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CONTACT AND QUASICONFORMAL MAPPINGS ON REAL MODEL FILIFORM GROUPS

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We prove that Carnot groups with real model filiform Lie algebras are not rigid. Consequently non-trivial smooth contact and quasiconformal mappings exist in abundance.

1. INTRODUCTION

A Carnot group is a connected, simply connected nilpotent Lie group, equipped with a left-invariant sub-Riemannian metric, defined on a left-invariant sub-bundle of the tangent bundle. The sub-bundle is called the horizontal bundle and the metric is called a Carnot-Caratheodory metric. Diffeomorphisms which preserve the horizontal bundle are called contact maps. Quasiconformal maps are defined with respect to the Carnot-Caratheodory metric which, in a weak sense, implies they must also be contact maps. Carnot groups are said to be rigid when the space of contact maps is finite dimensional, which appears to be the rule rather than the exception. The Euclidean spaces and the real and complex Heisenberg groups are the only established examples of a non-rigid groups. This paper establishes the non-rigidity of Carnot groups whose Lie algebra is real and model filiform.

Quasiconformal mappings on Carnot groups were first considered by Mostow [**6**]. In the proof of his celebrated rigidity theorem, Carnot groups arise as the one point compactifications of the boundaries of non-compact rank one symmetric spaces with negative sectional curvature, that is, the hyperbolic spaces H^n_K, where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and the 16-dimensional Cayley hyperbolic plane H^{16}_O. A homeomorphism $h : M \to N$ between negatively curved locally symmetric spaces of rank one lifts to a homeomorphism $\widetilde{h} : H^{n_1}_{K_1} \to H^{n_2}_{K_2}$, equivariant with respect to the action of the fundamental groups of M and N, which implies that $\widetilde{h}$ induces a quasiconformal map of the spheres at infinity. The rigidity theorem follows by showing that the induced quasiconformal map is conformal, thus implying $\widetilde{h}$ is equivalent to an isometry, $K_1 = K_2$ and $n_1 = n_2$.

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For $H^n_{\mathbb{R}}$, the sphere at infinity is $S^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$, hence the quasiconformal theory is relative to the Euclidean metric. It is well known that Euclidean spaces are not rigid, for example, see [8]. For $H^n_{\mathbb{C}}$ the sphere at infinity is the Heisenberg group H_n. In this setting Korányi and Reimann [4, 5] showed that the spaces of contact and quasiconformal maps are also infinite dimensional, and furthermore, Reimann and Ricci [10] proved similar results for the complexified Heisenberg group $\mathbb{C} \otimes H^3_{\mathbb{R}}$.

Pansu [7] established the rigidity of the remaining cases H^n_K, where $K = \mathbb{H}$ and H^{16}_O. These cases are particular examples of Heisenberg-type groups, and Reimann [9] established the rigidity of all Heisenberg-type groups with centre of dimension at least 3. Pansu's crucial observation was, that in a certain sense, contact and quasiconformal mappings are differentiable and that the derivative must be an isomorphism of the group. Usually, nilpotent Lie groups have few automorphisms, which is why Carnot groups are usually rigid.

2. Carnot Groups

An n-step *stratification* of a nilpotent Lie algebra $\mathfrak{g}$ is a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ such that $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$, where $j = 1, \ldots, n-1$, and $\mathfrak{g}_n$ is contained in the centre $Z(\mathfrak{g})$. A *Carnot group* is a connected, simply connected nilpotent Lie group G, with stratified Lie algebra equipped with an inner product such that $\mathfrak{g}_i \perp \mathfrak{g}_j$, $i \neq j$.

For simply connected nilpotent Lie groups, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism which becomes an isomorphism $(\mathfrak{g}, *) \to G$ when we define

$$X * Y = \exp^{-1}\bigl(\exp(X)\exp(Y)\bigr).$$

The Baker–Campbell–Hausdorff formula gives an explicit expression for $X * Y$. Choosing a basis for $\mathfrak{g}$ identifies $(\mathfrak{g}, *)$ with $\mathbb{R}^{\dim \mathfrak{g}}$ and $X * Y$ becomes polynomial of degree $\leqslant n-1$. A coordinate system of this type is said to be *normal of the first kind.*

In a similar fashion we obtain *normal coordinates of the second kind.* Given a basis $\{e_j\}_{j=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$ the map $\pi : \mathfrak{g} \to G$ given by

$$X = \sum_j x_j e_j \xrightarrow{\pi} \prod_j \exp(x_j e_j)$$

is a diffeomorphism, [11, p. 86], which becomes an isomorphism $(\mathfrak{g}, \odot) \to G$ when we define

$$X \odot Y = \pi^{-1}\bigl(\pi(X)\pi(Y)\bigr).$$

As before $X \odot Y$ becomes polynomial of degree $\leqslant n-1$.

Left translation, denoted $\tau_X Y$, is the analogue of translation in Euclidean spaces. Specifically $\tau_X Y = X * Y$ in coordinates of the first kind and $\tau_X Y = X \odot Y$ in coordinates of the second kind. An important feature of Carnot groups is an analogue of dilation.

For $t > 0$, the *dilation* $\delta_t : \mathfrak{g} \to \mathfrak{g}$ is given by $\delta_t(X) = \sum_{j=1}^{n} t^j X_j$ where $X = \sum_{j=1}^{n} X_j$, $X_j \in \mathfrak{g}_j$, which defines dilation on G via the coordinate systems.

Let

$$\{ e_{i,\alpha(i)} \mid i = 1, \dots, n, \ \alpha(i) = 1, \dots, \dim \mathfrak{g}_i \},$$

denote a basis of $\mathfrak{g}$ such that

$$\mathfrak{g}_i = \operatorname{span}\{ e_{i,\alpha(i)} \mid \alpha(i) = 1, \dots, \dim \mathfrak{g}_i \}$$

and let

$$\{\lambda_{i,\beta(i)} \mid i = 1, \dots, n, \ \beta(i) = 1, \dots, \dim \mathfrak{g}_i\} \subset \mathfrak{g}^*$$

denote the corresponding dual basis such that

$$\lambda_{i,\beta(i)}(e_{j,\alpha(j)}) = \begin{cases} 1 & \text{if } i = j \text{ and } \beta(i) = \alpha(i) \\ 0 & \text{otherwise.} \end{cases}$$

Since we identify $\mathfrak{g}$ with G, the $\lambda_{i,\alpha(i)}$ serve the dual role as coordinates on G and TG, that is, $V \in T_X G$ has coordinates

$$(\dots, \lambda_{i,\alpha(i)}(X), \dots, \dots, \lambda_{i,\alpha(i)}(V), \dots)$$

since

$$d\lambda_{i,\alpha(i)}(V) = \frac{d}{dt}\lambda_{i,\alpha(i)}(X + tV)\Big|_{t=0} = \frac{d}{dt}\{\lambda_{i,\alpha(i)}(X) + t\lambda_{i,\alpha(i)}(V)\}\Big|_{t=0}.$$

The fields $(X_{i,\alpha(i)})_X = \tau_{X*} e_{i,\alpha(i)}$ form a basis for the left-invariant fields and the corresponding dual forms are $(\theta_{i,\alpha(i)})_X = (\tau_X^{-1})^* d\lambda_{i,\alpha(i)}$. The stratification of the left-invariant fields is then given by

$$L_1 = \operatorname{span}\{X_{1,\alpha(1)} \mid \alpha(1) = 1, \dots \dim \mathfrak{g}_1\}$$

and

$$L_{j+1} = [L_1, L_j] = \operatorname{span}\{X_{j+1,\alpha(j+1)} \mid \alpha(j+1) = 1, \dots \dim \mathfrak{g}_{j+1}\}$$

where $j = 1, \dots, n-1$. Furthermore the basis

$$\{X_{i,\alpha(i)} \mid i = 1, \dots, n, \ \alpha(i) = 1, \dots, \dim \mathfrak{g}_i\}$$

is orthonormal relative to the inner product

$$\langle V, W \rangle_G = \sum_i \theta_{i,\alpha(i)}(V)\theta_{i,\alpha(i)}(W). \tag{2.1}$$

A curve γ in G is said to be horizontal if $\dot\gamma(t) \in L_1(\gamma(t))$. If $\Gamma_0(X, Y)$ denotes the set of horizontal curves joining X to Y then the *Carnot–Caratheodory* distance is

$$d_c(X, Y) = \inf_{\gamma \in \Gamma_0(X,Y)} \int \|\dot\gamma(t)\|_G \, dt,$$

where $\|\dot\gamma(t)\|_G = \sqrt{\langle\dot\gamma(t),\dot\gamma(t)\rangle_G}$. The theorem of Chow [**1**] implies that Carnot groups are path connected via horizontal curves and that d_c is a metric. By definition d_c is left-invariant, that is, $d_c(\tau_z X, \tau_z Y) = d_c(X,Y)$, and homogeneous with respect to dilation, that is, $d_c(\delta_t X, \delta_t Y) = t d_c(X,Y)$.

If $f : \Omega_1 \to \Omega_2$ is a diffeomorphism between open sets $\Omega_1, \Omega_2 \subseteq G$ and $V \in T_X G$, $X \in \Omega_1$ then

$$f_*(V) = \sum_i \sum_{\alpha(i)} \sum_j \sum_{\beta(j)} \lambda_{i,\alpha(i)}(f_* e_{j,\beta(j)}) \lambda_{j,\beta(j)}(V) e_{i,\alpha(i)} \tag{2.2}$$

$$= \sum_i \sum_{\alpha(i)} \sum_j \sum_{\beta(j)} \theta_{i,\alpha(i)}(f_* X_{j,\beta(j)}) \theta_{j,\beta(j)}(V) X_{i,\alpha(i)}, \tag{2.3}$$

and we use the notation Jf and Df to denote the matrices

$$\Big(\lambda_{i,\alpha(i)}\big(f_* e_{j,\beta(j)}\big)\Big) \quad \text{and} \quad \Big(\theta_{i,\alpha(i)}\big(f_* X_{j,\beta(j)}\big)\Big).$$

Note that the substitutions

$$(X_{i,\alpha(i)})_X = \tau_{X\,*} e_{i,\alpha(i)}$$

and

$$(\theta_{i,\alpha(i)})_X = (\tau_X^{-1})^* d\lambda_{i,\alpha(i)}$$

show that

$$Df(X) = J(\tau_{f(X)}^{-1} \circ f \circ \tau_X)(0). \tag{2.4}$$

If $f_*\big(L_1(X)\big) \subseteq L_1(X)$ then f is called a *contact* diffeomorphism, the trivial examples being left translation and dilation. For such a diffeomorphism, $\theta_{i,1}(f_* X_{1,\beta(1)}) = 0$ when $1 < i$ and $1 \leqslant \beta(1) \leqslant \dim L_1$, however more is true. Indeed if $f_*(L_1) \subseteq L_1$ then $f_*\big(L_j(X)\big) \subseteq L_1(X) \oplus \cdots \oplus L_j(X)$, since for any pair of smooth vector fields V and W we have $f_*\big[V(X), W(X)\big] = \big[f_* V(X), f_* W(X)\big]$. It follows that f is a contact diffeomorphism if and only if

$$\theta_{i,\alpha(i)}\big(f_* X_{j,\beta(j)}\big) = 0, \quad 1 \leqslant j < i. \tag{2.5}$$

3. Quasiconformal maps

The *metric definition* of quasiconformality is as follows. Let Ω_1 and Ω_2 be open subsets of G, let $f : \Omega_1 \to \Omega_2$ be a homeomorphism and define

$$H_f(X) = \limsup_{r\to 0} \frac{\sup\{d_c(f(X), f(Y)) : d_c(X,Y) = r\}}{\inf\{d_c(f(X), f(Y)) : d_c(X,Y) = r\}}.$$

Then f is said to be K-quasiconformal if H_f is bounded and

$$\operatorname{ess\,sup}\big|H_f(X)\big| = \|H_f\|_\infty \leqslant K.$$

The trivial examples are dilations and left translations which are 1-quasiconformal.

The analytic definition, due to Pansu [7], requires the notion of P-differentiability. A homeomorphism f is P-differentiable at $X \in G$, if for each $Y \in G$,

$$\phi(Y) = \lim_{t\to 0} \delta_{1/t} \circ \tau_{f(X)}^{-1} \circ f \circ \tau_X \circ \delta_t(Y)$$

converges locally uniformly in Y as $t \to 0$, and defines an isomorphism of G. The P-differential of f at X is the corresponding Lie algebra isomorphism ϕ_*.

The *analytic definition* is as follows. A homeomorphism $f : \Omega_1 \to \Omega_2$ is K-quasiconformal if and only if the horizontal distributional derivatives exist and belong to L^Q_{loc}, f is P-differentiable almost everywhere with the P-differential satisfying

$$\|\phi_*\|_X^Q \leqslant K \det Jf(X)$$

where $Q = \sum_i i \dim \mathfrak{g}_i$ is the *homogeneous dimension* of G,

$$\|\phi_*\|_X = \sup_{V \in L_1(X), \|V\|_G = 1} \|\phi_* V\|_G,$$

and $\det Jf(X)$ is the generalized Jacobian determinant of f.

LEMMA 1. *A diffeomorphism $f : \Omega_1 \to \Omega_2$ between open sets $\Omega_1, \Omega_2 \subseteq G$, is P-differentiable if and only if it is a contact diffeomorphism.*

PROOF: From (2.4) we have

$$\psi_X(Y) = \tau_{f(X)}^{-1} \circ f \circ \tau_X(Y) = Df(X)Y + E(X,Y) \tag{3.1}$$

hence if f is P-differentiable, then $\lim_{t\to 0} \delta_{1/t}\big(DF(X)\delta_t(Y)\big)$ exists, forcing $\theta_{i,\alpha(i)}(f_* X_{j,\alpha(j)}) = 0$ for $1 < i$ and $j < i$ which implies that f is a contact diffeomorphism.

Conversely if f is a contact diffeomorphism, then continuous differentiability implies that the map is locally Lipschitz with respect to the Carnot–Caratheodory metric. Let $B_l(Z_0)$ be a Carnot–Caratheodory ball with centre Z_0 and radius l. If $X, Y \in B_l(Z_0)$, then the triangle inequality implies that $d_c(X,Y) \leqslant 2l$. Let $U_l(X,Y) \subset \Gamma_0(X,Y)$ be the set of horizontal curves joining X to Y of length at most $2l$. If $Z = \gamma(t)$ for some $\gamma \in U_l(X,Y)$, then the triangle inequality implies that $d_c(Z_0, Z) \leqslant 3l$, hence

$$\bigcup_{X,Y \in B_l(Z_0)} U_l(X,Y) \subseteq B_{3l}(Z_0).$$

If $\gamma \in U_l(X,Y)$, then by (2.3) and the Schwarz inequality,

$$\begin{aligned}
\|f_*\dot\gamma(t)\|^2 &= \sum_{\alpha(1)} \Big(\sum_{\beta(1)} \theta_{1,\alpha(1)}(f_* X_{1,\beta(1)}) \theta_{1,\beta(1)}(\dot\gamma(t)) \Big)^2 \\
&\leqslant \Big(\sum_{\alpha(1)} \sum_{\beta(1)} \big(\theta_{1,\alpha(1)}(f_* X_{1,\beta(1)})\big)^2 \Big) \|\dot\gamma(t)\|^2 \\
&= C_f(\gamma(t)) \|\dot\gamma(t)\|^2.
\end{aligned}$$

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It now follows that, if $X, Y \in B_l(Z_0)$, then

$$d_c(f(X), f(Y)) \leqslant \inf_{\gamma \in U_l(X,Y)} \int \|f_* \dot\gamma(t)\|_G dt \leqslant \sup_{Z \in B_{3l}(Z_0)} \sqrt{C_f(Z)} d_c(X, Y),$$

hence f is Lipschitz.

Pansu's theorem, [7, Thm 2], states that Lipschitz maps are P-differentiable almost everywhere. Since f is Lipschitz and continuously differentiable, we conclude that it is P-differentiable everywhere. □

4. Model filiform groups

A stratified nilpotent Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ is *model filiform* if it has a basis $\{e_1, \dots, e_n\}$ such that

$$\begin{aligned} &[e_i, e_j] = 0, && i, j > 1 \\ &[e_i, e_n] = 0, && i \geqslant 1 \\ &[e_1, e_j] = e_{j+1}, && j = 2, \dots, n-1 \\ &\mathfrak{g}_1 = \operatorname{span}\{e_1, e_2\} && \\ &\mathfrak{g}_j = \operatorname{span}\{e_{j+1}\} && j = 2, \dots, n-1. \end{aligned}$$

A connected, simply connected, real nilpotent Lie group G is called model filiform if its Lie algebra is model filiform, moreover modelfiliform groups are Carnot groups.

In coordinates of the second kind, the elements of the corresponding connected, simply connected Lie group G take the form

$$\exp(x_1 e_1) \exp(x_2 e_2 + \cdots + x_n e_n).$$

The multiplication in terms of the identification with $\mathbb{R}^n$ can be found by solving

$$\exp(w_1 e_1) \exp\left(\sum_{j=1}^{n} w_j e_j\right) = \exp(x_1 e_1) \exp\left(\sum_{j=1}^{n} x_j e_j\right) \exp(y_1 e_1) \exp\left(\sum_{j=1}^{n} y_j e_j\right) \tag{4.1}$$

for w. To this end we use the following model of $\mathfrak{g}$ in $GL(n, \mathbb{R})$:

$$\sum x_j e_j \sim \begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 & x_n \\ 0 & 0 & x_1 & \cdots & 0 & x_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & x_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_1 & x_3 \\ 0 & 0 & 0 & \cdots & 0 & x_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It follows that

$$\exp(x_1 e_1) \sim \begin{pmatrix} 1 & x_1 & x_1^2/2! & x_1^3/3! & \cdots & x_1^{n-2}/(n-2)! & 0 \\ 0 & 1 & x_1 & x_1^2/2! & \cdots & x_1^{n-3}/(n-3)! & 0 \\ 0 & 0 & 1 & x_1 & \cdots & x_1^{n-4}/(n-4)! & 0 \\ 0 & 0 & 0 & 1 & \cdots & x_1^{n-5}/(n-5)! & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix},$$

say, and

$$\exp\Big(\sum_{j=1}^{n} x_j e_j\Big) \sim \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & x_n \\ 0 & 1 & 0 & 0 & \cdots & 0 & x_{n-1} \\ 0 & 0 & 1 & 0 & \cdots & 0 & x_{n-2} \\ 0 & 0 & 0 & 1 & \cdots & 0 & x_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & x_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & v(x) \\ 0 & 1 \end{pmatrix}.$$

Substituting these expressions into (4.1) gives

$$\begin{pmatrix} A(w) & A(w)v(w) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A(x)A(y) & A(x)A(y)v(y) + A(x)v(x) \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

From (4.2), we have

$$A(w) = A(x)A(y) \quad \text{and} \quad A(w)v(w) = A(x)A(y)v(y) + A(x)v(x).$$

It follows that

$$w_1 = x_1 + y_1 \quad \text{and} \quad v(w) = v(y) + A(y)^{-1}v(x),$$

hence, in terms of the identification with $\mathbb{R}^n$. multiplication takes the form

$$(x_1, \dots, x_n) \odot (y_1, \dots, y_n) = (w_1, \dots, w_n)$$

where

$$w_1 = x_1 + y_1, \quad w_2 = x_2 + y_2, \quad w_k = x_k + y_k + \sum_{j=1}^{k-2} \frac{(-1)^j}{j!} y_1^j x_{k-j}, \quad k \geqslant 3 \tag{4.3}$$

and $x^{-1} = (\tilde{x}_1, \dots, \tilde{x}_n)$ where

$$\tilde{x}_1 = -x_1, \quad \tilde{x}_2 = -x_2, \quad \tilde{x}_k = -\sum_{j=0}^{k-2} \frac{x_1^j x_{k-j}}{j!}, \quad k \geqslant 3.$$

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The left-invariant vector fields take the form

$$X_1 = \frac{\partial}{\partial x_1} - \sum_{k=3}^{n} x_{k-1} \frac{\partial}{\partial x_k} \quad \text{and} \quad X_j = \frac{\partial}{\partial x_j}, \quad j \geqslant 2$$

and the corresponding dual forms are

$$\theta_1 = dx_1 \qquad \theta_2 = dx_2 \qquad \theta_j = dx_j + x_{j-1} dx_1, \quad j \geqslant 3.$$

5. Contact flows

The main result of this paper is that contact and quasiconformal mappings on real model filiform Carnot groups are abundant. This is achieved by showing that the vector fields which generate contact and quasiconformal flows form infinite dimensional vector spaces.

Theorem 1. *A smooth vector field $V = \sum v_i X_i$ defined on $\Omega \subseteq G$ induces a local flow of contact mappings if and only if*

$$V = (-1)^{n-3}(X_2 X_1^{n-3} v_n) X_1 + \sum_{j=2}^{n} (-1)^{n-j} (X_1^{n-j} v_n) X_j, \tag{5.1}$$

where $v_n = F(x_1, x_{n-1}, x_n)$ and $F \in C^\infty(\Omega)$.

Proof: The flow f_t of a vector field V has the contact property $f_{t_*} L_1(x) \subseteq L_1(x)$ if and only if $[L_1, V] \subseteq L_1$, see [**3**, p. 33]. If $V = \sum v_i X_i$, then

$$\begin{aligned} [X_1, V] &= \sum_{i=1}^{n} (X_1 v_i) X_i + v_i [X_1, X_i] = (X_1 v_1) X_1 + \sum_{i=1}^{n-1} \Big(X_1 v_{i+1} + v_i \Big) X_{i+1} \\ [X_2, V] &= \sum_{i=1}^{n} (X_2 v_i) X_i + v_i [X_2, X_i] = \sum_{i=1}^{n} (X_2 v_i) X_i - v_1 X_3, \end{aligned}$$

hence $[L_1, V] \subseteq L_1$ if and only if

$$X_1 v_{i+1} + v_i = 0, \quad i = 2, \dots, n-1 \tag{5.2}$$

$$X_2 v_3 - v_1 = 0 \tag{5.3}$$

$$X_2 v_i = 0, \quad i = 4, \dots, n. \tag{5.4}$$

Formula (5.1) now follows from (5.2) and (5.3), moreover (5.2) and (5.4) show that

$$X_2 X_1^k v_n = 0, \quad k = 0, \dots, n-4. \tag{5.5}$$

Since

$$X_{2+k} = \Big(\mathrm{ad} X_1 \Big)^k X_2 = \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_1^{k-j} X_2 X_1^j, \tag{5.6}$$

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$X_{2+k}v_n = 0$, when $k = 0, \ldots, n-4$, hence $v_n = F(x_1, x_{n-1}, x_n)$ with $F \in C^\infty(\Omega)$.

Conversely, if $v_n = F(x_1, x_{n-1}, x_n)$, where $F \in C^\infty(\Omega)$ and V is defined as in (5.1), then the flow will consist of contact maps provided (5.2), (5.3) and (5.4) are satisfied. By definition, (5.2) and (5.3) are immediately satisfied. To see that (5.4) holds, observe that $X_1^{n-i}F$ depends on $x_1, x_{i-1}, \ldots, x_n$ only, hence

$$X_2 v_i = (-1)^{n-i} X_2 X_1^{n-i} F = 0, \quad i = 4, \ldots, n,$$

as required. ∎

The function F will be called the generator of V. Since the space of generators

$$\{F(x_1, x_{n-1}, x_n) \mid F \in C^\infty(\Omega)\}$$

is infinite dimensional we have the following result.

Corollary 1. *The smooth vector fields which induce contact flows form an infinite dimensional vector space.*

6. Differentials and Quasiconformality

To apply the analytic definition of quasiconformality, we need to compute the P-differential from the differential $Df(X) = \big(\theta_i(f_* X_j)\big)$. By (2.5), if f is a contact diffeomorphism, then

$$\theta_i(f_* X_j) = X_j f_i + f_{i-1} X_j f_1 = 0, \quad 3 \leqslant i \leqslant n, \quad j < i \tag{6.1}$$

and

$$\det Df(X) = \big(\theta_1(f_* X_1)\theta_2(f_* X_2) - \theta_1(f_* X_2)\theta_2(f_* X_1)\big) \prod_{i=3}^{n} \theta_i(f_* X_i) \neq 0. \tag{6.2}$$

Lemma 2. *If f is a contact diffeomorphism of $\Omega \subset G$ then*

(1) $X_j f_1 = 0, \quad j = 2, \ldots, n-2$

(2) $\theta_j(f_* X_j) = (X_1 f_1)^{j-2}(X_2 f_2), \quad j = 3, \ldots, n$

Proof: For $i = 4, \ldots, n$, the substitutions $j = i-1$ and $j = i-2$ in (6.1) give

$$X_{i-1} f_i + f_{i-1}(X_{i-1} f_1) = 0 \tag{6.3}$$

$$X_{i-2} f_i + f_{i-1}(X_{i-2} f_1) = 0. \tag{6.4}$$

Letting X_{i-2} act on (6.3) and X_{i-1} act on (6.4), we obtain

$$X_{i-2} X_{i-1} f_i + (X_{i-2} f_{i-1})(X_{i-1} f_1) + f_{i-1}(X_{i-2} X_{i-1} f_1) = 0 \tag{6.5}$$

$$X_{i-1} X_{i-2} f_i + (X_{i-1} f_{i-1})(X_{i-2} f_1) + f_{i-1}(X_{i-1} X_{i-2} f_1) = 0 \tag{6.6}$$

for $i = 4, \dots, n$. Subtracting (6.5) from (6.6), keeping in mind that X_{i-1} and X_{i-2} commute, gives

$$(6.7) \qquad (X_{i-1}f_{i-1})(X_{i-2}f_1) - (X_{i-2}f_{i-1})(X_{i-1}f_1) = 0, \quad i = 4 \dots n.$$

For $i = 4, \dots, n$, we can make the the substitution $X_{i-2}f_{i-1} = -f_{i-2}(X_{i-2}f_1)$, hence (6.7) becomes

$$(6.8) \ (X_{i-2}f_1)(X_{i-1}f_{i-1} + f_{i-2}X_{i-1}f_1) = (X_{i-2}f_1)\theta_{i-1}(f_*X_{i-1}) = 0, \quad i = 4, \dots, n.$$

Item (1) now follows from (6.2) and (6.8).

To prove item (2) we first recall [**3**, p. 36] that for any pair of smooth vector fields X, Y and any smooth 1-form ω on a manifold M,

$$(6.9) \qquad \omega\big([X, Y]\big) = X\omega(Y) - Y\omega(X) - 2d\omega(X, Y).$$

It follows that for $j = 3, \dots, n$,

$$\begin{aligned} \theta_j(f_*X_j) &= \theta_j\big([f_*X_1, f_*X_{j-1}]\big) \\ &= (f_*X_1)\theta_j(f_*X_{j-1}) - (f_*X_{j-1})\theta_j(f_*X_1) - 2d\theta_j(f_*X_1, f_*X_{j-1}) \\ &= -2d\theta_j(f_*X_1, f_*X_{j-1}) \\ &= -2dx_{j-1} \wedge dx_1(f_*X_1, f_*X_{j-1}) \\ &= -dx_{j-1}(f_*X_1)dx_1(f_*X_{j-1}) + dx_{j-1}(f_*X_{j-1})dx_1(f_*X_1) \\ (6.10) \qquad &= -(X_1f_{j-1})(X_{j-1}f_1) + (X_{j-1}f_{j-1})(X_1f_1). \end{aligned}$$

Moreover if $j > 3$, then we can make the substitution $X_1f_{j-1} = -f_{j-2}X_1f_1$ in (6.10), giving

$$\begin{aligned} \theta_j(f_*X_j) &= f_{j-2}(X_1f_1)(X_{j-1}f_1) + (X_{j-1}f_{j-1})(X_1f_1) \\ &= (X_1f_1)(f_{j-2}X_{j-1}f_1 + X_{j-1}f_{j-1}) \\ (6.11) \qquad &= (X_1f_1)\theta_{j-1}(f_*X_{j-1}). \end{aligned}$$

Putting $j = 3$ in (6.10) gives

$$\theta_3(f_*X_3) = (X_1f_1)(X_2f_2),$$

which together with (6.11) proves item (3). ∎

The P-differential takes the form

$$\phi_* \sim \begin{pmatrix} X_1f_1 & 0 & 0 & 0 & \cdots & 0 \\ X_1f_2 & X_2f_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & X_1f_1X_2f_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & X_1f_1^2X_2f_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & X_1f_1^{n-2}X_2f_2 \end{pmatrix},$$

hence $\|\phi_*\|_g^2$ is the largest eigenvalue of $M^{\mathrm{tr}}M$ where

$$M = \begin{pmatrix} X_1 f_1 & 0 \\ X_1 f_2 & X_2 f_2 \end{pmatrix}.$$

The eigenvalues of $M^{\mathrm{tr}}M$ are

$$\frac{1}{2}\Big\{(X_1f_1)^2+(X_1f_2)^2+(X_2f_2)^2 \pm \sqrt{\left[(X_1f_1)^2+(X_1f_2)^2+(X_2f_2)^2\right]^2 - 4(X_1f_1)^2(X_2f_2)^2}\Big\},$$

the homogeneous dimension of G is

$$Q = 1 + n(n-1)/2,$$

and, by (2.4),

$$\det Jf = (X_1f_1)^{Q-(n-1)}(X_2f_2)^{n-1}.$$

If

$$\alpha(n) = 2\Big(\frac{2+(n-1)(n-2)}{2+n(n-1)}\Big), \quad \beta(n) = \frac{4(n-1)}{2+n(n-1)} \quad \text{and} \quad \gamma(n) = 2\frac{(n-2)(n-3)}{2+n(n-1)},$$

then, by the analytic definition, f is K-quasiconformal if

$$\Delta(f) = \Delta_1(f) + \sqrt{\left(\Delta_1(f)\right)^2 - 4\Delta_2(f)} \leqslant 2K^{2/Q}, \tag{6.12}$$

where

$$\Delta_1(f) = \frac{(X_1f_1)^2+(X_1f_2)^2+(X_2f_2)^2}{(X_1f_1)^{\alpha(n)}(X_2f_2)^{\beta(n)}} \quad \text{and} \quad \Delta_2(f) = \Big(\frac{X_2f_2}{X_1f_1}\Big)^{\gamma(n)}.$$

7. Quasiconformal flows

Conditions on the generating function $F(x_1, x_{n-1}, x_n)$ which imply quasiconformality of the flow, f_s, are obtained by estimating $\Delta(f_s)$ in terms of $\Delta_1(f_s)$. For if $\Delta_1(f_s) \leqslant K'$, then the flow is at most $(K')^{Q/2}$-quasiconformal. Following the methods of [4], we obtain estimates on $\Delta_1(f_s)$ involving the generating function by integrating estimates on $\frac{d}{ds}\Delta_1(f_s)$.

Theorem 2. *A contact vector field V defined on $\Omega \subseteq G$ with generator $F(x_1, x_{n-1}, x_n) \in C^\infty(\Omega)$ induces a flow of quasiconformal maps if*

$$(\beta(n)-1)\Big[(n-1)X_1X_{n-1}F - X_nF\Big] + \sqrt{\Big[(n-1)X_1X_{n-1}F - X_nF\Big]^2 + (X_1^{n-1}F)^2}$$

is bounded.

PROOF: The flow equations are $\frac{d}{ds}f(s,x) = A(f(s,x))\theta\Big(V\big(f(s,x)\big)\Big)^{\text{tr}}$, where

$$A(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ -x_2 & 0 & 1 & 0 & \cdots & 0 \\ -x_3 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{n-1} & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is the transition matrix satisfying $dx(v) = A(x)\theta(v)$ and

$$A\big(f(s,x)\big)Df(s,x) = Jf(s,x)A(x). \tag{7.1}$$

Let

$$E = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $Df(s,x)E = EDf(s,x)E$, which together with (7.1) and the *variational equation*

$$\frac{d}{ds}Jf(s,x) = J\theta(V)\big(f(s,x)\big)Jf(s,x),$$

gives

$$\frac{d}{ds}M = EJ\theta(V)\big(f(s,x)\big)A\big(f(s,x)\big)EM = Y\big(f(s,x)\big)M,$$

where

$$Y(x) = \begin{pmatrix} X_1v_1(x) & 0 \\ X_1v_2(x) & X_2v_2(x) \end{pmatrix}.$$

It follows that

$$\frac{d}{ds}\Delta_1\big(f(s,x)\big) = \frac{d}{ds}\frac{M^{\text{tr}}M}{(\det Jf)^{2/Q}} = (\det Jf)^{-2/Q}M^{\text{tr}}SM,$$

where

$$S = Y^{\text{tr}} + Y - \frac{2}{Q}\Big(\frac{d}{ds}(\det Jf)\Big)(\det Jf)^{-1}I.$$

Using $\|\ \ \|_T$ to denote the trace norm, we obtain

$$(\det Jf)^{2/Q}\frac{d}{ds}\frac{\|M\|_T^2}{(\det Jf)^{2/Q}} = \text{Trace}(M^{\text{tr}}SM) \leqslant \lambda(S)\|M\|_T^2,$$

where $\lambda(S)$ denotes the largest eigenvalue of S. It follows from the previous inequality that

$$\frac{d}{ds}\ln\Delta_1\big(f(s,x)\big) \leqslant \lambda(S),$$

hence if $0 \leqslant \lambda(S) \leqslant k$, then the flow is quasiconformal.

Since

$$S = \begin{pmatrix} (2-\alpha(n))X_1v_1 - \beta(n)X_2v_2 & X_1v_2 \\ X_1v_2 & -\alpha(n)X_1v_1 + (2-\beta(n))X_2v_2 \end{pmatrix},$$

we have

$$\lambda(S) = \big(1-\alpha(n)\big)X_1v_1 + \big(1-\beta(n)\big)X_2v_2 + \sqrt{(X_1v_1 - X_2v_2)^2 + (X_1v_2)^2}.$$

From (5.1), (5.5) and (5.6), we have

$$\begin{aligned} v_1 &= (-1)^{n-3}X_2X_1^{n-3}F = X_{n-1}F \\ v_2 &= (-1)^{n-2}X_1^{n-2}F \\ X_2v_2 &= (-1)^{n-2}X_2X_1^{n-2}F = X_nF - (n-2)X_1X_{n-1}F, \end{aligned}$$

which when replaced in (7) gives

$$\lambda(S) = \big(\beta(n)-1\big)\Big[(n-1)X_1X_{n-1}F - X_nF\Big] + \sqrt{\Big[(n-1)X_1X_{n-1}F - X_nF\Big]^2 + (X_1^{n-1}F)^2},$$

moreover $0 < \beta(n) \leqslant 6/7$, when $n \geqslant 4$. □

For example, if $F(x_1, x_{n-1}, x_n) = g(x_1)$, then $\lambda(S) = \big|g^{(n-1)}(x_1)\big|$, hence if $\big|g^{(n-1)}(x_1)\big|$ is bounded then the flow is quasiconformal. Furthermore

$$V = \sum_{k=2}^{n}(-1)^{n-k}g^{(n-k)}(x_1)X_k,$$

and the flow takes the form

(7.2) $\quad f_1(t,x) = x_1, \quad$ and $\quad f_k(t,x) = (-1)^{n-k}g^{(n-k)}(x_1)t + x_k, \quad k = 2, \dots, n.$

Corollary 2. *The smooth vector fields which induce quasiconformal flows form an infinite dimensional vector space, furthermore global contact and quasiconformal transformations exist in abundance.*

8. Comments and Further Results

The results of this paper are motivated by the problem of classifying the rigid and non-rigid Carnot groups, that is, rigidity should reflect the non-commutativity of the group. As is demonstrated by the examples in the introduction, the known results which apply to this question concern mainly 2-step groups. Using MAPLE to calculate the contact vector fields, it seems, at least from an experimental point of view, that semi-direct products of the established non-rigid groups are rigid. It should also be pointed out that the non-rigidity of the four dimensional real model filiform group was observed in [**2**], where rigidity phenomena of the nilpotent parts of Iwasawa decompositions are considered. In particular, the four dimensional real model filiform group is the nilpotent part of the Iwasawa decomposition of $Sp(2,\mathbb{R})$. Again using MAPLE, it appears that the analogue for $Sp(n,\mathbb{R})$, $n > 4$, is rigid.

We mention here some results which will appear in future papers. An expanded version of Lemma 2 brings the Schwarzian derivative to bear on questions on the structure of the globally defined contact mappings of G. Using (7.2) as a guide, the results on global contact mapping give rise to explicit groups of contact and quasiconformal automorphisms of G, manufactured from the absolutely continuous functions of $\mathbb{R}$. These groups contain the 1-quasiconformal maps. We can also prove the following Liouville type theorem. A 1-quasiconformal map of $\Omega \subseteq G$ is a composition of left translation, dilation, the reflecting dilation δ_{-1} and the switch map given by

$$\sigma(x) = \big(-x_1, x_2, -x_3, x_4, \dots, (-1)^n x_n\big).$$

In the complexified setting we have analogues of theorems 1 and 2, however, due to Liouville's theorem on bounded entire functions, the vector space of vector fields which generate global quasiconformal flows is finite dimensional.

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