} \times \omega$ with $wt(A) \leq \varepsilon$, then $K(\sigma) \leq s$ for every $(\sigma, s) \in A$ (observe the lack of additive constant).*

PROOF. Fix U the universal prefix-free machine used to define K. We will generate an auxiliary c.e. set B with $\mathrm{wt}(B) \leq 1$. By the Recursion Theorem, we know a c.e. index for the set B we will enumerate. By the Machine Existence Theorem, this effectively gives us an index for a prefix-free machine V. From this we effectively obtain a string ρ with $U(\rho\tau) = V(\tau)$ for all τ, and so $K(\sigma) \leq t + |\rho|$ for all $(\sigma, t) \in B$.

Set $\varepsilon = 2^{-|\rho|}$, and define B by enumerating $(\sigma, s - |\rho|)$ whenever A enumerates (σ, s), provided this enumeration does not put $\mathrm{wt}(B)$ over 1. If $\mathrm{wt}(A) \leq \varepsilon$, then $(\sigma, s - |\rho|)$ is enumerated into B for every pair $(\sigma, s) \in A$, and so $K(\sigma) \leq (s - |\rho|) + |\rho| = s$, as desired. ⊣

§2. Hirschfeldt's Theorem. We prove Theorem 1.8. We will prove that

$$C^{\emptyset'}(\sigma) = \limsup_n C(\sigma \mid n_C^*).$$

Consider the Π_1^0 class of sequences

$$P = \{(m_0, m_1, \dots) \mid \forall n\, [2^n \leq m < 2^{n+1} \wedge C(m_n) \geq n]\}.$$

A simple counting argument shows that there is an appropriate m_n for every n, and so P is nonempty. Since there are only 2^n options for m_n, $P \subseteq 2^\omega$ under an appropriate effective identification. So there is a low infinite path $L = (m_0, m_1, \dots)$ by the Low Basis Theorem.

Recall that for $2^n \leq m < 2^{n+1}$, $C(m) \leq^+ n$, so fix the least d such that $\exists^\infty n\, C(m_n) = n + d$, and fix an N such that $C(m_n) \geq n + d$ for all $n \geq N$. Then

$$X = \{(m_n)_C^* : n \geq N \wedge C(m_n) = n + d\}$$

is L-c.e. and an infinite subset of N_V, where V is the universal machine defining C. Fix $(\rho_i)_{i \in \omega}$ an L-computable enumeration of X.

Then by relativizing Theorem 1.3, we have

$$C^{\emptyset'}(\sigma) =^+ C^{L'}(\sigma) =^+ \limsup_i C^L(\sigma \mid i).$$

Note that we can L-effectively pass between i and ρ_i, so $C^L(\sigma \mid i) =^+ C^L(\sigma \mid \rho_i)$, giving

$$C^{\emptyset'}(\sigma) =^+ \limsup_{\rho \in X} C^L(\sigma \mid \rho).$$

We also have

$$\limsup_{\rho \in X} C^L(\sigma \mid \rho) \leq^+ \limsup_{\rho \in X} C(\sigma \mid \rho),$$

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since oracles can only help;

$$\limsup_{\rho\in X} C(\sigma \mid \rho) \leq \limsup_{n^*_C} C(\sigma \mid n^*_C) \leq \limsup_{\rho\in M_V} C(\sigma \mid \rho) \leq \limsup_{\rho\in 2^{<\omega}} C(\sigma \mid \rho),$$

since $X \subseteq N_V \subseteq M_V \subseteq 2^{<\omega}$, and limit supremums over larger sets are larger; and finally

$$\limsup_{\rho\in 2^{<\omega}} C(\sigma \mid \rho) =^+ C^{\emptyset'}(\sigma),$$

by the unrelativized version of Theorem 1.3, after an effective identification of $2^{<\omega}$ with ω. Picking out the relevant bits, we see that

$$C^{\emptyset'}(\sigma) =^+ \limsup_{n} C(\sigma \mid n^*_C) =^+ \limsup_{\rho\in M_V} C(\sigma \mid \rho).$$

§3. No low hitting sets. We prove Theorem 1.9. Fix a universal prefix-free machine U, and suppose that $X = \lim_s X_s$ is an infinite Δ^0_2 subset of M_U. Fix ε as in Corollary 1.13; we will enumerate an appropriate set $A \subseteq 2^{<\omega} \times \omega$.

We describe how we code whether $n \in \emptyset'$. Fix $k \in \omega$ with

$$\frac{1}{k} < 2^{-(n+2)}\varepsilon.$$

To do this coding, we will define a k-colouring χ on $\mathrm{dom}(U)$. This colouring will be unique to n; the colourings for other values of n will have no interaction.

We declare that colour i is *small* if

$$\sum_{\chi(\sigma)=i} 2^{-|\sigma|} \leq \frac{1}{k}.$$

This has natural approximations: at a stage s, based on the finitely many strings we have so far coloured, a colour may still be small or may have already proven itself to be large. Note that since colours are disjoint, and we have k colours, there is always at least one small colour.

Suppose that σ enters $\mathrm{dom}(U)$ at stage s. Let

$$r_i = \min\{|\tau| \mid \chi(\tau) = i \wedge \tau \in X_s\}.$$

We regard this as infinite if there is no such τ. We fix a j maximizing r_j and colour $\chi(\sigma) = j$.

Suppose that n enters $\emptyset'$ at some stage $s+1$. Fix a single colour j which was small at stage s; we invalidate all the strings which had colour j at stage s. That is, for every $\sigma \in \mathrm{dom}(U)[s]$ with $\chi(\sigma) = j$, we enumerate $(U(\sigma), |\sigma| - 1)$ into A. Provided $\mathrm{wt}(A) \leq \varepsilon$, this will ensure that $\sigma \notin M_U$ for each such σ.

As j was small at stage s, the weight of these pairs is at most $\frac{2}{k} < 2^{-(n+1)}\varepsilon$. Thus, summing over the strategies for every n, $\mathrm{wt}(A) \leq \sum_n 2^{-(n+1)}\varepsilon = \varepsilon$, as required.

Claim 3.1. *X has members of every colour.*

Proof. Fix A the set of colours which occur in X, and suppose this is not all the colours.

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Fix a length n such that every colour in A occurs on a string $\tau \in X$ with $|\tau| \leq n$, and fix t sufficiently large such that X has converged on strings of length at most n by stage t, i.e., if $|\tau| \leq n$, then for all $s \geq t$, $X_s(\tau) = X(\tau)$.

Since X is infinite, it contains some τ which enters $\mathrm{dom}(U)$ at some stage $s > t$. At stage s, $r_i \leq n$ for every $i \in A$, whereas $r_i > n$ for every colour $i \notin A$. So $\chi(\tau)$ will be a colour not in A, contrary to choice of A. ⊣

We can now state our procedure for computing $\emptyset'(n)$ from X: in the colouring for n, search for a stage t such that for every colour i, some element of X has been given colour i by stage t; then output $\emptyset'_t(n)$.

As just argued, there is eventually some stage at which X intersects every colour, so this algorithm is total. Suppose first that $n \notin \emptyset'$. Then certainly $\emptyset'_t(n) = 0$, as desired.

Suppose instead that $n \in \emptyset'$, and fix the stage $s+1$ at which it enters. Fix the chosen colour j. Then no σ which received colour j at or before stage s belongs to M_U, and so cannot belong to X. Thus colour j witnesses $t \not\leq s$, giving $\emptyset'_t(n) = 1$, as desired.

This concludes the proof of Theorem 1.9.

The same method can be used to prove the following:

Corollary 3.2. *Suppose that X is an infinite Δ^0_2 set of hitting points for a Solovay function F. That is, a set S of points n where $F(n) =^+ K(n)$. Then $\emptyset' \leq_T S$.*

We remark that Corollary 3.2 improves a result of Bienvenu, Downey, Merkle, and Nies [1] who showed that the collection of *all* hitting points is Turing complete.

The reader should note that if $\sigma = m^*$, then σ must be weakly K-random in that $K(\sigma) \geq^+ |\sigma|$. The reason is that if $K(\sigma) << |\sigma|$ then using the KC Theorem, we can use σ^* to describe m, in a machine M we build. This would show that $K(m) << |\sigma| = |m^*|$, a contradiction. This brings in to focus the question of precisely which weakly K-random strings are minimal descriptions. By the Low Basis Theorem, there are infinite low collections of weakly K-random strings. At most finitely many can be minimal descriptions. Another consequence of Theorem 1.9 is the following[4].

Corollary 3.3. *If X is a Δ^0_2 collection of weakly K-random strings (that is, $K(\sigma) \geq^+ |\sigma|$), and $|X \cap N_U| = \infty$, then X computes $\emptyset'$.*

Proof. Fix d such that for every $\sigma \in X$, $K(\sigma) > |\sigma| - d$. For every n, let n^*_s be the natural stage s approximation to n^*. This may be undefined for small s, but it will eventually converge to the true n^*. Further, if $n^*_s\downarrow$ and $n^*_{s+1} \neq n^*_s$, then $|n^*_{s+1}| < |n^*_s|$.

Again fix ε as in Corollary 1.13. Fix k with $2^{-k} < \varepsilon$. For every n and s with $|n^*_s| > |n^*| + k + d$ (a c.e. event), we enumerate $(n, |n^*_s| - d)$ into A. Since for a fixed n there is at most one n^*_s of any given length, the weight of our requests is bounded by

$$\sum_n \sum_{i > |n^*|+k+d} 2^{-(i-d)} = 2^{-k} \sum_n 2^{-|n^*|} < 2^{-k}.$$

Thus $\mathrm{wt}(A) \leq \varepsilon$, and so $K(n^*_s) \leq |n^*_s| - d$ for every such n^*_s.

[4]More or less the same proof will also give this for intersections of Δ^0_2 sets of hitting points for Solovay functions, and Δ^0_2 subsets of M^*, this last one by the Coding Theorem, there are at most $O(1)$ many elements of M^* of length n^* for a fixed n.

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It follows that if $n_s^* \in X$, then since $K(n_s^*) > |n_s^*| - d$ by assumption, $|n_s^*| \leq |n^*| + k + d$, or $|n^*| \geq |n_s^*| - k - d$. As in the proof of Theorem 1.8, this allows X to enumerate an infinite sequence from N_U. Since every infinite c.e. set contains an infinite computable set, and this relativizes, we get that X computes an infinite $Y \subseteq N_U$. As X is Δ^0_2, Y is as well, so $Y \geq_T \emptyset'$, and thus $X \geq_T \emptyset'$. ⊣

As we mentioned in the Introduction, Theorem 1.9 cannot be improved to show that all infinite subsets of N_U compute $\emptyset'$.

THEOREM 3.4 (Joseph Miller, unpublished). *There is an infinite $X \subset N_U$ which does not compute $\emptyset'$.*

PROOF. Let P be a bounded Π^0_1 class of K-compression functions[5]. Since we have an a priori upper bound of $K(n) \leq^+ 2\log(n)$, we may take $P \subseteq 2^\omega$. Let F be a "weakly-low for K" path. That is, there are infinitely many n with $F(n) =^+ K(n)$. This can be shown to exist using the "low for Ω"-Basis Theorem[6] (Downey, Hirschfeldt, Miller, and Nies [6], Reimann and Slaman [15]) and the fact that low for Ω is equivalent to "weakly low for K" (see Downey and Hirschfeldt [5]).

Now fix the least c with $K(n) = F(n) + c$ for infinitely many n, and fix an m with $K(n) \geq F(n) + c$ for all $n \geq m$. F can enumerate an infinite subset of N_U: $\{n_s^* : n \geq m \wedge |n_s^*| = F(n) + c\}$. Thus F computes an infinite $X \subseteq N_U$ (again relativizing the fact that every infinite c.e. set has an infinite computable subset), Since F does not compute $\emptyset'$ (since it is weakly low for K), X also does not compute $\emptyset'$. ⊣

§4. Conditional complexity along semi-low sets. Semi-lowness has previously been studied for co-c.e. sets. We are interested in it for Δ^0_2 sets, in which case it is not entirely clear that the following is the correct definition[7], but it is the definition relevant to our current interest.

DEFINITION 4.1. Let $(W_e)_{e\in\omega}$ be a standard listing of c.e. sets. A set X is *semi-low* if the set $\{e : X \cap W_e \neq \emptyset\}$ is Δ^0_2 (i.e., $\leq_T \emptyset'$).

Recall Theorem 1.3:

$$K^{\emptyset'}(\sigma) =^+ \limsup_{n\in\omega} K(\sigma \mid n).$$

As we have seen, it can be helpful to consider $\limsup_{n\in X} K(\sigma \mid n)$ for X infinite. It is immediate that this is $\leq^+ K^{\emptyset'}(\sigma)$, as we are taking a limit supremum over a smaller set. It turns out that for semi-low sets, we have equality.

PROPOSITION 4.2. *If X is semi-low and infinite, then $K^{\emptyset'}(\sigma) =^+ \limsup_{n\in X} K(\sigma \mid n)$.*

[5]A K-compression function is an injective function $G : \omega \to \omega$ such that for all n, $G(n) \leq K(n)$. These were introduced by Nies, Stephan, and Terwijn [14] in their proof that 2-randomness is the same as infinitely often C-random.

[6]Every Π^0_1 class on 2^ω contains a $\emptyset'$-left c.e. real A relative to which $\Omega^A = \Omega$.

[7]An alternative definition would additionally require that $\{e : W_e \subseteq X\}$ is Δ^0_2; note that when X is co-c.e., this set is Π^0_1.

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PROOF. As one direction is immediate, it remains to show that

$$K^{\emptyset'}(\sigma) \leq^+ \limsup_{n \in X} K(\sigma \mid n).$$

We will work with request sets.

For each $n \in \omega$, define $A_n = \{(\sigma, s) : s \geq K(\sigma \mid n)\}$. We may think of A_n as the request set generating $K(\cdot|n)$. Observe that $\mathrm{wt}(A_n) < 2$ for all n.

Note that for any finite set $D \subset 2^{<\omega} \times \omega$ and any $m \in \omega$, the set $\{n \geq m : \mathrm{wt}(D \cup A_n) > 2\}$ is c.e. (it is even primitive recursive with appropriate assumptions on $K(\cdot|n)$, but this is not necessary). Indeed this is uniform, so we may fix a total computable function e such that $W_{e(D,m)} = \{n \geq m : \mathrm{wt}(D \cup A_n) > 2\}$, where D is given by a canonical index.

We will build a $\emptyset'$-enumerable request set B with $\mathrm{wt}(B) \leq 2$ and such that for all σ, if $s = \limsup_{n \in X} K(\sigma \mid n)$, then $(\sigma, s) \in B$. By Corollary 1.12 relativized to $\emptyset'$, this will suffice to prove the result.

Fix an effective listing $(\sigma_m, s_m)_{m \in \omega}$ of $2^{<\omega} \times \omega$ such that every pair is repeated infinitely many times on the list. We define B as follows:

- $B_0 = \emptyset$.
- Given B_m, fix $D = B_m \cup \{(\sigma_m, s_m)\}$. If $X \cap W_{e(D,m)} = \emptyset$, we let $B_{m+1} = D$; otherwise, we let $B_{m+1} = B_m$.

As X is semi-low, $\emptyset'$ can run this construction, and so B is $\emptyset'$-enumerable.

CLAIM 4.3. *For all $n \geq m$ with $n \in X$, $wt(B_m \cup A_n) \leq 2$, and thus $wt(B) \leq 2$.*

PROOF. Suppose not. Then as this clearly holds for B_0, we may fix $m+1$ the least value where the claim is violated. So there is some $n \geq m+1$ with $n \in X$, $\mathrm{wt}(B_m \cup A_n) \leq 2$ and $\mathrm{wt}(B_{m+1} \cup A_n) > 2$. As $B_{m+1} \neq B_m$, we must be in the case $X \cap W_{e(D,m)} = \emptyset$, with $B_{m+1} = B_m \cup \{(\sigma_m, s_m)\} = D$. But $n \in X \cap W_{e(D,m)}$, a contradiction.

That $\mathrm{wt}(B) \leq 2$ then follows from X being infinite. ⊣

CLAIM 4.4. *For any σ, if $s = \limsup_{n \in X} K(\sigma \mid n)$, then $(\sigma, s) \in B$.*

PROOF. Fix an n_0 such that for all $n \geq n_0$ with $n \in X$, $K(\sigma \mid n) \leq s$. Then for all $n \geq n_0$ with $n \in X$, $(\sigma, s) \in A_n$. Fix an $m \geq n_0$ such that $(\sigma, s) = (\sigma_m, s_m)$. Let $D = B_m \cup \{\sigma_m, s_m\}$. Then for all $n \geq m$ with $n \in X$, $D \cup A_n = B_m \cup A_n$, and $\mathrm{wt}(B_m \cup A_n) \leq 2$. So $X \cap W_{e(D,m)} = \emptyset$, and $(\sigma, s) \in B_{m+1}$ by construction. ⊣

This completes the proof. ⊣

§5. Conditional complexity along minimal codes. Fix a universal prefix-free machine U. We are interested in $\limsup_{\tau \in M_U} K(\sigma \mid \tau)$ and $\lim_{\tau \in N_U} K(\sigma \mid \tau)$. First we verify that these values are machine independent.

LEMMA 5.1. *If U and V are universal prefix-free machines, and $K(\cdot|\cdot)$ is defined from a third (unnamed) universal prefix-free machine, then*

$$\limsup_{\tau \in M_U} K(\sigma \mid \tau) =^+ \limsup_{\tau \in M_V} K(\sigma \mid \tau) =^+ \limsup_{\tau \in N_V} K(\sigma \mid \tau).$$

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PROOF. By symmetry, and the fact that $N_V \subseteq M_V$, it suffices to show

$$\limsup_{\tau \in M_U} K(\sigma \mid \tau) \leq^+ \limsup_{\tau \in N_V} K(\sigma \mid \tau).$$

By standard arguments, there is a constant c such that if $\tau \in M_U$, $\rho \in M_V$, and $U(\tau) = V(\rho)$, then $||\tau| - |\rho|| \leq c$.

For each $\tau \in 2^{<\omega}$ and $i \in \mathbb{Z}$ with $|i| \leq c$, let $\rho(\tau, i)$ be the first ρ located with $|\rho| = |\tau| + i$ and $U(\tau)\downarrow = V(\rho)\downarrow$, if such ρ exists. We define

$$B_\tau = \{(\sigma, s) : \exists i \; \rho(\tau, i)\downarrow \wedge s \geq K(\sigma \mid \rho(\tau, i))\}.$$

Then $\mathrm{wt}(B_\tau) \leq \sum_{|i| \leq c} \sum_\sigma 2 \cdot 2^{-K(\sigma|\rho(\tau,i))} \leq (2c+1) \cdot 2$, and thus these are uniformly given request sets.

It follows that $K(\sigma \mid \tau) \leq^+ K(\sigma \mid \rho(\tau, i))$ for all i with $\rho(\tau, i)\downarrow$. Note that if $\tau \in M_U$, then there is an i with $\rho(\tau, i)\downarrow \in N_V$. The claim follows. ⊣

PROPOSITION 5.2. *There is a universal prefix-free machine U and an infinite, semi-low set $X \subseteq N_U$.*

PROOF. Fix some universal prefix-free machine V. We define $U(0^3{}^\frown\tau) = V(\tau)$ for all τ, which makes U universal while giving us the freedom to do as we like on other neighborhoods.

Let $(N_s)_{s \in \omega}$ be the natural approximation to N_U. We will have semi-lowness requirements R_e and infiniteness requirements P_n. The strategy for each requirement will *claim* various strings in N_s, and each strategy will have a directive at every stage: *meet* or *avoid*. A string may only be claimed by a single strategy at a time, and a strategy will retain its claim on a string until either the string leaves N_s, or a higher priority strategy claims the string. In either case, the strategy will immediately relinquish its claim.

We will build a c.e. set $A \subseteq 2^{<\omega} \times \omega$. As we will argue, the sum $\sum 2^{-|\sigma|}$ over all strings σ which are ever claimed in the construction will be bounded by $1/2$. The first time a string σ is claimed by a strategy (i.e., it was unclaimed at all previous stages), we will immediately enumerate $(k, |\sigma| - 1)$ into A, where k is larger than any value yet seen in the construction. As the previous sum is bounded by $1/2$, $\mathrm{wt}(A) \leq 1$. By the Machine Existence Theorem, we uniformly obtain the index of a corresponding prefix-free machine Q such that for every such pair $(k, |\sigma| - 1) \in A$, there is a τ with $|\tau| = |\sigma| - 1$ and $Q(\tau) = k$.

We define $U(1^\frown\tau) = Q(\tau)$ for all τ. Suppose σ is first claimed at stage s, and so we enumerate $(k, |\sigma| - 1)$ into A for some large k. Then for the appropriate τ, $|1^\frown\tau| = |\sigma|$ and $U(1^\frown\tau) = k$. By the largeness of k, N_s contains no codes for k, so τ will belong to N_U unless V enumerates a sufficiently shorter code at some subsequent stage. The idea is that whenever a potential element of N_U is claimed, we ensure it is replaced with a new element of the same length or shorter.

This completes the description of U, apart from describing how strategies claim strings. We order our requirements $R_0, P_0, R_1, P_1, \ldots$. At stage s, we consider the first s requirements in order, implementing the following strategies.

Strategy for P_n:

P_n will always have the *meet* directive, and will claim at most one string at a time. At stage s, if it retains a claimed string from the previous stage (i.e., $s > 0$, P_n had

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a claimed string at stage $s-1$, that string remains in N_s, and that string has not been claimed by a higher priority strategy earlier in stage s), then we take no further action. Otherwise, if there is a string in N_s of length at least $2n+4$ and unclaimed by any higher priority strategy, P_n claims the least one (in some effective ordering). If P_n did not retain a claimed string, and there is no appropriate string to claim, we simply do nothing.

Strategy for R_e: The behaviour of R_e is a yo–yo: it continues to claim strings until the weight of the claimed strings surpasses $2^{-(2e+4)}$, at which point it stops and lets those strings bleed away. Once it has lost all of its claims, the strategy begins claiming new strings again. The details follow.

At stage s, let C be the set of strings which R_e retains the claim on from the previous stage: those strings which it claimed at stage $s-1$, which remain in N_s, and which were not claimed by a higher priority strategy earlier in stage s. If R_e was not considered at stage $s-1$ (possibly because $s=0$), then $C=\emptyset$.

Let $w=\sum_{\sigma\in C}2^{-|\sigma|}$. Our directive for R_e at stage s will depend on w and on R_e's directive at stage $s-1$:

- If $w=0$ (i.e., $C=\emptyset$), then R_e has the directive *avoid* at stage s.
- If $0<w\leq 2^{-(2e+4)}$, then R_e retains the same directive as it had at the previous stage ($w>0$ entails that R_e was considered at the previous stage).
- If $w>2^{-(2e+4)}$, then R_e has the directive *meet* at stage s.

If our directive for R_e at stage s is *meet*, we take no further action at this stage.

If our directive for R_e at stage s is *avoid*, and there is a string in $N_s\cap W_{e,s}$ of length at least $2e+4$ and unclaimed by any higher priority strategy, then R_e claims the least such string (in some effective ordering). Our action for R_e at this stage is then complete. Note that we claim no more than one string for R_e at each stage.

This completes the construction.

Definition of X:

Let X_s be the set of strings σ claimed by a strategy with the *meet* directive at stage s.

Observe that if $\sigma\in N_U$ is claimed by a strategy, then there are only two possibilities: that strategy may retain its claim on σ forever, or the claim on σ may pass to a higher priority strategy. As we will argue, each R_e strategy changes its directive only finitely many times. It follows that $X=\lim_s X_s$ is defined.

Verification:

First we must keep our promises.

CLAIM 5.3. *At any stage s, for the strategy R_e, the value $w=w(e,s)$ is at most* $2^{-(2e+3)}$.

PROOF. At each stage, R_e claims at most one string, and that string will always have length at least $2e+4$. So $w(e,s+1)-w(e,s)\leq 2^{-(2e+4)}$. Further, R_e only claims a string if $w(e,s)\leq 2^{-(2e+4)}$, so $w(e,s+1)$ is at most $2^{-(2e+4)}+2^{-(2e+4)}=2^{-(2e+3)}$. ⊣

CLAIM 5.4. *The sum $\sum 2^{-|\sigma|}$ over all strings σ which are ever claimed in the course of the construction is at most* $1/2$.

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PROOF. If a string σ is claimed during the construction, there are three possible fates for σ: 1) there is some n such that P_n claims σ at almost every stage; 2) there is some e such that R_e claims σ at almost every stage; 3) σ leaves N_U. We consider each case in turn.

Fix n. By construction, there is at most one string which is ultimately claimed by P_n, and such a string has length at least $2n+4$. So the sum $\sum 2^{-|\sigma|}$ over all strings σ ultimately claimed by P_n is bounded by $2^{-(2n+4)}$.

Fix e. Let $\hat{C}$ be the set of strings σ such that R_e claims σ at almost every stage. Then each $\sigma \in \hat{C}$ contributes to almost every $w(e,s)$, so $\sum_{\sigma\in\hat{C}} 2^{-|\sigma|} \leq \sup_s w(e,s) \leq 2^{-(2e+3)}$.

Finally, consider strings which leave N_U. We will split this case into two subcases, based on whether the given string extends 000, and so was introduced by our copying of V, or it extends 1, and so was introduced by our other actions. Since the extensions of 000 in the domain of U form an antichain, the sum $\sum 2^{-|\sigma|}$ over all strings in the first subcase is bounded by $1/8$.

Consider now the second subcase. Note that strings never leave N_U because of our action; instead, if σ leaves N_U, then there must be some $\tau \in N_V$ with $V(\tau) = U(\sigma)$ and $|\tau| < |\sigma| - 3$. As we always choose our values large, if σ_0 and σ_1 are distinct strings from this subcase, $U(\sigma_0) \neq U(\sigma_1)$, so the corresponding τs are distinct. So summing over the σ of this subcase, we have $\sum 2^{-|\sigma|} < \frac{1}{8}\sum_{\tau\in N_V} 2^{-|\tau|} < \frac{1}{8}$. Putting these all together, our desired sum is bounded by

$$6\sum_{n\in\omega} 2^{-(2n+4)} + \sum_{e\in\omega} 2^{-(2e+3)} + \frac{1}{8} + \frac{1}{8} \leq \frac{1}{2},$$

as desired. ⊣

CLAIM 5.5. *Each R_e changes its directive only finitely many times.*

PROOF. Suppose not. Then there is a sequence of stages $s_0 < s_1 < \ldots$ such that R_e has directive *avoid* at stage s_i, and has directive *meet* at stage $s_i + 1$, for every i. In order to switch from *meet* at stage $s_i + 1$ back to *avoid* at stage s_{i+1}, every string claimed by R_e at stage $s_i + 1$ must either be stolen by a higher priority strategy or leave N_U, both of which are irreversible. Thus the strings which contribute to $w(e, s_i + 1)$ must be entirely different from those which contribute to $w(e, s_j + 1)$ for $j \neq i$. But $w(e, s_i + 1) > 2^{-(2e+4)}$ for every i, and the strings which contribute to $w(e, s_i + 1)$ all belong to $\mathrm{dom}(U)$, a contradiction. ⊣

Thus we may speak of an R_j strategy's ultimate directive.

Our promises being met, the construction of X is as described. Now we verify that X has the desired properties.

CLAIM 5.6. *For each n, the strategy P_n ultimately claims a string which it never renounces its claim upon, and thus X is infinite.*

PROOF. Fix n. It suffices to argue that there is some string in N_U of length at least $2n+4$ which is never claimed by any strategy.

Fix s_0 such that N_{s_0} has converged on all strings of length less than $2n+4$. We build a sequence $\sigma_0, \sigma_1, \ldots \in N_U$:

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- Fix some $\sigma_0 \in N_U \setminus \mathrm{dom}(U_{s_0})$. Such a σ_0 must exist, as U is universal (and in particular, surjective).
- If σ_i is eventually claimed by some strategy, then the construction responds by enumerating a τ into $\mathrm{dom}(U)$ with $|\tau| = |\sigma_i|$ and $U(\tau)$ not any previously seen value. It may be that τ is not a minimal code, but there is some $\sigma_{i+1} \in N_U$ with $U(\sigma_{i+1}) = U(\tau)$, and σ_{i+1} enters $\mathrm{dom}(U)$ after the stage at which σ_i is claimed (by the largeness of $U(\tau)$).

Inductively, we see that $U(\sigma_i) \neq U(\sigma_j)$ for any $j < i$, and so the sequence $\sigma_0, \sigma_1, \ldots$ is injective. Further, $|\sigma_{i+1}| \leq |\sigma_i|$. As $\sum_{\sigma \in N_U} 2^{-|\sigma|} < 1$, this sequence must be finite, so there is some σ_i which is never claimed by any strategy. Since σ_i enters $\mathrm{dom}(U)$ after stage s_0, $|\sigma_i| \geq 2n + 4$ by choice of s_0. ⊣

CLAIM 5.7. *X is semi-low.*

PROOF. We give an algorithm for determining whether $X \cap W_e = \emptyset$, using oracle $\emptyset'$. First, $\emptyset'$ can determine a stage s_0 such that every R_j strategy with $j \leq e$ has settled on its ultimate directive, and such that each P_n strategy with $n < e$ has made its ultimate claim.

We may ignore those R_j with $j \leq e$ which have *avoid* as their ultimate directive. For the remaining, they have claimed some finitely many strings by stage s_0, and none will ever claim another string. With oracle $\emptyset'$, we can examine the entire finite collection to determine if there is a string σ among them which remains claimed by its current strategy forever, and with $\sigma \in W_e$.

We claim that there is such a σ if and only if $X \cap W_e \neq \emptyset$. In the one direction, if there is such a σ, then $\sigma \in X$ by construction, so $\sigma \in X \cap W_e$.

In the other direction, if there is no such σ, note that this implies that R_e's ultimate directive is *avoid*—if it were *meet*, then R_e's strings are amongst those examined, so it must eventually renounce its claim to all of them, resulting in R_e changing directive to *avoid*, contrary to choice of s_0. Now for any $\tau \in W_e \cap N_U$, we consider two cases: $|\tau| < 2e + 4$ and $|\tau| \geq 2e + 4$.

If $|\tau| < 2e + 4$, then τ is too short to be claimed by any strategy of lower priority than R_e, and by assumption τ cannot be ultimately claimed by any strategy of higher priority with ultimate directive *meet*. So $\tau \notin X$.

If $|\tau| \geq 2e + 4$, then τ will eventually be claimed by R_e, by construction, and so $\tau \notin X$. ⊣

This completes the proof. ⊣

COROLLARY 5.8. *For some, and hence any, universal prefix-free machine U,*

$$\limsup_{\tau \in N_U} K(\sigma \mid \tau) = \limsup_{\tau \in M_U} K(\sigma \mid \tau) =^+ K^{\emptyset'}(\sigma).$$

PROOF. As this is independent of choice of machine, let U and X be as in Proposition 5.2. Then

$$K^{\emptyset'}(\sigma) =^+ \limsup_{\tau \in X} K(\sigma \mid \tau) \leq \limsup_{\tau \in N_U} K(\sigma \mid \tau) \leq \limsup_{\tau \in M_U} K(\sigma \mid \tau) \leq \limsup_{\tau \in 2^{<\omega}} K(\sigma \mid \tau) =^+ K^{\emptyset'}(\sigma),$$

where the first equality is by Proposition 4.2, the last is by Theorem 1.3, and the inequalities are by subset. ⊣

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§6. Where the limsup's live, and finite strings as oracles. Here we collect some miscellaneous results about finite strings as oracles. The first is motivated by our (numerous!) failed attempts to prove Theorem 1.7 before we finally discovered the method of Sections 4 and 5.

One avenue we pursued was attempting to determine for which m does $K(\sigma \mid m)$ achieve $\limsup_n K(\sigma \mid n)$. A natural candidate is the nondeficiency stages: fix a computable enumeration $(a_m)_{m\in\omega}$ of $\emptyset'$, and define

$$E = \{m : (\forall n > m)\,[a_m < a_n]\}.$$

This is the basis for the method of *true stages* (see Montalbán [12] for a modern interpretation for higher level priority arguments, but the idea going back to Dekker [4], as per Soare [17, Chapter V 2.5]), where the elements of E are employed because they make correct guesses about $\emptyset'$ (as we shall see in a moment).

However, this turns out to be approaching from the wrong direction. Since $K^{\emptyset'}(\sigma) = \limsup_n K(\sigma \mid n)$, to find places where the limit supremum is achieved, we are not concerned with doing *as well as* $\emptyset'$—we are concerned with doing *no better than* $\emptyset'$. Thus we are looking not for n which are powerful, but for those which are weak.

The following result says that for $m \in E$, $K(\sigma \mid m)$ does much better than $\limsup_n K(\sigma \mid n)$.

Proposition 6.1.

$$\limsup_{m\in E} K(\sigma \mid m) =^+ K^{\emptyset^{(2)}}(\sigma).$$

Proof. For any m, define $\tau_m \in 2^{<\omega}$ to be the string of length a_m such that $\tau_m(x) = 1$ iff $x = a_n$ for some $n < m$. Note that $m \mapsto \tau_m$ is effective, so $K(\sigma \mid m) \leq^+ K(\sigma \mid \tau_m)$. Also, $\sigma_m \prec \emptyset'$ iff $m \in E$, so

$$\begin{aligned}\limsup_{m\in E} K(\sigma \mid m) &\leq^+ \limsup_{m\in E} K(\sigma \mid \emptyset'\restriction a_m)\\ &\leq \limsup_n K(\sigma \mid \emptyset'\restriction n)\\ &=^+ \limsup_n K^{\emptyset'}(\sigma \mid n).\end{aligned}$$

Conversely, $\emptyset'$ can compute the increasing enumeration of E, $E = \{b_0 < b_1 < \dots\}$, so $K^{\emptyset'}(\sigma \mid n) \leq^+ K(\sigma \mid b_n)$, giving

$$\limsup_{m\in E} K(\sigma \mid m) =^+ \limsup_n K^{\emptyset'}(\sigma \mid n).$$

By Theorem 1.3 relativized to $\emptyset'$, this is (up to an additive constant) $K^{\emptyset^{(2)}}(\sigma)$. ⊣

The reader might note the following somewhat paradoxical situation. The natural proof to show that $K^{\emptyset'}(\sigma) \leq^+ \limsup_n K(\sigma \mid n)$ is to fully approximate $K^{\emptyset'}(\sigma)[n]$ at each stage n, where both the computations and oracles are approximated for n stages. We would do this as part of the computation of $K^{\overline{n}}(\sigma)$ for some machine $M^{\overline{n}}(\sigma)$ via the Machine Existence Theorem as mentioned above. Therefore, for all stages $t > n$ it can only be that $K(\sigma \mid n)[t] \leq^+ K(\sigma \mid n)[n] \leq^+ K^{\emptyset'}(\sigma)[n]$. The true

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value of $K^{\emptyset'}(\sigma)$ must have been achieved at a true stage, but we see above, it does not happen at almost all true stages. Thus it must be achieved at infinitely many non-true stages s, but where $K^{\emptyset'}[s] =^+ K^{\emptyset'}(\sigma)$. We don't really understand the characteristics of such "almost true" stages s. We also point out that the limsups appear to be achieved for different s's for different σ's.

Theorem 1.3 says that for almost any string σ, almost any finite oracle can aid in the compression of σ. A priori, however, there is no reason to expect there to be a single finite oracle which aids in the compression of almost every string. Nevertheless, this is the case.

THEOREM 6.2. *For all e there is a string ρ such that for almost all τ, $K(\tau \mid \rho) < K(\tau) - e$. That is for $\xi = \overline{\rho}$, $K^{\xi}(\tau) < K(\tau) - e$.*

Our proof is based on the conditional complexity variant of the Machine Existence Theorem, which we first state.

PROPOSITION 6.3. *Suppose $A \subseteq 2^{<\omega} \times \omega \times 2^{<\omega}$ is a c.e. set such that for every $\tau \in 2^{<\omega}$, $\sum_{(\sigma,s,\tau)\in A} 2^{-s} \leq 1$. Then for all $(\sigma, s, \tau) \in A$, $K(\sigma \mid \tau) \leq^+ s$.*

PROOF OF THEOREM 6.2. We will enumerate a c.e. set A. Fix an effective bijection $\tau = \langle D, k\rangle$ between $\tau \in 2^{<\omega}$ and pairs $\langle D, k\rangle$ with $D \subset 2^{<\omega}$ finite and $k \in \omega$. For $\tau = \langle D, k\rangle$, for every $\sigma \in 2^{<\omega} \setminus D$, we enumerate $(\sigma, t - k, \tau)$ into A for every $t \geq K(\sigma)$, provided doing so does not cause $\sum_{(\sigma,s,\tau)\in A} 2^{-s}$ to exceed 1.

Fix the constant c such that $K(\sigma \mid \tau) \leq s + c$ for every $(\sigma, s, \tau) \in A$. As $\sum_{\sigma} 2^{-K(\sigma)} < 1$, there is some finite D such that $\sum_{\sigma\notin D} 2^{-K(\sigma)} < 2^{-(e+c+1)}$. Fix $\tau = \langle D, e + c\rangle$. Then $\sum_{\sigma\notin D}\sum_{t\geq K(\sigma} 2^{-(t-e-c)} < 1$, so $(\sigma, t - e - c, \tau)$ is successfully enumerated into A for all such σ and t. Thus $K(\sigma \mid \tau) \leq K(\sigma) - e - c + c = K(\sigma) - e$ for all $\sigma \notin D$. ⊣

Define ρ to be *e-compressing* if for almost all τ, $K(\tau \mid \rho) < K(\tau) - e$.

QUESTION 6.4. *What can be said about the set $C_e = \{\rho \mid \rho$ is e-compressing$\}$?*

QUESTION 6.5. *Is there a definition of k-randomness purely involving C?*

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SCHOOL OF MATHEMATICS AND STATISTICS
VICTORIA UNIVERSITY
PO BOX 600, WELLINGTON 6140, NEW ZEALAND
E-mail: rod.downey@vuw.ac.nz
E-mail: dan.turetsky@vuw.ac.nz

SCHOOL OF MATHEMATICS AND STATISTICS
HNP-LAMA, CENTRAL SOUTH UNIVERSITY
CHANGSHA, HUNAN 410083, CHINA
E-mail: g.jiayi.liu@gmail.com

DIVISION OF MATHEMATICAL SCIENCES
SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES
NANYANG TECHNOLOGICAL UNIVERSITY
21 NANYANG LINK, SINGAPORE 637371, SINGAPORE
E-mail: kmng@ntu.edu.sg

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