ON THE RADICAL OF THE GROUP ALGEBRA OF A p-NILPOTENT GROUP

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1. Introduction

In this note we give a basis for the radical of the group algebra of a p-nilpotent group over a field of characteristic p in terms of the ordinary representation theory of the group. We use our result to calculate the exponent of the radical for such a group.

Notation. Let p be a fixed prime, k an algebraically closed field of characteristic p and G a finite group. Denote by kG the group algebra of G over k and by $N = N(G)$ the radical of kG. We denote the radical of a general finite dimensional k-algebra A by *rad* A. Let G have order $|G|$. We assume throughout that p divides $|G|$, in which case $N \neq 0$. By a kG module we mean a left kG module.

2. Lemmas

We begin with two results which are perhaps of independent interest.

LEMMA 1. *Let H be a normal p'-subgroup of G and L an irreducible kH module. Write $E = \mathrm{End}_{kG}(L^G)$, $F = \mathrm{rad}\, E$ and $N = N(G)$. Then, using the natural (right) action of F on L^G,*

$$N^i \cdot L^G = L^G \cdot F^i \text{ for all } i \geqq 1.$$

PROOF. We may take $L = kHe$ for some primitive kH idempotent e, and $L^G = kGe$. Write $1 = e_1 + e_2 + \cdots + e_n$, a sum of primitive kH idempotents, with $e = e_1$.

$$\begin{aligned} N \cdot L^G = Ne &= kGNe \\ &= kGeNe + kGe_2Ne + \cdots + kGe_nNe \end{aligned}$$

as left kG modules, where the sum is not necessarily direct.

Now $e_i kGe \cong \mathrm{Hom}_{kG}(kGe_i, kGe)$ as k-spaces under the map

$$a \to \varphi \in \mathrm{Hom}_{kG}(kGe_i, kGe)$$

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where $b\varphi = ba$ for all b in kGe_i. We use this fact to show that $Ne = kGeNe$.

Let f_i be the primitive central kH idempotent corresponding to e_i, $1 \leqq i \leqq n$. Denote by $N_G(f_i)$ the group of elements of G commuting with f_i and by T_i a left transversal for $N_G(f_i)$ in G. Then $F_i = \sum_{g \in T_i} f_i^g$ is a central kG idempotent. Now if f_1 and f_i are not conjugate in G, $F_iF_1 = 0$. Hence

$$e_i kGe = e_i f_i F_i kGF_1 f_1 e_1 = 0.$$

Suppose f_1 and f_i are conjugate in G, say $f_1 = f_i^g$. Now

$$e_i^g f_1 = (e_i f_i)^g = e_i^g.$$

Hence e_i^g and e are in the same kH block kHf_i. Since H is a p'-group we may use ordinary representation theory to deduce that $kHe \cong kHe_i^g$. Thus

$$kGe \cong kGe_i^g \cong kGe_i.$$

We claim that in this case $e_iNe = e_ikGeNe$. For there is an $a \in e_ikGe$ such that the map $\varphi : kGe_i \to kGe$ given by $x\varphi = xa$ is an isomorphism. Hence there is a $b \in ekGe_i$ such that $y\varphi^{-1} = yb$ for all $y \in kGe$. Hence $xab = x$ for all x in kGe_i. Thus

$$e_i = e_i ab = (e_i a)b = ab.$$

Let $c \in e_iNe$.

$$c = e_i c = a(bc) \in e_i kGeNe.$$

Thus $e_iNe \subset e_ikGeNe$. Since the reverse inclusion is obvious we have equality. Hence

$$kGe_iNe = kGe_ikGeNe \subset kGeNe,$$
$$Ne = kGeNe = (kGe)(eNe).$$

Now by [1] 54.6 we know that eNe and F are identical as rings. Hence $N \cdot L^G = L^G \cdot F$. Thus our result holds for $i = 1$.

Suppose $N^j \cdot L^G = L^G \cdot F^j$ for all $j \leqq i$, i.e.

$$N^j e = (kGe)(eNe)^j. \tag{1}$$

Multiplying (1) on the left by N gives

$$N^{j+1} e = (Ne)^{j+1}, \tag{2}$$

whereas multiplying (1) on the right by Ne gives

$$(N^j e)(Ne) = (kGe)(eNe)^{j+1}. \tag{3}$$

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Thus $N^{i+1}e = (Ne)^{i+1}$, using (2) with $j = i$

$$= (Ne)^i(Ne)$$
$$= (N^i e)(Ne), \text{ using (2) with } j = i-1$$
$$= (kGe)(eNe)^{i+1}, \text{ using (3) with } j = i.$$

Therefore $N^{i+1} \cdot L^G = L^G \cdot F^{i+1}$. The result follows by induction.

DEFINITION. If H is normal in G and L is a kH module, the *stabilizer* $S = S(L)$ of L in G is defined by

$$S = \{g \in G; L^g \cong L\}.$$

LEMMA 2. *In the situation of Lemma* 1, *if* S *is the stabilizer of* L *in* G, $N^i \cdot L^G = kG \cdot N(S)^i \cdot L^S$ *for all* $i \geqq 1$.

PROOF. Let $g_1, \cdots, g_s$ be a left transversal for H in S and $g_1, \cdots, g_n$ a left transversal for H in G.

$$L^S = \bigoplus_{i=1}^{s} g_i \otimes L \text{ is a } kS \text{ submodule of}$$

$$L^G = \bigoplus_{i=1}^{n} g_i \otimes L \text{ and } L \text{ is a } kH \text{ submodule of } L^S.$$

Let $\theta \in \mathrm{End}_{kS}(L^S)$ and define $\varphi : \mathrm{End}_{kS}(L^S) \to \mathrm{End}_{kG}(L^G)$ by putting $\varphi(\theta) = \theta'$, where

$$(g_i \otimes l)\theta' = g_i(l\theta), \; l \in L, \; i = 1, \cdots, n,$$

and extending θ' linearly to L^G. It is well known and easy to prove that φ is an isomorphism of rings such that θ and $\varphi(\theta)$ have the same action on L.

Thus

$$N^i \cdot L^G = \Big(\sum_{j=1}^{n} g_j \otimes L\Big)(\text{rad End}_{kG}(L^G))^i, \text{ by Lemma 1,}$$
$$= \sum_{1}^{n} g_j(L \cdot (\text{rad End}_{kG}(L^G))^i)$$
$$= \sum_{1}^{n} g_j(L \cdot (\text{rad End}_{kS}(L^S))^i)$$
$$\subset kG \cdot N(S)^i L^S, \text{ by Lemma 1.}$$

The reverse inclusion is proven similarly. Hence the result follows.

3. *p*-nilpotent groups

Let G be a p-nilpotent group with Sylow p-subgroup P and normal p-complement H. Let e be a primitive idempotent of kH and put $L = kHe$. Suppose L has stabilizer $S = HQ$ in G, where Q is a Sylow p-subgroup of S. Now

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$$E = \mathrm{End}_{kS}(L^S) \cong \bigoplus_{q \in Q} \mathrm{Hom}_{kH}(L, q \otimes L). \tag{4}$$

We know from [3] that there is a unique kS module X such that $X|_H = L$. Let X afford the representation ρ of S with respect to the k-basis W.

For each $q \in Q$ the map $T_q : L \to q \otimes L$ given by

$$lT_q = q \otimes \rho(q^{-1})l, \; l \in L,$$

is a kH-isomorphism.

Therefore $\{T_q; q \in Q\}$ is a k-basis for the right hand side of (4). E therefore has k-basis $\{\eta_q; q \in Q\}$, where η_q is defined by

$$\begin{aligned}(q' \otimes l)\eta_q &= q'(lT_q) \\ &= q'q \otimes \rho(q^{-1})l, \; q' \in Q, \; l \in L.\end{aligned}$$

Now $\eta_{q'}\eta_q = \eta_{q'q}$. Hence $E \cong kQ$. Thus *rad* E has basis $\{\eta_1 - \eta_q; q \in Q - \{1\}\}$
Define $\eta(q, l) = 1 \otimes l - q \otimes \rho(q^{-1})l$, $q \in Q$, $l \in L$.

THEOREM 1. *The set* $\{\eta(q, l); q \in Q - \{1\}, l \in W\}$ *is a k-basis for* $N(S)L^S$.

PROOF. $N(S)L^S = L^S \cdot \mathrm{rad}\, E$. Now

$$\begin{aligned}(q' \otimes l)(\eta_1 - \eta_q) &= q' \otimes l - q'q \otimes \rho(q^{-1})l \\ &= -\eta(q', \rho(q')l) + \eta(q'q, \rho(q')l), \quad \text{and} \\ (1 \otimes l)(\eta_1 - \eta_q) &= \eta(q, l).\end{aligned}$$

Hence the result follows.

We can now give an explicit expression for $N(G)$. For let $1 = e_1 + \cdots + e_n$ be a decomposition of $1 \in kH$ into primitive orthogonal idempotents. Write $L_i = kHe_i$ and let L_i have stabilizer S_i in G. Let S_i have Sylow p-subgroup Q_i. Then

$$\begin{aligned}&kG = \oplus kGe_i = \oplus L_i^G \text{ as left } kG \text{ modules and} \\ &N = \sum N \cdot L_i^G \\ &\quad = \sum kG \cdot N(S) \cdot L_i^{S_i}, \text{ which can be calculated.}\end{aligned}$$

DEFINITION. The exponent of $N(G)$ is the least integer n such that $N(G)^n = 0$.

THEOREM 2. *If G is p-nilpotent and P is a Sylow p-subgroup of G then* $N(G)$ *and* $N(P)$ *have the same exponent.*

PROOF. We use the previous notation.

Consider the idempotent $f = \sum_{h \in H} h/|H|$ of kG. It is easy to show that $kGf \cong kP$ as algebras. Hence

$$\begin{aligned}N(G)^n = 0 &\Rightarrow (\mathrm{rad}\,(kGf))^n = 0 \\ &\Rightarrow (\mathrm{rad}\, kP)^n = 0 \\ &\Rightarrow N(P)^n = 0.\end{aligned}$$

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Conversely, let $N(P)^n = 0$. We have that

$$\begin{aligned} N(G)^n &= \sum_i N(G)^n L_i^G \\ &= \sum_i kG \cdot N(S_i)^n L_i^{S_i} \text{ by Lemma 2,} \\ &= \sum_i kG \cdot L_i^{S_i}\{\text{rad End}_{kS_i}(L_i^{S_i})\}^n \text{ by Lemma 1.} \end{aligned}$$

Now Q_i is contained in some Sylow p-subgroup P_1 of G, so

$$N(Q_i)^n \subset N(P_1)^n = 0.$$

Since

$$\text{End}_{kS_i}(L_i^{S_i}) \cong kQ_i$$

we have

$$\{\text{rad End}_{kS_i}(L_i^{S_i})\}^n = 0.$$

Therefore $N(G)^n = 0$.

REMARKS. If G is a group of p-length one then G contains a normal p-nilpotent subgroup K such that G/K is a p'-group. By results of Highman [2] and Villamayor [4] we have that $N(G) = kG \cdot N(K)$. Theorem 2 therefore holds for groups of p-length one. Similar calculations can be carried out in the case of a general p-soluble group. However Theorem 2 does not hold in such a case. The exponent of $N(G)$ may be greater than or less than that of $N(P)$.

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