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RESEARCH ARTICLE

Topological Noetherianity of the infinite half-spin representations

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Abstract
We prove that the infinite half-spin representations are topologically Noetherian with respect to the infinite spin group. As a consequence, we obtain that half-spin varieties, which we introduce, are defined by the pullback of equations at a finite level. The main example for such varieties is the infinite isotropic Grassmannian in its spinor embedding, for which we explicitly determine its defining equations.

Contents

1. Introduction

1.1. Purpose of this paper and main theorem

The purpose of this paper is to study certain varieties X_n that live in the half-spin representations of the even spin groups $\mathrm{Spin}(2n)$ with n varying. In particular, we will show that these varieties are defined, for all n, by pulling back the equations for a single X_{n_0} along suitable contraction maps. The simplest instance of such a variety is the Grassmannian of n-dimensional isotropic spaces in a $2n$-dimensional orthogonal space. In this case, we use earlier work [16] by the last two authors to show that n_0 can be taken equal to 4; see Theorem 6.1.

But the *half-spin varieties* that we introduce go far beyond the maximal isotropic Grassmannian. Indeed, this class of varieties is preserved under linear operations such as joins and tangential varieties, and under finite unions and arbitrary intersections. Consequently, any variety obtained from several copies of the maximal isotropic Grassmannian by such operations is defined by equations of some degree bounded independently of n. We stress, though, that these results are of a purely topological/set-theoretic nature. It is not true, for instance, that one gets the entire ideal of the maximal isotropic Grassmannian of n-spaces in a $2n$-space by pulling back equations for X_4 along the maps that we define.

Our main results about half-spin varieties are Theorem 5.6, which establishes a descending chain condition for these, and Corollary 5.8, which implies the results mentioned above. These results follow from a companion result in infinite dimensions, which is a little easier to state here. We will construct a direct limit $\mathrm{Spin}(V_\infty)$ of all spin groups; here, $V_\infty = \bigcup_n V_n$ is a countable-dimensional vector space with basis $e_1, f_1, e_2, f_2, e_3, f_3, \ldots$ and a bilinear form determined by $(e_i|e_j) = (f_i|f_j) = 0$ and $(e_i|f_j) = \delta_{ij}$. Furthermore, we will construct a direct limit $\bigwedge^+_\infty E_\infty$ of all even half-spin representations. This space has as basis all formal infinite products

$$e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots ,$$

where $\{i_1 < i_2 < \ldots\}$ is a cofinite subset of the positive integers. The group $\mathrm{Spin}(V_\infty)$ acts naturally on this space, and hence on its dual $(\bigwedge^+_\infty E_\infty)^*$, which we regard as the spectrum of the symmetric algebra on $\bigwedge^+_\infty E_\infty$. Our main theorem is as follows.

Theorem 1.1. *The scheme* $(\bigwedge^+_\infty E_\infty)^*$ *is topologically* $\mathrm{Spin}(V_\infty)$*-Noetherian. That is, every chain*

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$$

of $\mathrm{Spin}(V_\infty)$*-stable reduced closed subschemes stabilises.*

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1.2. Relations to the literature

Our work is primarily motivated by earlier work by the second and third author on *Plücker varieties*, which live in exterior powers $\bigwedge^n K^{p+n}$ with both p and n varying. The results in [6] on Plücker varieties are analoguous to the results we establish here for half-spin varieties, and the main result in [12] is an exact analogue of Theorem 1.1 for the dual infinite wedge, acted upon by the infinite general linear group.

On the one hand, we now have much better tools available to study these kind of questions than we had at the time of [6] – notably the topological Noetherianity of polynomial functors [5] and their generalisation to algebraic representations [7]. But on the other hand, spin representations are much more intricate than polynomial functors, and a part of the current paper will be devoted to establishing the precise relationship between the infinite half-spin representation and algebraic representations of the infinite general linear group, so as to use those tools.

This paper fits in a general programme that asks for which sequences of representations of increasing groups one can expect Noetherianity results. This seems to be an extremely delicate question. Indeed, while Theorem 1.1 establishes Noetherianity of the dual infinite *half*-spin representation, we do not know whether the dual infinite spin representation is $\mathrm{Spin}(V_\infty)$-Noetherian; see Remark 4.9. Similarly, we do not know whether a suitable inverse limit of exterior powers $\bigwedge^n V_n$ is $\mathrm{SO}(V_\infty)$-Noetherian – and there are many more natural sequences of representations for which we do not yet have satisfactory results.

In the context of secant varieties, we point out the work by Sam on Veronese varieties: the k-th secant variety of the d-th Veronese embedding of $\mathbb{P}(K^n)$ is defined *ideal-theoretically* by finitely many types of equations, independently of n – and in particular in bounded degree [14]. Furthermore, a similar statement holds for the p-th syzygies for any fixed p [15]. Similar results for ordinary Grassmannians were established by Laudone in [9]. It would be very interesting to know whether their techniques apply to secant varieties of the maximal isotropic Grassmannian in its spinor embedding. Our results here give a weaker set-theoretic statement, but for a more general class of varieties.

After establishing Noetherianity, it would be natural to try and study additional geometric properties of $\mathrm{Spin}(V_\infty)$-stable subvarieties of the dual infinite half-spin representation. Perhaps there is a theory there analogous to the theory of GL-varietes [1, 2]. However, we are currently quite far from any such deeper understanding!

1.3. Organisation of this paper

In §2, we recall the construction of the (finite-dimensional) half-spin representations. We mostly do this in a coordinate-free manner, only choosing – as one must – a maximal isotropic subspace of an orthogonal space for the construction. But for the construction of the infinite half-spin representation, we will need explicit formulas, and these are derived in §2, as well.

In §3, we first describe the embedding of the maximal isotropic Grassmannian in the projectivised half-spin representation. Then, we define suitable contraction and multiplication maps, which we show preserve the cones over these isotropic Grassmannians. Finally, we use these maps to construct the infinite-dimensional half-spin representations.

In §4, we prove Theorem 1.1 (see Theorem 4.1); and in §5, we state and prove the main results about half-spin varieties discussed above. Finally, in §6, we prove the universality of the isotropic Grassmannian of 4-spaces in an 8-dimensional space. We do so by relating the half-spin representations via the Cartan map to the exterior power representations and using results from [16].

2. Finite spin representations and the spin group

In this section, we collect some preliminaries on spin groups and their defining representations. Throughout, we will assume that K is an algebraically closed field of characteristic 0. We follow [11] in our set-up; for more general references on spin groups and their representations, see [10, 13].

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2.1. The Clifford algebra

Let V be a finite-dimensional vector space over K endowed with a quadratic form q. The *Clifford algebra* $\mathrm{Cl}(V,q)$ of V is the quotient of the tensor algebra $T(V) = \bigoplus_{d\geq 0} V^{\otimes d}$ by the two-sided ideal generated by all elements

$$v \otimes v - q(v) \cdot 1, \ v \in V. \tag{2.1}$$

This is also the two-sided ideal generated by

$$v \otimes w + w \otimes v - 2(v|w) \cdot 1, \ v, w \in V, \tag{2.2}$$

where $(\cdot|\cdot)$ denotes the bilinear form associated to q defined by $(v|w) := \frac{1}{2}(q(v+w) - q(v) - q(w))$.

The Clifford algebra is a functor from the category of vector spaces equipped with a quadratic form to the category of (unital) associative algebras. That is, any linear map $\varphi\colon (V,q) \to (V',q')$ with $q'(\varphi(v)) = q(v)$ for all $v \in V$ induces a homomorphism of associative algebras $\mathrm{Cl}(\varphi)\colon \mathrm{Cl}(V,q) \to \mathrm{Cl}(V',q')$. If φ is an inclusion $V \subseteq V'$, then $\mathrm{Cl}(\varphi)$ is injective, and hence, $\mathrm{Cl}(V,q)$ is a subalgebra of $\mathrm{Cl}(V',q')$.

The decomposition of $T(V)$ into the even part $T^+(V) := \bigoplus_{d \text{ even}} V^{\otimes d}$ and the odd part $T^-(V) := \bigoplus_{d \text{ odd}} V^{\otimes d}$ induces a decomposition $\mathrm{Cl}(V,q) = \mathrm{Cl}^+(V,q) \oplus \mathrm{Cl}^-(V,q)$, turning $\mathrm{Cl}(V,q)$ into a $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra. Note that, via the commutator on $\mathrm{Cl}(V,q)$, the even Clifford algebra $\mathrm{Cl}^+(V,q)$ is a Lie subalgebra of $\mathrm{Cl}(V,q)$.

The anti-automorphism of $T(V)$ determined by $v_1 \otimes \cdots \otimes v_d \mapsto v_d \otimes \cdots \otimes v_1$ preserves the ideal in the definition of $\mathrm{Cl}(V,q)$ and therefore induces an anti-automorphism $x \mapsto x^*$ of $\mathrm{Cl}(V,q)$.

2.2. The Grassmann algebra as a $\mathrm{Cl}(V)$-module

From now on, we will write $\mathrm{Cl}(V)$ for $\mathrm{Cl}(V,q)$ when q is clear from the context. If $q = 0$, then $\mathrm{Cl}(V) = \bigwedge V$, the Grassmann algebra of V. If $E \subseteq V$ is an isotropic subspace – that is, a subspace for which $q|_E = 0$ – then this fact allows us to identify $\bigwedge E$ with the subalgebra $\mathrm{Cl}(E)$ of $\mathrm{Cl}(V)$.

For general q, $\mathrm{Cl}(V)$ is not isomorphic as an algebra to $\bigwedge V$, but $\bigwedge V$ is naturally a $\mathrm{Cl}(V)$-module as follows. For $v \in V$, define $o(v) : \bigwedge V \to \bigwedge V$ (the 'outer product') as the linear map

$$o(v)\omega := v \wedge \omega$$

and $\iota(v) : \bigwedge V \to \bigwedge V$ (the 'inner product') as the linear map determined by

$$\iota(v) w_1 \wedge \cdots \wedge w_k := \sum_{i=1}^{k} (-1)^{i-1} (w_i \mid v) w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_k.$$

Here, and elsewhere in the paper, $\widehat{\cdot}$ indicates a factor that is left out. Now $v \mapsto \iota(v) + o(v)$ extends to an algebra homomorphism $\mathrm{Cl}(V) \to \mathrm{End}(\bigwedge V)$. To see this, it suffices to consider $v, w_1, \ldots, w_k \in V$ and verify

$$(\iota(v) + o(v))^2 w_1 \wedge \cdots \wedge w_k = (v|v) w_1 \wedge \cdots \wedge w_k.$$

We write $a \bullet \omega$ for the outcome of $a \in \mathrm{Cl}(V)$ acting on $\omega \in \bigwedge V$. Using induction on the degree of a product, the linear map $\mathrm{Cl}(V) \to \bigwedge V, a \mapsto a \bullet 1$ is easily seen to be an isomorphism of vector spaces. In particular, $\mathrm{Cl}(V)$ has dimension $2^{\dim V}$.

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2.3. *Embedding* $\mathfrak{so}(V)$ *into the Clifford algebra*

From now on, we assume that q is nondegenerate and write $\mathrm{SO}(V) = \mathrm{SO}(V, q)$ for the special orthogonal group of q. Its Lie algebra $\mathfrak{so}(V)$ consists of linear maps $V \to V$ that are skew-symmetric with respect to $(\cdot|\cdot)$ – that is, those $A \in \mathrm{End}(V)$ such that $(Av|w) = -(v|Aw)$ for all $v, w \in V$. We have a unique linear map $\psi : \bigwedge^2 V \to \mathrm{Cl}^+(V)$ with $\psi(u \wedge v) = uv - vu$, and ψ is injective. A straightforward computation shows that the image L of ψ is closed under the commutator in $\mathrm{Cl}(V)$, and hence a Lie subalgebra. We claim that L is isomorphic to $\mathfrak{so}(V)$. Indeed, for $u, v, w \in V$, we have

$$[\psi(u \wedge v), w] = [[u, v], w] = 4(v|w)u - 4(u|w)v.$$

We see, first, that $V \subseteq \mathrm{Cl}(V)$ is preserved under the adjoint action of L; and second, that L acts on V via skew-symmetric linear maps, so that L maps into $\mathfrak{so}(V)$. Since every map in $\mathfrak{so}(V)$ is a linear combination of the linear maps above, and since $\dim(L) = \dim(\mathfrak{so}(V))$, the map $L \to \mathfrak{so}(V)$ is an isomorphism. We will identify $\mathfrak{so}(V)$ with the Lie subalgebra $L \subseteq \mathrm{Cl}(V)$ via the inverse of this isomorphism, and we will identify $\bigwedge^2 V$ with $\mathfrak{so}(V)$ via the map $u \wedge v \mapsto (w \mapsto (v|w)u - (u|w)v)$. The concatenation of these identifications is the linear map $\frac{1}{4}\psi$.

2.4. *The half-spin representations*

From now on, we assume that $\dim(V) = 2n$. We believe that all our results hold *mutatis mutandis* also in the odd-dimensional case, but we have not checked the details. A *maximal* isotropic subspace U of V is an isotropic subspace which is maximal with respect to inclusion. Since K is algebraically closed, q has maximal Witt index, so that every maximal isotropic subspace of V has dimension n.

The spin representation of $\mathfrak{so}(V)$ is constructed as follows. Let F be a maximal isotropic subspace of V and let $f_1, \dots, f_n$ be a basis of F. Define $f := f_1 \cdots f_n \in \mathrm{Cl}(F)$; this element in $\mathrm{Cl}(F) = \bigwedge F$ is well defined up to a scalar. Then the left ideal $\mathrm{Cl}(V) \cdot f$ is a left module for the associative algebra $\mathrm{Cl}(V)$, and hence for its Lie subalgebra $\mathfrak{so}(V)$. This ideal is called the *spin representation* of $\mathfrak{so}(V)$. As $\mathrm{Cl}(V)$ is $\mathbb{Z}/2\mathbb{Z}$-graded, the spin representation splits into a direct sum of two subrepresentations for $\mathrm{Cl}^+(V)$, and hence for $\mathfrak{so}(V) \subseteq \mathrm{Cl}^+(V)$ – namely, $\mathrm{Cl}^+(V) \cdot f$ and $\mathrm{Cl}^-(V) \cdot f$. These representations are called the *half-spin representations* of $\mathfrak{so}(V)$.

2.5. *Explicit formulas*

We will need more explicit formulas for the action of $\mathfrak{so}(V)$ on the half-spin representations. To this end, let E be another isotropic n-dimensional subspace of V such that $V = E \oplus F$. Then the map

$$\bigwedge E = \mathrm{Cl}(E) \to \mathrm{Cl}(V)f, \quad \omega \mapsto \omega f$$

is a linear isomorphism, and we use it to identify $\bigwedge E$ with the spin representation. We write $\rho : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge E)$ for the corresponding representation. It splits as a direct sum of the half-spin representations $\rho_+ : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge^+ E)$ and $\rho_- : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge^- E)$, where $\bigwedge^+ E = \bigoplus_{d \text{ even}} \bigwedge^d E$ and $\bigwedge^- E = \bigoplus_{d \text{ odd}} \bigwedge^d E$.

In this model of the spin representation, the action of $v \in E \subseteq \mathrm{Cl}(V)$ on the spin representation $\bigwedge E$ is just the outer product on $\bigwedge E : o(v) : \bigwedge E \to \bigwedge E,\ \omega \mapsto v \wedge \omega$, while the action of $v \in F \subseteq \mathrm{Cl}(V)$ is twice the inner product on $\bigwedge E$:

$$2\iota(v) w_1 \wedge \cdots \wedge w_k = 2 \sum_{i=1}^{k} (-1)^{i-1} (v|w_i) w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_k.$$

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The factor 2 and the alternating signs come from the following identity in $\mathrm{Cl}(V)$:

$$vv_i = 2(v|v_i) - v_i v \text{ for } v \in F \text{ and } v_i \in E.$$

For a general $v \in V$, we write $v = v' + v''$ with $v' \in E$, $v'' \in F$. Then the action of V on $\bigwedge E$ is given by

$$v \mapsto o(v') + 2\iota(v'').$$

We now compute the linear maps by means of which $\mathfrak{so}(V)$ acts on $\bigwedge E$. To this end, recall that a pair $e, f \in V$ is called *hyperbolic* if e, f are isotropic and $(e|f) = 1$. Given the basis $f_1, \ldots, f_n$ of F, there is a unique basis $e_1, \ldots, e_n$ of E so that $(e_i|f_j) = \delta_{ij}$; then $e_1, \ldots, e_n, f_1, \ldots, f_n$ is called a *hyperbolic basis* of V. Now the element $e_i \wedge e_j \in \mathfrak{so}(V)$ acts on $\bigwedge E \simeq \mathrm{Cl}(V)f$ via the linear map

$$\frac{1}{4}(o(e_i)o(e_j) - o(e_j)o(e_i)) = \frac{1}{2}o(e_i)o(e_j);$$

the element $f_i \wedge f_j$ acts via the linear map

$$\frac{1}{4}(4\iota(f_i)\iota(f_j) - 4\iota(f_j)\iota(f_i)) = 2\iota(f_i)\iota(f_j);$$

and the element $e_i \wedge f_j$ acts via the linear map

$$\frac{1}{4}(o(e_i)2\iota(f_j) - 2\iota(f_j)o(e_i)) = \frac{1}{2}(o(e_i)\iota(f_j) - \iota(f_j)o(e_i)).$$

In particular, $\omega_0 := e_1 \wedge \cdots \wedge e_n \in \bigwedge E$ is mapped to 0 by all elements $e_i \wedge e_j$ and all elements $e_i \wedge f_j$ with $i \neq j$, and it is mapped to $\frac{1}{2}\omega_0$ by all $e_i \wedge f_i$.

2.6. Highest weights of the half-spin representations

Recall, for example, from [8, Chapter IV, pages 140–141], that in the basis $e_1, \ldots, e_n, f_1, \ldots, f_n$, matrices in $\mathfrak{so}(V)$ have the form

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \text{ with } B^T = -B, \text{ and } C^T = -C.$$

Here, the (e_i, e_j)-entry of A is the coefficient of $e_i \wedge f_j$, the (e_i, f_j)-entry of B is the coefficient of $e_i \wedge e_j$, and the (f_i, e_j)-entry of C is the coefficient of $f_i \wedge f_j$.

The diagonal matrices $e_i \wedge f_i$ span a Cartan subalgebra of $\mathfrak{so}(V)$ with standard basis consisting of $h_i := e_i \wedge f_i - e_{i+1} \wedge f_{i+1}$ for $i = 1, \ldots, n-1$ and $h_n := e_{n-1} \wedge f_{n-1} + e_n \wedge f_n$ (this last element is forgotten in the basis of the Cartan algebra on [8, page 141]).

Now $(e_i \wedge e_j)\omega_0 = (e_i \wedge f_j)\omega_0 = 0$ for all $i \neq j$. Furthermore, the elements $h_1, \ldots, h_{n-1}$ map ω_0 to 0, while h_n maps ω_0 to ω_0. Thus, the Borel subalgebra maps the line $K\omega_0$ into itself, and ω_0 is a highest weight vector of the *fundamental weight* $\lambda_0 := (0, \ldots, 0, 1)$ relative to the standard basis. Summarising, $\omega_0 \in \bigwedge E$ generates a copy of the irreducible $\mathfrak{so}(V)$-module V_{λ_0} with highest weight λ_0. Clearly, the $\mathfrak{so}(V)$-module generated by ω_0 is contained in $\bigwedge^+ E$ if n is even, and contained in $\bigwedge^- E$ when n is odd. One can also show that both half-spin representations are irreducible; hence, one of them is a copy of V_{λ_0}. For the other half-spin representation, consider the element

$$\omega_1 := e_1 \wedge \cdots \wedge e_{n-1} \in \bigwedge E.$$

This element is mapped to zero by $e_i \wedge e_j$ for all $i \neq j$ and by $e_i \wedge f_j$ for all $i < j$. It is further mapped to 0 by $h_1, \ldots, h_{n-2}, h_n$, and to ω_1 by h_{n-1}. For example, we have

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$$
\begin{aligned}
h_n\omega_1 &= \frac{1}{2}(o(e_{n-1})\iota(f_{n-1}) - \iota(f_{n-1})o(e_{n-1}) + o(e_n)\iota(f_n) - \iota(f_n)o(e_n))e_1 \wedge \cdots \wedge e_{n-1} \\
&= \frac{1}{2}(1-0+0-1)\omega_1 = 0, \text{ and similarly} \\
h_{n-1}\omega_1 &= \frac{1}{2}(1-0-0+1)\omega_1 = \omega_1.
\end{aligned}
$$

Hence, ω_1 generates a copy of V_{λ_1}, the irreducible $\mathfrak{so}(V)$-module of highest weight $\lambda_1 := (0,\dots,0,1,0)$; this is the other half-spin representation.

2.7. *The spin group*

Let $\rho : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge E)$ be the spin representation. The spin group $\mathrm{Spin}(V)$ can be defined as the subgroup of $\mathrm{GL}(\bigwedge E)$ generated by the one-parameter subgroups $t \mapsto \exp(t\rho(X))$, where X runs over the root vectors $e_i \wedge e_j, f_i \wedge f_j$ and $e_i \wedge f_j$ with $i \neq j$. Note that $\rho(X)$ is nilpotent for each of these root vectors, so that $t \mapsto \exp(t\rho(X))$ is an *algebraic* group homomorphism $K \to \mathrm{GL}(\bigwedge E)$. It is a standard fact that the subgroup generated by irreducible curves through the identity in an algebraic group is itself a connected algebraic group; see [3, Proposition 2.2]. So $\mathrm{Spin}(V)$ is a connected algebraic group, and one verifies that its Lie algebra is isomorphic to the Lie algebra generated by the root vectors X (i.e., to $\mathfrak{so}(V)$).

By construction, the (half-)spin representations $\bigwedge E$, $\bigwedge^+ E$ and $\bigwedge^- E$ are representations of $\mathrm{Spin}(V)$. We use the same notation $\rho : \mathrm{Spin}(V) \to \mathrm{GL}(\bigwedge E)$, $\rho_+ \colon \mathrm{Spin}(V) \to \mathrm{GL}(\bigwedge^+ E)$, and $\rho_- \colon \mathrm{Spin}(V) \to \mathrm{GL}(\bigwedge^- E)$ for these as for the corresponding Lie algebra representations.

Remark 2.1. The algebraic group $\mathrm{Spin}(V)$ is usually constructed as a subgroup of the unit group $\mathrm{Cl}^\times(V)$ as follows: consider first

$$\Gamma(V) = \{x \in \mathrm{Cl}^\times(V) \mid xVx^{-1} = V\},$$

sometimes called the Clifford group. Then $\mathrm{Spin}(V)$ is the subgroup of $\Gamma(V)$ of elements of *spinor norm* 1; that is, $xx^* = 1$, where x^* denotes the involution defined in Section 2.1. In this model of the spin group, it is easy to see that it admits a $2:1$ covering $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ – namely, the restriction of the homomorphism $\Gamma(V) \to \mathrm{O}(V)$ given by associating to $x \in \Gamma(V)$ the orthogonal transformation $w \mapsto xwx^{-1}$. For more details, see [13]. Since our later computations involve the Lie algebra $\mathfrak{so}(V)$ only, the definition of $\mathrm{Spin}(V)$ above suffices for our purposes.

The half-spin representations are *not* representations of the group $\mathrm{SO}(V)$; this can be checked, for example, by showing that the highest weights λ_0 and λ_1 are not in the weight lattice of $\mathrm{SO}(V)$.

2.8. *Two actions of* $\mathfrak{gl}(E)$ *on* $\bigwedge E$

The definition of the (half-)spin representation(s) of $\mathfrak{so}(V)$ and $\mathrm{Spin}(V)$ as $\mathrm{Cl}^{(\pm)}(V)f$ involves only the quadratic form q and the choice of a maximal isotropic space $F \subseteq V$. Consequently, any linear automorphism of V that preserves q and maps F into itself also acts on $\mathrm{Cl}^{(\pm)}(V)f$. These linear automorphisms form the stabiliser of F in $\mathrm{SO}(V)$, which is the parabolic subgroup whose Lie algebra consists of the matrices in $\mathrm{SO}(V)$ that are block lower triangular in the basis $e_1,\dots,e_n,f_1,\dots,f_n$. So, while $\mathrm{SO}(V)$ does not act naturally on the (half-)spin representation(s), this stabiliser does.

In particular, in our model $\bigwedge^{(\pm)} E$ of the (half-)spin representation(s), the group $\mathrm{GL}(E)$, embedded into $\mathrm{SO}(V)$ as the subgroup of block diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & -a^T \end{bmatrix}$$

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acts on $\bigwedge E$ in the natural manner. We stress that this is *not* the action obtained by integrating the action of $\mathfrak{gl}(E) \subseteq \mathfrak{so}(V)$ on $\bigwedge E$ *regarded as the spin representation*. Indeed, the standard action of $e_i \wedge f_j \in \mathfrak{gl}(E)$ on $\omega := e_{i_1} \wedge \cdots \wedge e_{i_k} \in \bigwedge^k E$ yields

$$\sum_{l=1}^{k} e_{i_1} \wedge \cdots \wedge e_i(f_j|e_{i_l}) \wedge \cdots \wedge e_{i_k} = \begin{cases} 0 \text{ if } j \notin \{i_1, \ldots, i_k\} \\ (-1)^{l-1} e_i \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_l}} \wedge \cdots \wedge e_{i_k} \text{ if } j = i_l. \end{cases}$$

However, in the spin representation, the action is given by the linear map $\frac{1}{2}(o(e_i)\iota(f_j) - \iota(f_j)o(e_i))$. If $j \neq i$ and $j \notin \{i_1, \ldots, i_k\}$, then

$$o(e_i)\iota(f_j)\omega = \iota(f_j)o(e_i)\omega = 0.$$

If $j \neq i$ and $j = i_l$, then

$$o(e_i)\iota(f_j)\omega = (-1)^{l-1} e_i \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_l}} \wedge \cdots \wedge e_{i_k} = -\iota(f_j)o(e_i)\omega.$$

We conclude that for $i \neq j$, the action of $e_i \wedge f_j$ is the same in both representations. However, if $i = j$, then

$$\frac{1}{2}(o(e_i)\iota(f_i) - \iota(f_i)o(e_i))\omega = \begin{cases} -\frac{1}{2}\omega \text{ if } i \notin \{i_1, \ldots, i_k\}, \text{ and} \\ \frac{1}{2}(-1)^{l-1} e_i \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_l}} \wedge \cdots \wedge e_{i_k} = \frac{1}{2}\omega \text{ if } i = i_l. \end{cases}$$

We conclude that if $\tilde{\rho} : \mathfrak{gl}(E) \to \mathrm{End}(\bigwedge E)$ is the standard representation of $\mathfrak{gl}(E)$, then the restriction of the spin representation $\rho : \mathfrak{so}(V) \to \mathrm{End}(\bigwedge E)$ to $\mathfrak{gl}(E)$ *as a subalgebra of* $\mathfrak{so}(V)$ satisfies

$$\rho(A) = \tilde{\rho}(A) - \frac{1}{2}\operatorname{tr}(A)\,\mathrm{Id}_{\bigwedge E}. \tag{2.3}$$

At the group level, this is to be understood as follows. The pre-image of $\mathrm{GL}(E) \subseteq \mathrm{SO}(V)$ in $\mathrm{Spin}(V)$ is isomorphic to the connected algebraic group

$$H := \left\{(g, t) \in \mathrm{GL}(E) \times K^* \mid \det(g) = t^2\right\}$$

for which $(g, t) \mapsto g$ is a $2 : 1$ cover of $\mathrm{GL}(E)$, and the restriction of ρ to H satisfies $\rho(g, t) = \tilde{\rho}(g) \cdot t^{-1}$ – a 'twist of the standard representation by the inverse square root of the determinant'.

3. The isotropic Grassmannian and infinite spin representations

3.1. The isotropic Grassmannian in its spinor embedding

As before, let V be a $2n$-dimensional vector space over K endowed with a nondegenerate quadratic form. The (maximal) *isotropic Grassmannian* $\mathrm{Gr}_{\mathrm{iso}}(V, q)$ parametrizes all maximal isotropic subspaces of V. It has two connected components, denoted $\mathrm{Gr}^+_{\mathrm{iso}}(V)$ and $\mathrm{Gr}^-_{\mathrm{iso}}(V)$. The goal of this subsection is to introduce the isotropic Grassmann cone, which is an affine cone over $\mathrm{Gr}_{\mathrm{iso}}(V, q)$ in the spin representation.

Fix a maximal isotropic subspace $F \subseteq V$ and as before, set $f := f_1 \cdots f_n \in \mathrm{Cl}(V)$, where $f_1, \ldots, f_n$ is any basis of F. Now let $H \subseteq V$ be another maximal isotropic space. Then we claim that the space

$$S_H := \{\omega \in \mathrm{Cl}(V)f \mid v \cdot \omega = 0 \text{ for all } v \in H\} \subseteq \mathrm{Cl}(V)f \tag{3.1}$$

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is 1-dimensional. Indeed, we may find a hyperbolic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that $f_1, \ldots, f_k$ span $H \cap F$, $f_1, \ldots, f_n$ span F, and $e_{k+1}, \ldots, e_n, f_1, \ldots, f_k$ span H. We call this hyperbolic basis *adapted to H and F*. Then the element

$$\omega_H := e_{k+1} \cdots e_n f_1 \cdots f_k f_{k+1} \cdots f_n \in \mathrm{Cl}(V)f$$

lies in S_H since $e_i \omega_H = f_j \omega_H = 0$ for all $i > k$ and $j \leq k$. Conversely, if $\mu \in S_H$, then write

$$\mu = \sum_{l=0}^{n} \sum_{i_1 < \ldots < i_l} c_{\{i_1, \ldots, i_l\}} e_{i_1} \cdots e_{i_l} f.$$

If $c_I \neq 0$ for some I with $I \not\supseteq \{k+1, \ldots, n\}$, then for any $j \in \{k+1, \ldots, n\} \setminus I$, we find that $e_j \mu \neq 0$. So all I with $c_I \neq 0$ contain $\{k+1, \ldots, n\}$. If some I with $c_I \neq 0$ further contains an $i \leq k$, then $f_i \mu$ is nonzero. Hence, S_H is spanned by ω_H, as claimed. In what follows, by slight abuse of notation, we will write ω_H for any nonzero vector in S_H.

The space H can be uniquely recovered from ω_H via

$$H = \{v \in V \mid v \cdot \omega_H = 0\}.$$

Indeed, we have already seen $\subseteq$. For the converse, observe that the vectors $e_i \omega_H, f_j \omega_H$ with $i \leq k$ and $j > k$ are linearly independent.

The map that sends $H \in \mathrm{Gr}_{\mathrm{iso}}(V, q)$ to the projective point representing it, that is,

$$H \mapsto [\omega_H] \in \mathbb{P}(\mathrm{Cl}(V)f),$$

is therefore injective, and it is called the *spinor embedding* of the isotropic Grassmannian (see [11]). The *isotropic Grassmann cone* is defined as

$$\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V, q) := \bigcup_H \langle \omega_H \rangle \subseteq \mathrm{Cl}(V)f,$$

where the union is taken over all maximal isotropic subspaces $H \subseteq V$. We denote by $\widehat{\mathrm{Gr}}^{\pm}_{\mathrm{iso}}(V, q) := \widehat{\mathrm{Gr}}_{\mathrm{iso}}(V, q) \cap \mathrm{Cl}^{\pm}(V)f$ the cones over the connected components of the isotropic Grassmannian in its spinor embedding.

3.2. Contraction with an isotropic vector

Let $e \in V$ be a nonzero isotropic vector. Then $V_e := e^{\perp}/\langle e \rangle$ is equipped with a natural nondegenerate quadratic form, and there is a rational map $\mathrm{Gr}_{\mathrm{iso}}(V) \to \mathrm{Gr}_{\mathrm{iso}}(V_e)$ that maps an n-dimensional isotropic space H to the image in V_e of the $(n-1)$-dimensional isotropic space $H \cap e^{\perp}$ (this is defined if $e \notin H$, which by maximality of H is equivalent to $H \not\subseteq e^{\perp}$). This map is the restriction to $\mathrm{Gr}_{\mathrm{iso}}(V)$ of the rational map $\mathbb{P}(\bigwedge^n V) \to \mathbb{P}(\bigwedge^{n-1} V_e)$ induced by the linear map ('contraction with e'):

$$c_e : \bigwedge^n V \to \bigwedge^{n-1} V_e, \quad v_1 \wedge \cdots \wedge v_n \mapsto \sum_{i=1}^{n} (-1)^{i-1} (e|v_i) \overline{v_1} \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \overline{v_n},$$

where $\overline{v_i}$ is the image of v_i in $V/\langle e \rangle$. Note first that this map is the inner product $\iota(e)$ followed by a projection. Furthermore, *a priori*, the codomain of this map is the larger space $\bigwedge^{n-1}(V/\langle e \rangle)$, but one may choose $v_1, \ldots, v_n$ such that $(e|v_i) = 0$ for $i > 1$, and then it is evident that the image is indeed in $\bigwedge^{n-1} V_e$.

We want to construct a similar contraction map at the level of the spin representation. For reasons that will become clear in a moment, we restrict our attention first to a map between two half-spin

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representations, as follows. Assume that $e \notin F$, and choose a basis $f_1, \dots, f_n$ of F such that $(e|f_i) = \delta_{in}$. As usual, write $f := f_1 \cdots f_n$, and write $\overline{f} := \overline{f}_1 \cdots \overline{f}_{n-1}$, so that $\mathrm{Cl}^+(V_e)\overline{f}$ is a half-spin representation of $\mathfrak{so}(V_e)$.

Then we define the map

$$\pi_e : \mathrm{Cl}^+(V)f \to \mathrm{Cl}^+(V_e)\overline{f}, \quad \pi_e(af) := \text{ the image of } \frac{1}{2}((-1)^{n-1}eaf + afe) \text{ in } \mathrm{Cl}(V_e)\overline{f},$$

where the implicit claim is that the expression on the right lies in $\mathrm{Cl}(e^\perp)f_1 \cdots f_{n-1}$, so that its image in $\mathrm{Cl}(V_e)\overline{f}$ is well defined (note that the projection $e^\perp \to V_e$ induces a homomorphism of Clifford algebras), and that this image lies in the left ideal generated by $\overline{f}$. To verify this claim, and to derive a more explicit formula for the map above, let $e_1, \dots, e_n = e$ be a basis of an isotropic space E complementary to F. Then it suffices to consider the case where $a = e_{i_1} \cdots e_{i_k}$ for some $i_1 < \dots < i_k$. We then have

$$\begin{aligned} eaf &= ee_{i_1} \cdots e_{i_k} f_1 \cdots f_n \\ &= \begin{cases} 0 \text{ if } i_k = n, \text{ and} \\ 2(-1)^{k+n-1} e_{i_1} \cdots e_{i_k} f_1 \cdots f_{n-1} + (-1)^{k+n} e_{i_1} \cdots e_{i_k} f_1 \cdots f_n e \text{ otherwise.} \end{cases} \end{aligned}$$

Multiplying by $(-1)^{n-1}$ and using that k is even, the latter expression becomes

$$2e_{i_1} \cdots e_{i_k} f_1 \cdots f_{n-1} - afe.$$

Hence, we conclude that

$$\pi_e(e_{i_1} \cdots e_{i_k} f) = \begin{cases} 0 & \text{if } i_k = n, \text{and} \\ \overline{e}_{i_1} \cdots \overline{e}_{i_k} \overline{f} & \text{otherwise.} \end{cases}$$

In short, in our models $\bigwedge^+ E$ and $\bigwedge^+(E/\langle e \rangle)$ for the half-spin representations of $\mathfrak{so}(V)$ and $\mathfrak{so}(V_e)$, π_e is just the reduction-mod-e map. We leave it to the reader to check that the reduction-mod-e map $\bigwedge^- E \to \bigwedge^-(E/\langle e \rangle)$ arises in a similar fashion from the map

$$\pi_e : \mathrm{Cl}(V)^- f \to \mathrm{Cl}(V_e)^- \overline{f}, \quad \pi_e(af) := \text{ the image of } \frac{1}{2}((-1)^n eaf + afe) \text{ in } \mathrm{Cl}(V_e)\overline{f}.$$

We will informally call the maps π_e 'contraction with e'. Together, they define a map on $\mathrm{Cl}(V)f$ which we also denote by π_e.

Proposition 3.1. *The contraction map* $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V)\overline{f}$ *is a homomorphism of* $\mathrm{Cl}(e^\perp)$*-representations.*

Proof. Let $v \in e^\perp$ and consider $a \in \mathrm{Cl}^-(V)$. Then $va \in \mathrm{Cl}^+(V)$, and hence, $\pi_e(vaf)$ is the image in $\mathrm{Cl}(V_e)\overline{f}$ of

$$\frac{1}{2}((-1)^{n-1}evaf + vafe) = \frac{1}{2}((-1)^n veaf + vafe) = v\frac{1}{2}((-1)^n eaf + afe),$$

where we have used $(v|e) = 0$ in the first equality. The right-hand side clearly equals $\overline{v}$ times the image of $\pi_e(af)$ in $\mathrm{Cl}(V_e)\overline{f}$. □

3.3. Multiplying with an isotropic vector

In a sense dual to the contraction maps, $c_e : \bigwedge^n V \to \bigwedge^{n-1} V_e$ are multiplication maps defined as follows. Let $e, h \in V$ be isotropic with $(e|h) = 1$; such a pair is called a *hyperbolic pair*. We then have

$V = \langle e, h\rangle \oplus \langle e, h\rangle^\perp$, and the map from the second summand to $V_e = e^\perp/\langle e\rangle$ is an isometry. We use this isometry to identify V_e with the subspace $\langle e, h\rangle^\perp$ of V and write s_e for the corresponding inclusion map. Then we define

$$m_h : \bigwedge^{n-1} V_e \to \bigwedge^n V, \quad \overline{v}_1 \wedge \cdots \wedge \overline{v}_{n-1} \mapsto h \wedge \overline{v}_1 \wedge \cdots \wedge \overline{v}_{n-1},$$

which is just the outer product $o(h)$. The projectivisation of this map sends $\mathrm{Gr}_{\mathrm{iso}}(V_e)$ isomorphically to the closed subset of $\mathrm{Gr}_{\mathrm{iso}}(V)$ consisting of all H containing h. We further observe that

$$c_e \circ m_h = \mathrm{id}_{\bigwedge^{n-1} V_e}.$$

We define a corresponding multiplication map at the level of spin representations as follows: first, we assume that $h \in F$, and choose a basis $f_1, \ldots, f_n = h$ of F such that $(e|f_i) = \delta_{in}$. As usual, we set $f = f_1 \cdots f_n$ and $\overline{f} = \overline{f}_1 \cdots \overline{f}_{n-1}$. Then we define

$$\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f, \quad \tau_h(a\overline{f}) := a\overline{f} f_n = af.$$

Note that, for $a \in \mathrm{Cl}(V_e)$, we have

$$\pi_e(\tau_h(a\overline{f})) = \pi_e(af) = a\overline{f},$$

where the last identity can be seen verified in the model $\bigwedge E$ for the spin representation, where π_e is the reduction-mod-e map, and τ_h is just the inclusion $\bigwedge E/\langle e\rangle \to \bigwedge E$ corresponding to the inclusion $V_e \to V$. So $\pi_e \circ \tau_h = \mathrm{id}_{\mathrm{Cl}(V_e)\overline{f}}$. We will informally call τ_h the multiplication map with h.

Proposition 3.2. *The multiplication map $\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$ is a homomorphism of $\mathrm{Cl}(V_e)$-representations, where $\mathrm{Cl}(V_e)$ is regarded a subalgebra of $\mathrm{Cl}(V)$ via the section $s_e : V_e \to V$.*

Proof. Let $v \in V_e$ and let $a \in \mathrm{Cl}(V_e)$. Then

$$\tau_h(va\overline{f}) = va\overline{f} f_n = vaf,$$

as desired. □

Corollary 3.3. *Both the map $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$ and the map $\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$ are $\mathrm{Spin}(V_e)$-equivariant, where $\mathrm{Spin}(V_e)$ is regarded as a subgroup of $\mathrm{Spin}(V)$ via the orthogonal decomposition $V = V_e \oplus \langle e, h\rangle$.*

Proof. Propositions 3.1 and 3.2 imply that both maps are homomorphisms of $\mathfrak{so}(V_e)$-representations. Since $\mathrm{Spin}(V_e)$ is generated by one-parameter subgroups corresponding to nilpotent elements of $\mathfrak{so}(V_e)$, π_e and τ_h are $\mathrm{Spin}(V_e)$-equivariant. □

3.4. Properties of the isotropic Grassmannian

The goal of this subsection is to collect properties of the isotropic Grassmann cone that will later motivate the definition of a *(half-)spin variety* (see Section 5). We fix a maximal isotropic subspace $F \subseteq V$ and a hyperbolic pair (e, h) with $h \in F$ and $e \notin F$ and identify $V_e = e^\perp/\langle e\rangle$ with the subspace $\langle e, h\rangle^\perp$ of V. We choose any basis $f_1, \ldots, f_n$ of F with $f_n = h$ and $(e|f_i) = 0$ for $i < n$ and write $f := f_1 \cdots f_n \in \mathrm{Cl}(V)$ and $\overline{f} := \overline{f_1} \cdots \overline{f_{n-1}} \in \mathrm{Cl}(V_e)$.

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Proposition 3.4. *The isotropic Grassmann cone in* $\mathrm{Cl}(V)f$ *has the following properties:*

1. $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V) \subseteq \mathrm{Cl}(V)f$ *is Zariski-closed and* $\mathrm{Spin}(V)$*-stable.*
2. *Let* $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$ *be the contraction defined in §3.2. Then for every maximal isotropic subspace* $H \subseteq V$*, we have*

$$\pi_e(S_H) \subseteq S_{H_e},$$

where $H_e \subseteq V_e$ *is the image of* $e^\perp \cap H$ *in* V_e*.*

3. *Let* $\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$ *be the map defined in §3.3. Then for every maximal isotropic* $H' \subseteq V_e$*, we have*

$$\tau_h(S_{H'}) = S_{H' \oplus \langle h \rangle}.$$

In particular, the contraction and multiplication map π_e *and* τ_h *preserve the isotropic Grassmann cones – that is,*

$$\pi_e\big(\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V)\big) \subseteq \widehat{\mathrm{Gr}}_{\mathrm{iso}}(V_e) \quad \text{and} \quad \tau_h\big(\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V_e)\big) \subseteq \widehat{\mathrm{Gr}}_{\mathrm{iso}}(V).$$

Proof of Proposition 3.4. 1. This is well known. Indeed, the isotropic Grassmann cone is the union of the cones over the two connected components, and these cones are the union of $\{0\}$ with the orbits of the highest weight vectors ω_0 and ω_1. These minimal orbits are always Zariski closed. For more detail, see [13, Theorem 1, p.428].

2. Let ω_H be a spanning element of S_H. Then for all $v \in e^\perp \cap H$, we have

$$\overline{v} \cdot \pi_e(\omega_H) = \pi_e(v \cdot \omega_H) = \pi_e(0) = 0,$$

where the first equality follows from Proposition 3.1. Hence, $\pi_e(\omega_H)$ lies in S_{H_e}.

3. Let $\omega_{H'}$ be a spanning element of $S_{H'}$. Then for all $v \in H'$, we have

$$v \cdot \tau_h(\omega_{H'}) = \tau_h(v \cdot \omega_{H'}) = \tau_h(0) = 0,$$

where the first equality holds by Proposition 3.2. Furthermore, we have

$$h \cdot \tau_h(\omega_{H'}) = h \cdot \omega_{H'} f_n = 0,$$

where we used the definition of τ_h and $h = f_n$. Thus, $\tau_h(\omega_{H'})$ lies in $S_{H' \oplus \langle h \rangle}$. The equality now follows from the fact that τ_h is injective. □

Remark 3.5. If $h \in H$, then $H = H_e \oplus \langle h \rangle$, and since $\pi_e \circ \tau_h$ is the identity on $\mathrm{Cl}(V_e)\overline{f}$, we find that

$$\pi_e(S_H) = \pi_e(\tau_h(S_{H_e})) = S_{H_e}$$

(i.e., equality holds in (2) of Proposition 3.4). Later, we will see that equality holds under the weaker condition that $e \notin H$, while $\pi_e(S_H) = \{0\}$ when $e \in H$. These statements can also be checked by direct computations, but some care is needed since for e, H, F in general position, one cannot construct a hyperbolic basis adapted to H and F that moreover contains e.

3.5. The dual of contraction

Let $e \notin F \subseteq V$ be an isotropic vector. We want to compute the dual of the contraction map $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$; indeed, we claim that this is essentially the map

$$\psi_e : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$$

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defined by its restriction $\mathrm{Cl}^{\pm}(V_e)\overline{f} \to \mathrm{Cl}^{\mp}(V)f$ as

$$\psi_e(\overline{b}\cdot\overline{f}_1\cdots\overline{f_{n-1}}) := \pm e b f_1\cdots f_n,$$

where the sign is + on $\mathrm{Cl}^{+}(V_e)\overline{f}$ and − on $\mathrm{Cl}^{-}(V_e)\overline{f}$. The reason for the 'flip' of the choice of half-spin representation in the dual will become obvious below. Observe that ψ_e is well defined and, given a basis $e_1,\ldots,e_n = e$ of an isotropic space complementary to F such that $e_1,\ldots,e_n,f_1,\ldots,f_n$ is a hyperbolic basis, maps $\overline{e_J}\,\overline{f}$ to $e_{J\cup\{n\}}f$.

Proposition 3.6. *The following diagram*

$$\begin{tikzcd}
(\mathrm{Cl}(V_e)\overline{f})^* \arrow[r, "\pi_e^*"] \arrow[d, "\simeq"'] & (\mathrm{Cl}(V)f)^* \arrow[d, "\simeq"] \\
\mathrm{Cl}(V_e)\overline{f} \arrow[r, "\psi_e"'] & \mathrm{Cl}(V)f
\end{tikzcd}$$

can be made commuting via a $\mathrm{Spin}(V_e)$*-module isomorphism on the left vertical arrow and a* $\mathrm{Spin}(V)$*-module isomorphism on the right vertical arrow.*

Remark 3.7. The statement of Proposition 3.6 holds true when replacing $\mathrm{Cl}(V)f$ by either one of the two half-spin representations by considering the correct 'flip'. For example, if $n = \dim F$ is even, and $e_1,\ldots,e_n,f_1,\ldots,f_n$ is a hyperbolic basis as above, then in the $\bigwedge E$-model, the correct grading is

$$\begin{tikzcd}
\left(\bigwedge^+ E_{n-1}\right)^* \arrow[r, "\pi_{e_n}^*"] \arrow[d, "\simeq"'] & \left(\bigwedge^+ E_n\right)^* \arrow[d, "\simeq"] \\
\bigwedge^- E_{n-1} \arrow[r, "\psi_{e_n}"'] & \bigwedge^+ E_n.
\end{tikzcd}$$

To prove Proposition 3.6, we consider the bilinear form β on the spin representation $\mathrm{Cl}(V)f$ defined as in [13] as follows: for $af, bf \in \mathrm{Cl}(V)f$, it turns out that $(af)^*bf = f^*a^*bf$, where $*$ denotes the anti-automorphism from §2.1, is a scalar multiple of f. The scalar is denoted $\beta(af, bf)$. We have the following properties:

Lemma 3.8 [13, p. 430]. *Let β be the bilinear form defined as above.*

1. *The form β is nondegenerate and* $\mathrm{Spin}(V)$*-invariant.*
2. *β is symmetric if $n \equiv 0, 1 \bmod 4$, and it is skew-symmetric if $n \equiv 2, 3 \bmod 4$.*
3. *The two half-spin representations are self-dual via β if n is even, and each is the dual of the other if n is odd.*

In the proof of Proposition 3.6, we will use a hyperbolic basis $e_1,\ldots,e_n,f_1,\ldots,f_n$ with $e_n = e$. For a subset $I = \{i_1 < \ldots < i_k\} \subseteq [n]$, set $e_I := e_{i_1}\cdots e_{i_k} \in \mathrm{Cl}(E) \simeq \bigwedge E$, where E is the span of the e_i. We have seen in §2.5 that the spin representation has as a basis the elements $e_I f$ with I running through all subsets of $[n]$.

Proof of Proposition 3.6. Consider the bilinear forms β on $\mathrm{Cl}(V)f$ and β_e on $\mathrm{Cl}(V_e)\overline{f}$ as defined above. By Lemma 3.8, the spin representations $\mathrm{Cl}(V)f$ and $\mathrm{Cl}(V_e)\overline{f}$ are self-dual via β and β_e, respectively. Thus, it suffices to prove, for $a \in \mathrm{Cl}(V)$ and $b \in \mathrm{Cl}(e^\perp)$, that

$$\beta_e(\pi_e(af), \overline{bf}) = \frac{(-1)^{n-1}}{2}\beta(af, \psi_e(\overline{bf})).$$

We may assume that $a = e_I$, $b = e_J$ with $I \subseteq [n]$, $J \subseteq [n-1]$.

In the $\bigwedge E$-model, π_e is the mod-e map, and hence, the left-hand side is zero if $n \in I$. If $n \notin I$, then the left-hand side equals the coefficient of $\overline{f}$ in $\overline{f}^* \overline{e_I}^* \overline{e_J} \overline{f}$. This is nonzero if and only if $[n-1]$ is the disjoint union of I and J, and then it is 2^{n-1} times a sign corresponding to the number of swaps needed to move the factors $\overline{f_i}$ of $\overline{f}^*$ to just before the corresponding factor $\overline{e_i}$ in either $\overline{e_I}^*$ or $\overline{e_J}$.

Apart from the factor $\frac{(-1)^{n-1}}{2}$, the right-hand side is the coefficient of f in $f^* e_I e_J e_n f$. This is nonzero if and only if $[n]$ is the disjoint union of the sets $\{n\}, J, I$, and in that case, it is 2^n times a sign corresponding to the number of swaps needed to move the factors f_i of f^* to the corresponding factor e_i in either e_I or e_J or (in the case of f_n) to just before the factor e_n. The latter contributes $(-1)^{n-1}$, and apart from this factor, the sign is the same as on the left-hand side. □

3.6. *Two infinite spin representations*

Let V_∞ be the countable-dimensional vector space with basis $e_1, f_1, e_2, f_2, \ldots$, and equip V_∞ with the quadratic form for which this is a hyperbolic basis (i.e., $(e_i|e_j) = (f_i|f_j) = 0$ and $(e_i|f_j) = \delta_{ij}$ for all i, j). We write E_∞ and F_∞ for the subspaces of V_∞ spanned by the e_i and the f_i, respectively.

Let V_n be the subspace of V_∞ spanned by $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$, with the restricted quadratic form. We further set $E_n := V_n \cap E_\infty$ and $F_n := V_n \cap F_\infty$. We define the *infinite spin group* as $\mathrm{Spin}(V_\infty) := \varinjlim_n \mathrm{Spin}(V_n)$, where $\mathrm{Spin}(V_{n-1})$ is embedded into $\mathrm{Spin}(V_n)$ as the subgroup that fixes $\langle e_n, f_n\rangle$ element-wise. Similarly, we write $\mathrm{GL}(E_\infty) := \varinjlim_n \mathrm{GL}(E_n)$ and H for the preimage of $\mathrm{GL}(E_\infty)$ in $\mathrm{Spin}(V_\infty)$. We use the notation $\mathfrak{so}(V_\infty)$ and $\mathfrak{gl}(E_\infty)$ for the corresponding direct limits of the Lie algebras $\mathfrak{so}(V_n)$ and $\mathfrak{gl}(E_n)$. Here, the direct limits are taken in the categories of abstract groups and Lie algebras, respectively.

The previous paragraphs give rise to various $\mathrm{Spin}(V_{n-1})$-equivariant maps between the spin representations of $\mathrm{Spin}(V_{n-1})$ and $\mathrm{Spin}(V_n)$. First, contraction with e_n,

$$\pi_{e_n} : \mathrm{Cl}(V_n) f_1 \cdots f_n \to \mathrm{Cl}(V_{n-1}) f_1 \cdots f_{n-1},$$

and second, multiplication with f_n,

$$\tau_{f_n} : \mathrm{Cl}(V_{n-1}) f_1 \cdots f_{n-1} \to \mathrm{Cl}(V_n) f_1 \cdots f_n.$$

We have that these satisfy $\pi_{e_n} \circ \tau_{f_n} = \mathrm{id}$. Third, the map

$$\psi_{e_n} : \mathrm{Cl}(V_{n-1}) f_1 \cdots f_{n-1} \to \mathrm{Cl}(V_n) f_1 \cdots f_n$$

is dual to π_{e_n} in the sense of Proposition 3.6.

Definition 3.9. The *direct (infinite) spin representation* is the direct limit of all spaces $\mathrm{Cl}(V_n) f_1 \cdots f_n$ along the maps ψ_{e_n}. The *inverse (infinite) spin representation* is the inverse limit of all spaces $\mathrm{Cl}(V_n) f_1 \cdots f_n$ along the maps π_{e_n}.

Since the maps ψ_{e_n}, π_{e_n} are $\mathrm{Spin}(V_{n-1})$-equivariant, both of these spaces are $\mathrm{Spin}(V_\infty)$-modules. As the dual of a direct limit is the inverse limit of the duals, and since the maps ψ_{e_n} and π_{e_n} are dual to each other by Proposition 3.6, the inverse spin representation is the dual space of the direct spin representation.

In our model $\bigwedge E_n$ of $\mathrm{Cl}(V_n) f_1 \cdots f_n$, the map ψ_{e_n} is just the right multiplication

$$\bigwedge E_{n-1} \to \bigwedge E_n, \ \omega \mapsto \omega \wedge e_n.$$

Hence, the direct spin representation has as a basis formal infinite products

$$e_{i_1} \wedge e_{i_2} \wedge \ldots =: e_I,$$

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where $I = \{i_1 < i_2 < \ldots\}$ is a cofinite subset of $\mathbb{N}$. We will write $\bigwedge_\infty E_\infty$ for this countable-dimensional vector space. The action of the Lie algebra $\mathfrak{so}(V_\infty)$ of $\mathrm{Spin}(V_\infty)$ on this space is given via the explicit formulas from §2.5. In particular, the span of the e_I with $|\mathbb{N} \setminus I|$ even (respectively, odd) is a $\mathrm{Spin}(V_\infty)$-submodule, and $\bigwedge_\infty E_\infty$ is the direct sum of these (irreducible) modules.

Remark 3.10. The reader may wonder why we do not introduce the direct spin representation as the direct limit of all $\mathrm{Cl}(V) f_1 \cdots f_n$ along the maps τ_{f_n}. This would make the ordinary Grassmann algebra $\bigwedge E_\infty$ a model for the direct spin representation, instead of the slightly more complicated-looking space $\bigwedge_\infty E_\infty$. However, the maps dual to the τ_{f_n} correspond to contraction maps with $f_n \in F$, which we have not discussed and which interchange even and odd half-spin representations. We believe that our theorem below goes through for this different setting, as well, but we have not checked the details.

3.7. *Four infinite half-spin representations*

Keeping in mind that the maps ψ_{e_n} interchange the even and odd subrepresentations, we define the *direct (infinite) half-spin representations* $\bigwedge_\infty^\pm E_\infty$ to be the direct limit

$$\bigwedge\nolimits_\infty^{\pm} E_\infty = \varinjlim\Big(\bigwedge\nolimits^{\pm} E_0 \to \bigwedge\nolimits^{\mp} E_1 \to \bigwedge\nolimits^{\pm} E_2 \to \bigwedge\nolimits^{\mp} E_3 \to \bigwedge\nolimits^{\pm} E_4 \to \cdots\Big)$$

along the maps ψ_{e_n}. For the sake of readability, we will abbreviate this by

$$\bigwedge\nolimits_\infty^{\pm} E_\infty = \varinjlim_n \bigwedge\nolimits^{\pm(-1)^n} E_n, \tag{3.2}$$

where $\pm(-1)^n$ denotes $\pm$ if n is even and $\mp$ if n is odd. In terms of the basis e_I introduced in §3.6, the half-spin representation $\bigwedge_\infty^+ E_\infty$ is spanned by all e_I with $|\mathbb{N} \setminus I|$ even, and $\bigwedge_\infty^- E_\infty$ by those with $|\mathbb{N} \setminus I|$ odd. The *inverse (infinite) half-spin representations* are defined as the duals of the direct (infinite) half-spin representations. Using the isomorphisms from Remark 3.7, we observe

$$\Big(\bigwedge\nolimits_\infty^{\pm} E_\infty\Big)^* = \varprojlim_n \Big(\bigwedge\nolimits^{\pm(-1)^n} E_n\Big)^* \simeq \varprojlim_n \bigwedge\nolimits^{\pm} E_n. \tag{3.3}$$

So the inverse (infinite) half-spin representations can be identified with the inverse limits of the half-spin representations $\bigwedge^\pm E_n$ along the projections π_{e_n}.

We can enrich the inverse spin representation to an affine scheme whose coordinate ring is the symmetric algebra on $\bigwedge_\infty E_\infty$, recalling the following remark.

Remark 3.11. Let K be any field (not necessarily algebraically closed) and W any K-vector space (not necessarily finite dimensional). Then there are canonical identifications

$$W^* = \mathrm{Spec}\big(\mathrm{Sym}(W)\big)(K) \subseteq \big\{\text{closed points in } \mathrm{Spec}\big(\mathrm{Sym}(W)\big)\big\}.$$

So $\mathrm{Spec}\big(\mathrm{Sym}(W)\big)$ can be seen as an enrichment of W^* to an affine scheme. If W is a linear representation for a group G, then G acts via K-algebra automorphisms on $\mathrm{Sym}\, W$ and hence via K-automorphisms on the affine scheme corresponding to W^*. For $W = \bigwedge_\infty^\pm E_\infty$, this construction extends the natural $\mathrm{Spin}(V_\infty)$-action on the vector space $\varprojlim_n \bigwedge^\pm E_n \simeq W^*$ to the corresponding affine scheme.

By abuse of notation, we will write $(\bigwedge_\infty E_\infty)^*$ also for the scheme itself, and similarly for the inverse half-spin representations $\big(\bigwedge_\infty^\pm E_\infty\big)^*$. Later, we will also write $\bigwedge^\pm E_n$ for the affine scheme $\mathrm{Spec}\Big(\mathrm{Sym}\Big(\bigwedge^{\pm(-1)^n} E_n\Big)\Big)$ by identifying $\bigwedge^\pm E_n \cong \Big(\bigwedge^{\pm(-1)^n} E_n\Big)^*$ as in Equation(3.3).

4. Noetherianity of the inverse half-spin representations

In this section, we prove our main theorem.

Theorem 4.1. *The inverse half-spin representation* $(\bigwedge_\infty^+ E_\infty)^*$ *is topologically Noetherian with respect to the action of* $\mathrm{Spin}(V_\infty)$. *That is, every descending chain*

$$\left(\bigwedge\nolimits_\infty^+ E_\infty\right)^* \supseteq X_1 \supseteq X_2 \supseteq \dots$$

of closed, reduced $\mathrm{Spin}(V_\infty)$*-stable subschemes stabilises, and the same holds for the other inverse half-spin representation.*

Recall that the action of $\mathrm{Spin}(V_\infty)$ on the inverse half-spin representation (as an affine scheme) is given by K-automorphisms, as described in Remark 3.11. We write R for the symmetric algebra on the direct spin representation $\bigwedge_\infty E_\infty$, so the inverse spin representation is $\mathrm{Spec}(R)$. Similarly, we write $R^\pm$ for the symmetric algebras on the direct half-spin representations, so $R^\pm$ is the coordinate ring of $\varprojlim_n \bigwedge^\pm E_n$, respectively.

Let us briefly outline the proof strategy. We will proceed by induction on the minimal degree of an equation defining a closed subset X. Starting with such an equation p, we show that there exists a partial derivative $q := \frac{\partial p}{\partial e_I}$ such that the principal open $X[1/q]$ is topologically H_n-Noetherian, where H_n is the subgroup of $\mathrm{Spin}(V_\infty)$ defined below. For that, we use that the H_n-action corresponds to a 'twist' of the usual $\mathrm{GL}(E_\infty)$-action, as observed in Section 2.8 (for the exact formula see (2.3)); this allows us to apply the main result of [7]. Finally, for those points which are contained in the vanishing set of the $\mathrm{Spin}(V_\infty)$-orbit of q, we can apply induction, as the minimal degree of a defining equation has been lowered by 1.

4.1. Shifting

Let G_n be the subgroup of G that fixes $e_1, \dots, e_n, f_1, \dots, f_n$ element-wise. Note that G_n is isomorphic to G; at the level of the Lie algebras, the isomorphism from G to G_n is given by the map

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 0 & 0 \\ 0 & C & 0 & -A^T \end{bmatrix}$$

where the widths of the blocks are n, ∞, n, ∞, respectively. We write H_n for $H \cap G_n$, where $H \subseteq \mathrm{Spin}(V_\infty)$ is the subgroup corresponding to the subalgebra $\mathfrak{gl}(E_\infty) \subseteq \mathfrak{so}(V_\infty)$. Then H_n is the pre-image in $\mathrm{Spin}(V_\infty)$ of the subgroup $\mathrm{GL}(E_\infty)_n \subseteq \mathrm{GL}(E_\infty)$ of all g that fix $e_1, \dots, e_n$ element-wise and maps the span of the e_i with $i > n$ into itself. The Lie algebra of H_n and of $\mathrm{GL}(E_\infty)_n$ consists of the matrices above on the right with $B = C = 0$.

4.2. Acting with the general linear group on E

For every fixed $k \in \mathbb{Z}_{\geq 0}$, the Lie algebra $\mathfrak{gl}(E_\infty) \subseteq \mathfrak{so}(V_\infty)$ preserves the linear space

$$\left(\bigwedge\nolimits_\infty E_\infty\right)_k := \langle \{e_I : |\mathbb{N} \setminus I| = k\} \rangle,$$

and hence, so does the corresponding subgroup $H \subseteq \mathrm{Spin}(V_\infty)$. We let $R_{\leq \ell} \subseteq R$ be the subalgebra generated by the spaces $(\bigwedge_\infty E_\infty)_k$ with $k \leq \ell$. Crucial in the proof of Theorem 4.1 is the following result.

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Proposition 4.2. *For every choice of nonnegative integers ℓ and n,* $\mathrm{Spec}(R_{\leq\ell})$ *is topologically H_n-Noetherian; that is, every descending chain*

$$\mathrm{Spec}(R_{\leq\ell}) \supseteq X_1 \supseteq X_2 \supseteq \dots$$

of H_n-stable closed and reduced subschemes stabilizes.

The key ingredient in the proof of Proposition 4.2 is the main result of [7]. In order to apply their result, we need to do some preparatory work. We will start with the following lemma.

Lemma 4.3. *Every H_n-stable closed subscheme of* $\mathrm{Spec}(R_{\leq\ell})$ *is also stable under the group* $\mathrm{GL}(E_\infty)_n$ *acting in the natural manner on $\bigwedge_\infty E_\infty$ and its dual, and vice versa.*

Proof. Equation (2.3) implies that $\mathfrak{gl}(E_\infty) \subseteq \mathfrak{so}(V_\infty)$ acts on $\bigwedge_\infty E_\infty$ via

$$\rho(A) = \tilde{\rho}(A) - \frac{1}{2}\,\mathrm{tr}(A)\,\mathrm{id}_{\bigwedge_\infty E_\infty},$$

where $\tilde{\rho}$ is the standard representation of $\mathfrak{gl}(E_\infty)$ on $\bigwedge_\infty E_\infty$. An H_n-stable closed subscheme X of $\mathrm{Spec}(R_{\leq\ell})$ is given by an H_n-stable ideal I in the symmetric algebra $R_{\leq\ell}$. Such an I is then also stable under the action of the Lie algebra $\mathfrak{gl}(E_\infty)_n$ of H_n by derivations that act on variables in $\bigoplus_{k=0}^{\ell}(\bigwedge_\infty E_\infty)_k$ via ρ.

We claim that I is a homogeneous ideal. Indeed, for $f \in I$, choose $m > n$ such that all variables in f (which are basis elements e_I) contain the basis element e_m of E_∞. Let $A \in \mathfrak{gl}(E_\infty)_n$ be the diagonal matrix with 0's everywhere except a 1 on position (m, m). Then $\rho(A)$ maps each variable in f to $\frac{1}{2}$ times itself. Hence, by the Leibniz rule, $\rho(A)$ scales the homogeneous part of degree d in f by $\frac{d}{2}$. Since I is preserved by $\rho(A)$, it follows that I contains all homogeneous components of f, and hence, I is a homogeneous ideal.

Now let $B \in \mathfrak{gl}(E_\infty)_n$ and $f \in I$ be arbitrary. By the previous paragraph, we can assume f to be homogeneous of degree d, and we then have

$$\rho(B)f = \tilde{\rho}(B)f - \frac{d}{2}\,\mathrm{tr}(B)f,$$

and since I is $\rho(B)$-stable, we deduce $\tilde{\rho}(B)f \in I$. This completes the proof in one direction. The proof in the opposite direction is identical. □

Remark 4.4. By the proof above, any $\mathrm{Spin}(V_\infty)$-stable closed subscheme X of $(\bigwedge_\infty E_\infty)^*$ is an affine cone.

Following [7], the *restricted dual* $(E_\infty)_*$ of E_∞ is defined as the union $\bigcup_{n\geq 1}(E_n)^*$. We will denote by $\varepsilon^1, \varepsilon^2, \dots$ the basis of $(E_\infty)_*$ that is dual to the canonical basis $e_1, e_2, \dots$ of E_∞ given by $\varepsilon^i(e_j) = \delta_{ij}$.

Lemma 4.5. *There is an* $\mathrm{SL}(E_\infty)$*-equivariant isomorphism*

$$\bigwedge\nolimits_\infty E_\infty \longrightarrow \bigwedge (E_\infty)_*,$$

which restricts to an isomorphism

$$\Big(\bigwedge\nolimits_\infty E_\infty\Big)_k \longrightarrow \bigwedge\nolimits^k (E_\infty)_*.$$

We will use this isomorphism to regard $\bigwedge_\infty E_\infty$ as the restricted dual of the Grassmann algebra $\bigwedge E_\infty$. We stress, though, that this isomorphism is not $GL(E_\infty)$-equivariant.

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Proof. We have a natural bilinear map

$$\bigwedge E_\infty \times \bigwedge_\infty E_\infty \to \bigwedge_\infty E_\infty, \quad (\omega, \omega') \mapsto \omega \wedge \omega'.$$

If $I \subseteq \mathbb{N}$ is finite and $J \subseteq \mathbb{N}$ is cofinite, then $e_I \wedge e_J$ is 0 if $I \cap J \neq \emptyset$ and $\pm e_{I \cup J}$ otherwise, where the sign is determined by the permutation required to order the sequence I, J. We then define a perfect pairing γ between the two spaces by

$$\gamma(\omega, \omega') := \text{the coefficient of } e_{\mathbb{N}} \text{ in } \omega \wedge \omega'.$$

The map $\Phi_\gamma : \bigwedge_\infty E_\infty \to \bigwedge (E_\infty)_*, \ \omega' \mapsto \gamma(\cdot, \omega')$ induced by γ is the isomorphism given by $e_I \mapsto \pm \varepsilon^{I^c}$, where $I^c \subseteq \mathbb{N}$ is the complement of I and $\varepsilon^J := \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}$ for a finite set $J = \{j_1, \ldots, j_k\}$. Note that $\gamma(A \cdot \omega, A \cdot \omega') = \det(A)\gamma(\omega, \omega')$ for all $A \in \mathrm{GL}(E_\infty)$, and hence, γ is $\mathrm{SL}(E_\infty)$-invariant. Therefore, the isomorphism Φ_γ is $\mathrm{SL}(E_\infty)$-equivariant. □

Lemma 4.6. *An ideal $I \subseteq \mathrm{Sym}(\bigwedge(E_\infty)_*)$ is $\mathrm{SL}(E_\infty)$-stable if and only if it is $\mathrm{GL}(E_\infty)$-stable. The same holds for $\mathrm{SL}(E_\infty)_n$ and $\mathrm{GL}(E_\infty)_n$.*

Proof. Assume that I is $\mathrm{SL}(E_\infty)$-stable. Let $f \in I$ and $A \in \mathrm{GL}(E_\infty)$ be arbitrary. Choose $m = m(f, A) \in \mathbb{N}$ large enough so that $f \in \mathrm{Sym}(\bigwedge(E_m)^*)$ and A is the image of some $A_m \in \mathrm{GL}(E_m)$. Define $A_{m+1} \in \mathrm{GL}(E_{m+1})$ as the map given by $A_{m+1}(e_i) = A_m(e_i)$ for $i \leq m$ and $A_{m+1}(e_{m+1}) = (\det(A_m))^{-1}(e_{m+1})$, and let A' be the image of A_{m+1} in $\mathrm{SL}(E_\infty)$. Then the action of A_m and A_{m+1} agree on $(E_m)^*$. Hence, they also agree on $\mathrm{Sym}(\bigwedge(E_m)^*)$. So $A \cdot f = A' \cdot f \in I$ since I was assumed to be $\mathrm{SL}(E_\infty)$-stable and $A' \in \mathrm{SL}(E_\infty)$. As $f \in I$ and $A \in \mathrm{GL}(E_\infty)$ were arbitrary, this shows that I is $GL(E_\infty)$-stable. □

Proof of Proposition 4.2. First, we claim that $\mathrm{Spec}\left(\mathrm{Sym}\left(\bigoplus_{k=0}^{\ell} \bigwedge^k (E_\infty)_*\right)\right)$ is topologically $\mathrm{GL}(E_\infty)_n$-Noetherian. Indeed, the standard $\mathrm{GL}(E_\infty)$-representation of the space $\bigoplus_{k=0}^{\ell} \bigwedge^k (E_\infty)_*$ is an algebraic representation, and this also remains true when we act with $\mathrm{GL}(E_\infty)$ via its isomorphism into $\mathrm{GL}(E_\infty)_n$. Hence, the claim follows from [7, Theorem 2]. Let $(X_i)_{i \in \mathbb{N}} \subseteq \mathrm{Spec}(R_{\leq \ell})$ be a descending chain of H_n-stable, closed, reduced subschemes. By Lemma 4.3, every X_i is also $\mathrm{GL}(E_\infty)_n$-stable. By Lemma 4.5, there is an $\mathrm{SL}(E_\infty)_n$-equivariant isomorphism $\mathrm{Spec}(R_{\leq \ell}) \cong \mathrm{Spec}\left(\mathrm{Sym}\left(\bigoplus_{k=0}^{\ell} \bigwedge^k (E_\infty)_*\right)\right)$. Let $X_i' \subseteq \mathrm{Spec}\left(\mathrm{Sym}\left(\bigoplus_{k=0}^{\ell} \bigwedge^k (E_\infty)_*\right)\right)$ be the closed, reduced, $\mathrm{SL}(E_\infty)$-stable subscheme corresponding to X_i under this isomorphism. Using Lemma 4.6, we see that the subschemes X_i' are also $\mathrm{GL}(E_\infty)_n$-stable. Therefore, the chain $(X_i')_{i \in \mathbb{N}}$ stabilizes by our first claim. Consequently, also the chain $(X_i)_{i \in \mathbb{N}}$ stabilizes. □

Before we come to the proof of Theorem 4.1, let us recall the action of $f_i \wedge f_j \in \mathfrak{so}(V_\infty)$ on $\bigwedge_\infty^+ E_\infty$ and its symmetric algebra R^+ in explicit terms. Recall from Section 3.6 that a basis for $\bigwedge_\infty^+ E_\infty$ is given by $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots$, where $I = \{i_1 < i_2 < \cdots\} \subseteq \mathbb{N}$ is cofinite and $|\mathbb{N} \setminus I|$ even. Then we have

$$(f_i \wedge f_j) e_I = \begin{cases} (-1)^{c_{i,j}(I)} e_{I \setminus \{i,j\}} & \text{if } i, j \in I, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where $c_{i,j}(I)$ depends on the position of i, j in I. (Note that there is no factor 4, since in our identification of $\bigwedge^2 V$ to the Lie subalgebra L of $\mathrm{Cl}(V)$ we had a factor $\frac{1}{4}$.) The corresponding action of $f_i \wedge f_j$ on polynomials in R^+ is as a derivation.

4.3. *Proof of Theorem 4.1*

Let $R^+ \subseteq R$ be the symmetric algebra on the direct half-spin representation $\bigwedge_\infty^+ E_\infty$, so that $\mathrm{Spec}(R^+)$ is the inverse half-spin representation $(\bigwedge_\infty^+ E_\infty)^*$. We prove topological $\mathrm{Spin}(V_\infty)$-Noetherianity of $\mathrm{Spec}(R^+)$; the corresponding statement for $\mathrm{Spec}(R^-)$ is proved in exactly the same manner.

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For a closed, reduced $\mathrm{Spin}(V_\infty)$-stable subscheme X of $\mathrm{Spec}(R^+)$, we denote by $\delta_X \in \{0, 1, 2, \dots, \infty\}$ the lowest degree of a nonzero polynomial in the ideal $I(X) \subseteq R^+$ of X. Here, we consider the natural grading on $R^+ = \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$, where the elements of $\bigwedge_\infty^+ E_\infty$ all have degree 1.

We proceed by induction on δ_X to show that X is topologically Noetherian; we may, therefore, assume that this is true for all Y with $\delta_Y < \delta_X$. We have $\delta_X = \infty$ if and only if $X = \mathrm{Spec}(R^+)$. Then a chain

$$\mathrm{Spec}(R^+) = X \supseteq X_1 \supseteq X_2 \supseteq \dots$$

of $\mathrm{Spin}(V_\infty)$-closed subsets is either constant or else there exists an i with $\delta_{X_i} < \infty$. Hence, it suffices to prove that X is Noetherian under the additional assumption that $\delta_X < \infty$. At the other extreme, if $\delta_X = 0$, then X is empty and there is nothing to prove. So we assume that $0 < \delta_X < \infty$ and that all Y with $\delta_Y < \delta_X$ are $\mathrm{Spin}(V_\infty)$-Noetherian.

Let $p \in R^+$ be a nonzero polynomial in the ideal of X of degree δ_X. By Remark 4.4, since X is a cone, p is in fact homogeneous of degree δ_X. Let e_I be a variable appearing in p such that $k := |I^c|$ is maximal among all variables in p; note that k is even. Then choose $n \geq k+2$ even such that all variables of p are contained in $\bigwedge^+ E_n$ (i.e., they are of the form e_J with $J \supseteq \{n+1, n+2, \dots\}$).

Now act on p with the element $f_{i_1} \wedge f_{i_2} \in \mathfrak{so}(V_\infty)$ with $i_1 < i_2$ the two smallest elements in I. Since X is $\mathrm{Spin}(V_\infty)$-stable, the result p_1 is again in the ideal of X. Furthermore, p_1 has the form

$$p_1 = \pm e_{I \setminus \{i_1, i_2\}} \cdot q + r_1,$$

where $q = \frac{\partial p}{\partial e_I}$ contains only variables e_J with $|J^c| \leq k$ and where r_1 does not contain $e_{I\setminus\{i_1,i_2\}}$ but may contain other variables e_J with $|J^c| = k+2$ (namely, those with $i_1, i_2 \notin J$ for which $e_{J\cup\{i_1,i_2\}}$ appears in p).

If $n = k+2$, then $I \setminus \{i_1, i_2\} = \{n+1, n+2, \dots\}$, and since all variables e_J in p_1 satisfy $J \supseteq \{n+1, n+2, \dots\}$, $e_{I\setminus\{i_1,i_2\}}$ is the only variable e_J in p_1 with $|J^c| = k+2$. If $n > k+2$, then we continue in the same manner, now acting with $f_{i_3} \wedge f_{i_4}$ on p_1, where $i_3 < i_4$ are the two smallest elements in $I \setminus \{i_1, i_2\}$. We write p_2 for the result, which is now of the form

$$p_2 = \pm e_{I\setminus\{i_1,i_2,i_3,i_4\}} \cdot q + r_2,$$

where q is the same polynomial as before and r_2 does not contain the variable $e_{I\setminus\{i_1,i_2,i_3,i_4\}}$ but may contain other variables e_J with $|J^c| = k+4$.

Iterating this construction, we find the polynomial

$$p_\ell = \pm e_{\{n+1,n+2,\dots\}} \cdot q + r_\ell$$

in the ideal of X, where $\ell = (n-k)/2$, q is the same polynomial as before and r_ℓ only contains variables e_J with $|J^c| < n$. Let $Z := X[1/q]$ be the open subset of X where q is nonzero.

Lemma 4.7. *For every variable e_J with $|J^c| \geq n$, the ideal of Z in the localisation $R^+[1/q]$ contains a polynomial of the form $e_J - s/q^d$ for some $d \in \mathbb{Z}_{\geq 0}$ and some $s \in R^+_{\leq n-2}$.*

Proof. We proceed by induction on $|J^c| =: m$. By successively acting on p_ℓ with the elements $f_n \wedge f_{n+1}, f_{n+2} \wedge f_{n+3}, \dots, f_{m-1} \wedge f_m$, we find the polynomial

$$\pm e_{\{m+1,m+2,\dots\}} \cdot q + r$$

in the ideal of X, where r contains only variables e_L with $|L^c| < m$. Now act with elements of $\mathfrak{gl}(E_\infty)$ to obtain an element

$$\pm e_J \cdot q + \tilde{r},$$

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where $\tilde{r}$ still contains only variables e_L with $|L^c| < m$. Inverting q, this can be used to express e_J in such variables e_L. By the induction hypothesis, all those e_L admit an expression, on Z, as a polynomial in $R^+_{\leq n-2}$ times a negative power of q. Then the same holds for e_J. □

Lemma 4.8. *The open subscheme $Z = X[1/q]$ is stable under the group H_n and H_n-Noetherian.*

Proof. By Lemma 4.3, X is stable under $\mathrm{GL}(E_\infty)_n$. The polynomial q is homogeneous and contains only variables e_J with $J \supseteq \{n+1, n+2, \ldots\}$. Every $g \in \mathrm{GL}(E_\infty)_n$ scales each such variable with $\det(g)$, and hence, maps q to a scalar multiple of itself. We conclude that Z is stable under $\mathrm{GL}(E_\infty)_n$, and hence by (a slight variant of) Lemma 4.3 also under H_n.

By Lemma 4.7, the projection dual to the inclusion $R^+_{\leq n-2}[1/q] \subseteq R^+[1/q]$ restricts on Z to a closed embedding, and this embedding is H_n-equivariant. By Proposition 4.2, the image of Z is H_n-Noetherian, and hence, so is Z itself. □

Proof of Theorem 4.1. Let

$$X \supseteq X_1 \supseteq \ldots$$

be a chain of reduced, $\mathrm{Spin}(V_\infty)$-stable closed subschemes. Let $Y \subseteq X$ be the reduced closed subscheme defined by the orbit $\mathrm{Spin}(V_\infty) \cdot q$. Since q has degree $\delta_X - 1$, we have $\delta_Y < \delta_X$, and hence, Y is $\mathrm{Spin}(V_\infty)$-Noetherian by the induction hypothesis. It follows that the chain

$$Y \supseteq (Y \cap X_1)^{\mathrm{red}} \supseteq \ldots$$

is eventually stable. However, the chain

$$Z \supseteq (Z \cap X_1)^{\mathrm{red}} \supseteq \ldots$$

consists of reduced, H_n-stable closed subschemes of Z; hence, it is eventually stable by Lemma 4.8.

Now pick a (not necessarily closed) point $P \in X_i$ for $i \gg 0$. If $P \in Y \cap X_i$, then $P \in Y \cap X_{i-1}$ by the first stabilisation. However, if $P \notin Y \cap X_i$, then there exists a $g \in \mathrm{Spin}(V_\infty)$ such that $gP \in Z$. Then gP lies in $X_i \cap Z$, which by the second stabilisation equals $X_{i-1} \cap Z$; hence, $P = g^{-1}(gP)$ lies in X_{i-1}, as well. We conclude that the chain $(X_i)_i$ of closed, reduced subschemes of X stabilises. Hence, the inverse half-spin representation $(\bigwedge^+_\infty E_\infty)^*$ is topologically $\mathrm{Spin}(V_\infty)$-Noetherian. □

Remark 4.9. While the proof of Theorem 4.1 for the even half-spin case is easily adapted to a proof for the odd half-spin case, we *do not know whether the spin representation $(\bigwedge_\infty E_\infty)^*$ itself is topologically* $\mathrm{Spin}(V_\infty)$*-Noetherian!* Also, despite much effort, we have not succeeded in proving that the inverse limit $\varprojlim_n \bigwedge^n V_n$ along the contraction maps c_{e_n} is topologically $\mathrm{SO}(V_\infty)$-Noetherian. Indeed, the situation is worse for this question: like the inverse spin representation, this limit is the dual of a countable-dimensional module that splits as a direct sum of two $\mathrm{SO}(V_\infty)$-modules – and here, we do not even know whether the dual of *one* of these modules is topologically Noetherian!

5. Half-spin varieties and applications

In this section, we introduce the notion of half-spin varieties and reformulate our main result Theorem 4.1 in this language. We start by fixing the necessary data determining the half-spin representations of $\mathrm{Spin}(V)$.

Notation 5.1. *As shorthand, we write $\mathcal{V} = (V, q, F) \in \mathcal{Q}$ to refer to a triple where*

1. *V is an even-dimensional vector space over K,*
2. *q is a nondegenerate symmetric quadratic form on V, and*
3. *F is a maximal isotropic subspace of V.*

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An isomorphism $\mathcal{V} \to \mathcal{V}' = (V', q', F')$ of such triples is a linear bijection $\varphi : V \to V'$ with $q'(\varphi(v)) = q(v)$ and $\varphi(F) = F'$.

Given a triple $\mathcal{V}$, we have half-spin representations $\mathrm{Cl}^{\pm}(V)f$, where $f = f_1 \cdots f_n$ with $f_1, \dots, f_n$ a basis of F (recall that the left ideal $\mathrm{Cl}(V)f$ does not depend on this basis). Half-spin varieties are $\mathrm{Spin}(V)$-invariant subvarieties of these half-spin representations that are preserved by the contraction maps π_e from §3.2 and the multiplication maps τ_h from §3.3. The precise definition below is inspired by that of a Plücker variety in [6]. It involves a uniform choice of either even or odd half-spin representations. For convenience of notation, we will only explicitly work with the even half-spin representations, but all further results are valid for the odd counterparts as well.

Definition 5.2 (Half-spin variety). A *half-spin variety* is a rule X that assigns to each triple $\mathcal{V} = (V, q, F) \in \mathcal{Q}$ a closed, reduced subscheme $X(\mathcal{V}) \subseteq \mathrm{Cl}^{+}(V)f$ such that

1. $X(\mathcal{V})$ is $\mathrm{Spin}(V)$-stable;
2. for any isomorphism $\varphi : \mathcal{V} \to \mathcal{V}'$, the map $\mathrm{Cl}^{+}(\varphi)$ maps $X(\mathcal{V})$ into $X(\mathcal{V}')$;
3. for any isotropic $e \in V$ with $e \notin F$, if we set $V' := e^{\perp}/\langle e \rangle$, q' the induced form on V', F' the image of $F \cap e^{\perp}$ in V', and $\mathcal{V}' := (V', q', F')$, then the contraction map $\pi_e : \mathrm{Cl}^{+}(V)f \to \mathrm{Cl}^{+}(V')f'$ maps $X(\mathcal{V})$ into $X(\mathcal{V}')$; and
4. for any $\mathcal{V} = (V, q, F)$, if we construct a triple $\mathcal{V}'$ by setting $V' := V \oplus \langle e, h \rangle$, q' the quadratic form that restricts to q on V, makes the direct sum orthogonal, and makes e, h a hyperbolic basis, if we set $f' := f \cdot h$, then the map $\tau_h : \mathrm{Cl}^{+}(V)f \to \mathrm{Cl}^{+}(V')f'$ maps $X(\mathcal{V})$ into $X(\mathcal{V}')$.

Example 5.3. The following are examples of half-spin varieties.

1. Trivially, $X(\mathcal{V}) := \mathrm{Cl}^{+}(V)f$, $X(\mathcal{V}) := \{0\}$ and $X(\mathcal{V}) := \emptyset$ define half-spin varieties.
2. The even component of the cone over the isotropic Grassmannian, $X(\mathcal{V}) := \widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V, q)$ is a half-spin variety by Proposition 3.4.
3. For two half-spin varieties X and X', their *join* $X + X'$ defined by
$$(X + X')(\mathcal{V}) := \overline{\{x + x' \mid x \in X(\mathcal{V}), x' \in X'(\mathcal{V})\}}$$
is a half-spin variety. In particular, secant varieties to half-spin varieties are again half-spin varieties.
4. The intersection of two half-spin varieties X and X' is a half-spin variety, which is defined by $(X \cap X')(\mathcal{V}) := X(\mathcal{V}) \cap X'(\mathcal{V})$.

Similar as in §3.6, we will use the following notation: for every $n \in \mathbb{N}$, we consider the vector space $V_n = \langle e_1, \dots, e_n, f_1, \dots, f_n \rangle$ together with the quadratic form q_n whose corresponding bilinear form $(\cdot|\cdot)$ satisfies

$$(e_i|e_j) = 0, \quad (f_i|f_j) = 0 \quad \text{and} \quad (e_i|f_j) = \delta_{ij}.$$

Furthermore, let $E_n = \langle e_1, \dots, e_n \rangle$ and $F_n = \langle f_1, \dots, f_n \rangle$; these are maximal isotropic subspaces of V_n. We will denote the associated tuple by $\mathcal{V}_n = (V_n, q_n, F_n)$.

Remark 5.4. A half-spin variety X is completely determined by the values $X(\mathcal{V}_n)$; that is, if X and X' are half-spin varieties such that $X(\mathcal{V}_n) = X'(\mathcal{V}_n)$ for all $n \in \mathbb{N}$, then $X(\mathcal{V}) = X'(\mathcal{V})$ for all tuples $\mathcal{V}$.

We now want to associate to each half-spin variety X an infinite-dimensional scheme X_∞ embedded inside the inverse half-spin representation $(\bigwedge^{+}_{\infty} E_\infty)^*$ as follows. Since $V_n = E_n \oplus F_n$, we can use the isomorphism from §2.5 to embed $X(\mathcal{V}_n)$ as a reduced subscheme of $\bigwedge^{+} E_n$ (recall from §3.7 that we view $\bigwedge^{+} E_n$ as the affine scheme with coordinate ring $\mathrm{Sym}(\bigwedge^{+(-1)^n} E_n)$). We abbreviate $X_n := X(\mathcal{V}_n) \subseteq \bigwedge^{+} E_n$.

For $N \geq n$, let $\pi_{N,n} : \bigwedge^{+} E_N \to \bigwedge^{+} E_n$, resp. $\tau_{n,N} : \bigwedge^{+} E_n \to \bigwedge^{+} E_N$ be the maps induced by canonical projection $E_N \to E_n$, resp. by the injection $E_n \hookrightarrow E_N$. Note that $\tau_{n,N}$ is a section of $\pi_{N,n}$.

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Recall that $(\bigwedge_\infty^+ E_\infty)^* = \varprojlim_n \bigwedge^+ E_n$. We denote the structure maps by $\pi_{\infty,n} : (\bigwedge_\infty^+ E_\infty)^* \to \bigwedge^+ E_n$ and by $\tau_{n,\infty} : \bigwedge^+ E_n \to (\bigwedge_\infty^+ E_\infty)^*$ the inclusion maps induced by $\tau_{n,N}$.

From the definition of a half-spin variety, it follows that

$$\pi_{N,n}(X_N) \subseteq X_n \quad \text{and} \quad \tau_{n,N}(X_n) \subseteq X_N. \tag{5.1}$$

Hence, the inverse limit

$$X_\infty := \varprojlim_n X_n$$

is well defined, and a closed, reduced, $\mathrm{Spin}(V_\infty)$-stable subscheme of $(\bigwedge_\infty^+ E_\infty)^*$. In order to see this, write $R_n := \mathrm{Sym}(\bigwedge^{+(-1)^n} E_n)$ and $R_\infty := \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$. Let $I_n \subseteq R_n$ be the radical ideal associated to $X_n \subseteq \mathrm{Spec}(R_n)$ (i.e., $X_n = V(I_n) = \mathrm{Spec}(R_n/I_n)$). As $\mathrm{Spec}(\cdot)$ is a contravariant equivalence of categories, it holds that

$$X_\infty := \varprojlim_n X_n = \varprojlim_n \mathrm{Spec}(R_n/I_n) = \mathrm{Spec}(\varinjlim_n (R_n/I_n)).$$

So X_∞ corresponds to the ideal $I_\infty := \varinjlim_n I_n \subseteq R_\infty$. As all $I_n \subseteq R_n$ are radical, so is $I_\infty \subseteq R_\infty$, and therefore, X_∞ is a reduced subscheme.

It follows from Equation (5.1) that

$$\pi_{\infty,n}(X_\infty) \subseteq X_n \quad \text{and} \quad \tau_{n,\infty}(X_n) \subseteq X_\infty. \tag{5.2}$$

Lemma 5.5. *The mapping*

$$X \mapsto X_\infty$$

is injective. That is, if X and X' are half-spin varieties such that $X_\infty = X'_\infty$, then $X = X'$ (i.e., $X(\mathcal{V}) = X'(\mathcal{V})$ for all tuples $\mathcal{V}$).

Proof. Note that, for all $n \in \mathbb{N}$, it holds that

$$X_n = \pi_{\infty,n}(X_\infty).$$

Indeed, the inclusion $\supseteq$ is contained in Equation (5.2), and the other direction $\subseteq$ follows from the fact that $\tau_{n,\infty} : X_n \to X_\infty$ is a section of $\pi_{\infty,n}$. Hence, if $X_\infty = X'_\infty$, then

$$X_n = \pi_{\infty,n}(X_\infty) = \pi_{\infty,n}(X'_\infty) = X'_n.$$

By Remark 5.4, this shows that $X = X'$. □

For two half-spin varieties X and X', we will write $X \subseteq X'$ if $X(\mathcal{V}) \subseteq X'(\mathcal{V})$ for all $\mathcal{V} = (V, q, F)$. Theorem 4.1 then implies the following.

Theorem 5.6 (Noetherianity of half-spin varieties). *Every descending chain of half-spin varieties*

$$X^{(0)} \supseteq X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \supseteq \dots$$

stabilizes; that is, there exists $m_0 \in \mathbb{N}$ such that $X^{(m)} = X^{(m_0)}$ for all $m \geq m_0$.

Proof. Note that the mapping $X \mapsto X_\infty$ is order preserving; that is, if $X \subseteq X'$, then $X_\infty \subseteq X'_\infty$. Hence, a chain

$$X^{(0)} \supseteq X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \supseteq \dots$$

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of half-spin varieties induces a chain

$$X_\infty^{(0)} \supseteq X_\infty^{(1)} \supseteq X_\infty^{(2)} \supseteq X_\infty^{(3)} \supseteq \dots$$

of closed, reduced, $\mathrm{Spin}(V_\infty)$-stable subschemes in $(\bigwedge_\infty^+ E_\infty)^*$. By Theorem 4.1, we know that $(\bigwedge_\infty^+ E_\infty)^*$ is topologically $\mathrm{Spin}(V_\infty)$-Noetherian. Hence, the chain $X_\infty^{(m)}$ stabilizes. But then, by Lemma 5.5, also the chain of half-spin varieties $X^{(m)}$ stabilizes. This completes the proof. □

As a consequence, we obtain the next results, which state how X_∞ is determined by the data coming from some finite level of X.

Theorem 5.7. *Let X be a half-spin variety. Then there exists $n_0 \in \mathbb{N}$ such that*

$$X_\infty = V\big(\mathrm{rad}(\mathrm{Spin}(V_\infty) \cdot I_{n_0})\big),$$

where $\mathrm{rad}(\mathrm{Spin}(V_\infty) \cdot I_{n_0}) \subseteq \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$ is the radical ideal generated by the $\mathrm{Spin}(V_\infty)$-orbits of the ideal $I_{n_0} \subseteq \mathrm{Sym}(\bigwedge^{+(-1)^{n_0}} E_{n_0})$ defining $X_{n_0} \subseteq \bigwedge^+ E_{n_0}$.

Proof. For each $n \in \mathbb{N}$, set $J_n := \mathrm{rad}(\mathrm{Spin}(V_\infty) \cdot I_n) \subseteq \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$. We denote by $I_\infty \subseteq \mathrm{Sym}(\bigwedge_\infty^+ E_\infty)$ the ideal associated to X_∞. This ideal is $\mathrm{Spin}(V_\infty)$-stable, radical and it holds that $I_\infty = \varinjlim_n I_n$. Thus, $\bigcup_n J_n = I_\infty$. Since $(J_n)_{n\in\mathbb{N}}$ is an increasing chain of closed $\mathrm{Spin}(V_\infty)$-stable radical ideals, by Theorem 4.1, there exists $n_0 \in \mathbb{N}$ such that $J_n = J_{n_0}$ for all $n \geq n_0$. Therefore, $I_\infty = \bigcup_n J_n = J_{n_0}$ and hence, $X_\infty = V(I_\infty) = V(J_{n_0})$. □

Corollary 5.8. *Let X be a half-spin variety. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, it holds that*

$$X_n = V(\mathrm{rad}(\mathrm{Spin}(V_n) \cdot I_{n_0})).$$

Proof. Take n_0 as in Theorem 5.7. Then the statement follows from that theorem and [4, Lemma 2.1]. To apply that lemma, we must check condition (*) in that paper – namely, that for $q \geq n \geq n_0$ and $g \in \mathrm{Spin}(V_q)$, we can write

$$\pi_{q,n_0} \circ g \circ \tau_{n,q} = g'' \circ \tau_{m,n_0} \circ \pi_{n,m} \circ g'$$

for suitable $m \leq n_0$ and $g' \in \mathrm{Spin}(V_n)$ and $g'' \in \mathrm{Spin}(V_{n_0})$. In fact, since half-spin varieties are affine cones, it suffices that this identity holds up to a scalar factor. It also suffices to prove this for g in an open dense subset U of $\mathrm{Spin}(V_q)$, because the equations for X_{n_0} pulled back along the map on the left for $g \in U$ imply the equations for all g. We will prove this, with $m = n_0$, using the Cartan map in Lemma 6.6 below. □

6. Universality of $\widehat{\mathrm{Gr}}_{\mathrm{iso}}^+(4, 8)$ and the Cartan map

6.1. Statement

In [16], the last two authors showed that in even dimension, the isotropic Grassmannian in its Plücker embedding is set-theoretically defined by pulling back equations coming from $\widehat{\mathrm{Gr}}_{\mathrm{iso}}(4, 8)$. Using the *Cartan map*, we can translate this into a statement about the isotropic Grassmannian in its spinor embedding and prove the following result.

Theorem 6.1. *For all $n \geq 4$, we have*

$$\widehat{\mathrm{Gr}}_{\mathrm{iso}}^+(V_n) = V(\mathrm{rad}(\mathrm{Spin}(V_n) \cdot I_4)),$$

where I_4 is the ideal of polynomials defining $\widehat{\mathrm{Gr}}_{\mathrm{iso}}^+(V_4) \subseteq \mathrm{Cl}^+(V_4)f$.

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In other words, the bound n_0 from Corollary 5.8 can be taken equal to 4 for the cone over the isotropic Grassmannian. We give the proof of Theorem 6.1 in §6.5 using properties of the Cartan map that will be established in the following sections.

6.2. Definition of the Cartan map

When we regard $e_1 \wedge \cdots \wedge e_n$ as an element of the n-th exterior power $\bigwedge^n V$ of the standard representation V of $\mathfrak{so}(V)$, then it is a highest weight vector of weight $(0, \dots, 0, 2) = 2\lambda_0$, where λ_0 is the fundamental weight introduced in §2.6 and the highest weight of the half-spin representation $\mathrm{Cl}^{(-1)^n}(V)f$. Similarly, the element $e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1} \wedge f_n \in \bigwedge^n V$ is a highest weight vector of weight $(0, \dots, 0, 2, 0) = 2\lambda_1$, where λ_1 is the highest weight of the other half-spin representation. So $\bigwedge^n V$ contains copies of the irreducible representations $V_{2\lambda_0}, V_{2\lambda_1}$ of $\mathfrak{so}(V)$; in fact, it is well known to be the direct sum of these. To compare our results in this paper about spin representations with earlier work by the last two authors about exterior powers, we will need the following considerations.

Consider any connected, reductive algebraic group G, with maximal torus T and Borel subgroup $B \supseteq T$. Let λ be a dominant weight of G, let V_λ be the corresponding irreducible representation, and let $v_\lambda \in V_\lambda$ be a nonzero highest-weight vector (which is unique up to scalar multiples). Then the symmetric square $S^2 V_\lambda$ contains a one-dimensional space of vectors of weight 2λ, spanned by $v_{2\lambda} := v_\lambda^2$. This vector is itself a highest-weight vector, and hence generates a copy of $V_{2\lambda}$; this is sometimes called the *Cartan component* of $S^2 V_\lambda$. By semisimplicity, there is a G-equivariant linear projection $\pi : S^2 V_\lambda \to V_{2\lambda}$ that restricts to the identity on $V_{2\lambda}$. The map

$$\widehat{\nu_2} : V_\lambda \to V_{2\lambda}, \quad v \mapsto \pi(v^2)$$

is a nonzero polynomial map, homogeneous of degree 2, and hence induces a rational map $\nu_2 : \mathbb{P}V_\lambda \to \mathbb{P}V_{2\lambda}$. Note that this is the composition of the quadratic Veronese embedding and the projection π. We will refer to ν_2 and to $\widehat{\nu_2}$ as the *Cartan map*.

Lemma 6.2. *The rational map ν_2 is a morphism and injective.*

We thank J. M. Landsberg for help with the following proof.

Proof. To show that ν_2 is a morphism, we need to show that $\pi(v^2)$ is nonzero whenever v is. Now the set Q of all $[v] \in \mathbb{P}V_\lambda$ for which $\pi(v^2)$ is zero is closed and B-stable. Hence, if $Q \neq \emptyset$, then by Borel's fixed point theorem, Q contains a B-fixed point. But the only B-fixed point in $\mathbb{P}V_\lambda$ is $[v_\lambda]$, and v_λ is mapped to the nonzero vector $v_{2\lambda}$. Hence, $Q = \emptyset$.

Injectivity is similar but slightly more subtle. Assume that there exist distinct $[v], [w]$ with $\nu_2([v]) = \nu_2([w])$. Then $\{[v], [w]\}$ represents a point in the Hilbert scheme H of two points in $\mathbb{P}V_\lambda$. Now the locus Q of points S in H such that $\nu_2(S)$ is contained in a single reduced point is closed in H, as it is the projection to H of the closed subvariety

$$\{(S, [u]) \mid \nu_2(S) \subseteq \{[u]\}\} \subseteq H \times \mathbb{P}V_{2\lambda}$$

and since $\mathbb{P}V_{2\lambda}$ is projective. Since H is a projective scheme, Q is projective. Hence, Q contains a B-fixed point S_0. This scheme S_0 cannot consist of two distinct reduced points: if it did, then either both points would be B-fixed, but there is only one B-fixed point, or else they would be a B-orbit, but this is impossible since B is connected. Therefore, the reduced subscheme of S_0 is $\{[v_\lambda]\}$, and S_0 represents the point $[v_\lambda]$ together with a nonzero tangent direction in $T_{[v_\lambda]}\mathbb{P}V_\lambda = V_\lambda / K v_\lambda$, represented by $w \in V_\lambda$. Furthermore, B-stability of S_0 implies that the B-module generated by w equals $\langle w, v_\lambda \rangle_K$. That S_0 lies in Q means that

$$\pi((v_\lambda + \epsilon w)^2) = v_{2\lambda} \mod \epsilon^2.$$

We find that $\pi(v_\lambda w) = 0$, so that the G-module generated by $v_\lambda w \in S^2V$ does not contain $V_{2\lambda}$. But since v_λ is (up to a scalar) fixed by B, the B-module generated by $v_\lambda w$ equals v_λ times the B-module generated by w, and hence contains $v_\lambda^2 = v_{2\lambda}$, a contradiction. □

Observe that ν_2 maps the unique closed orbit $G \cdot [v_\lambda]$ in $\mathbb{P}V_\lambda$ isomorphically to the unique closed orbit $G \cdot [v_{2\lambda}]$ – both are isomorphic to G/P, where $P \supseteq B$ is the stabiliser of the line Kv_λ and of the line $Kv_{2\lambda}$. In our setting above, where $G = \mathrm{Spin}(V)$ and $\lambda \in \{\lambda_0, \lambda_1\}$, the closed orbit $G \cdot [v_{2\lambda}]$ is one of the two connected components of the Grassmannian $\mathrm{Gr}_{\mathrm{iso}}(V)$ of n-dimensional isotropic subspaces of V, in its Plücker embedding; and the closed orbit in the projectivised half-spin representation $\mathbb{P}V_\lambda$ is the same component of the isotropic Grassmannian but now in its spinor embedding.

In what follows, we will need a more explicit understanding both of the embedding of the isotropic Grassmannian in the projectivised (half-)spin representations and of the map $\widehat{\nu_2}$. These are treated in the next two paragraphs.

6.3. *The map $\widehat{\nu_2}$ from the spin representation to the exterior power*

In §6.2, we argued the existence of $\mathrm{Spin}(V)$-equivariant quadratic maps from the half-spin representations to the two summands of $\bigwedge^n V$. In [11], these two maps are described jointly as

$$\widehat{\nu_2} : \mathrm{Cl}(V)f \to \bigwedge^n V, \quad af \mapsto \text{ the component in } \bigwedge^n V \text{ of } (afa^*) \bullet 1 \in \bigwedge V,$$

where $\bullet$ stands for the $\mathrm{Cl}(V)$-module structure of $\bigwedge V$ from §2.2 and a^* refers to the anti-automorphism of the Clifford algebra from §2.1.

Lemma 6.3. *The map $\hat{\nu}_2$ maps the isotropic Grassmann cone in its spinor embedding to the isotropic Grassmann cone in its Plücker embedding, that is,*

$$\hat{\nu}_2\big(\widehat{\mathrm{Gr}}_{\mathrm{iso}}(V)\big) = \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V),$$

where $\widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V)$ is the isotropic Grassmann cone in its Plücker embedding (see [16, Definition 3.7]).

Proof. Let $H \subseteq V$ be a maximal isotropic subspace that intersects F in a k-dimensional space. Choose a hyperbolic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ adapted to H and F, so that $H = \langle e_{k+1}, \ldots, e_n, f_1, \ldots, f_k\rangle$ is represented by the vector $\omega_H := e_{k+1} \cdots e_n f \in \widehat{\mathrm{Gr}}_{\mathrm{iso}}(V)$ where $f = f_1 \cdots f_n$; see §3.1. Set $a := e_{k+1} \cdots e_n$. Now

$$\begin{aligned} afa^* &= e_{k+1} \cdots e_n f_1 \cdots f_n e_n \cdots e_{k+1} \\ &= e_{k+1} \cdots e_n f_1 \cdots f_{n-1}(2 - e_n f_n) e_{n-1} \cdots e_{k+1} \\ &= 2e_{k+1} \cdots e_n f_1 \cdots f_{n-1} e_{n-1} \cdots e_{k+1} \\ &= \ldots \\ &= 2^{n-k} e_{k+1} \cdots e_n f_1 \cdots f_k, \end{aligned}$$

where we have used the definition of $\mathrm{Cl}(V)$ (in the first step), the fact that the second copy of e_n is perpendicular to all elements before it and multiplies to zero with the first copy of e_n (in the second step), and have repeated this another $n - k - 1$ times in the last step. We now find that

$$(afa^*) \bullet 1 = 2^{n-k} e_{k+1} \wedge \cdots \wedge e_n \wedge f_1 \wedge \cdots \wedge f_k,$$

so that $(afa^*) \bullet 1$ lies in one of the two summands of $\bigwedge^n V$ and spans the line representing the space H in the Plücker embedding. This shows that $\hat{\nu}_2$ maps the isotropic Grassmann cone in its spinor embedding to the isotropic Grassmann cone in its Plücker embedding, as desired. □

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Remark 6.4. While the spin representation $\mathrm{Cl}(V)f$ depends only on the space F – since F determines f up to a scalar, which does not affect the left ideal $\mathrm{Cl}(V)f$ – the map $\widehat{\nu_2}$ actually depends on f itself: for $\tilde{f} := tf$ with $t \in K^*$, the map $\widehat{\nu_2}$ constructed from $\tilde{f}$ sends $af = (t^{-1}a)\tilde{f}$ to $t^{-1}a\tilde{f}t^{-1}a^* = t^{-1}afa^*$, so the new $\widehat{\nu_2}$ is t^{-1} times the old map.

6.4. Contraction and the Cartan map commute

Recall from §6.2 that we have quadratic maps $\widehat{\nu_2}$ from the half-spin representations to the two summands of $\bigwedge^n V$; together, these form a quadratic map $\widehat{\nu_2}$ which we discussed in §6.3. By abuse of terminology, we call this, too, the Cartan map. Given an isotropic vector $e \in V$ that is not in F, we write $\widehat{\nu_2}$ also for the Cartan map $\mathrm{Cl}(V_e)\overline{f} \to \bigwedge^{n-1} V_e$ (notation as in §3.2). Recall from §3.2 the contraction map $c_e : \bigwedge^n V \to \bigwedge^{n-1} V_e$ and its spin analogue $\pi_e : \mathrm{Cl}(V)f \to \mathrm{Cl}(V_e)\overline{f}$. Also, for a fixed $h = f_n \in F$ with $(e|h) = 1$, recall from §3.3 the multiplication map $m_h : \bigwedge^{n-1} V_e \to \bigwedge^n V$ and its spin analogue $\tau_h : \mathrm{Cl}(V_e)\overline{f} \to \mathrm{Cl}(V)f$. The relations between these maps are as follows.

Proposition 6.5. *The following diagrams essentially commute:*

$$\begin{tikzcd}
\mathrm{Cl}(V)f \arrow[r, "\pi_e"] \arrow[d, "\widehat{\nu_2}"'] & \mathrm{Cl}(V_e)\overline{f} \arrow[d, "\widehat{\nu_2}"] \\
\bigwedge^n V \arrow[r, "c_e"'] & \bigwedge^{n-1} V_e
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathrm{Cl}(V_e)\overline{f} \arrow[r, "\tau_h"] \arrow[d, "\widehat{\nu_2}"'] & \mathrm{Cl}(V)f \arrow[d, "\widehat{\nu_2}"] \\
\bigwedge^{n-1} V_e \arrow[r, "m_h"'] & \bigwedge^n V.
\end{tikzcd} \tag{6.1}$$

More precisely, one can rescale the restrictions of c_e to the two $\mathfrak{so}(V)$-submodules of $\bigwedge^n V$ each by ± 1 in such a manner that the diagram commutes, and similarly for m_h.

Naturally, we could have chosen the scalars in the definition of c_e (or, using a square root of -1, in that of π_e) such that the diagram literally commutes. However, we have chosen the scalars such that c_e has the most natural formula and π_e, τ_h have the most natural formulas in our model $\bigwedge E$ for the spin representation.

Proof. We may choose a hyperbolic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that $e = e_n$ and $f_1, \ldots, f_n$ is a basis of F. We write $f := f_1 \cdots f_n$ and $\overline{f} := \overline{f}_1 \cdots \overline{f}_{n-1}$.

Since the vertical maps are quadratic, it is not sufficient to show commutativity on a spanning set. We therefore consider

$$a := \sum_{I \subseteq [n]} c_I e_I,$$

where, for $I = \{i_1 < \ldots < i_k\}$, we write $e_I := e_{i_1} \cdots e_{i_k}$. We then have

$$\pi_e(af) = \sum_{I : n \notin I} c_I \overline{e_I}\,\overline{f} =: \overline{a}\,\overline{f}$$

and

$$\widehat{\nu_2}(\overline{a}\,\overline{f}) = \text{the component in } \bigwedge^{n-1} V_e \text{ of } \sum_{I,J : n \notin I \cup J} (c_I c_J \overline{e_I}\,\overline{f}\,\overline{e_J}^*) \bullet 1 \in \bigwedge V_e.$$

Now note that, since $\overline{f}$ has $n-1$ factors, if I, J do not have the same parity, then acting with $\overline{e_I}\,\overline{f}\,\overline{e_J}^*$ on 1 yields a zero contribution in $\bigwedge^{n-1} V_e$. Hence, the sum above may be split into two sums, one of which is

$$\text{the component in } \bigwedge^{n-1} V_e \text{ of } \sum_{I,J : |I|,|J| \text{ even},\, n \notin I \cup J} (c_I c_J \overline{e_I}\,\overline{f}\,\overline{e_J}^*) \bullet 1. \tag{6.2}$$

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However, consider

$$\widehat{\nu_2}(af) = \text{the component in } \bigwedge^n V \text{ of } \sum_{I,J} (c_I c_J e_I f e_J^*) \bullet 1 \in \bigwedge V.$$

For the same reason as above, this splits into two sums, and we want to compare the following expression to (6.2):

$$c_e(\text{the component in } \bigwedge^n V \text{ of } \sum_{I,J:|I|,|J| \text{ even}} (c_I c_J e_I f e_J^*) \bullet 1). \tag{6.3}$$

Now recall that the action of $e = e_n \in V \subseteq \mathrm{Cl}(V)$ on $\bigwedge V$ is via $o(e) + \iota(e)$, while c_e is ι_e followed by projection to $\bigwedge^{n-1} V_e$. Hence, to compute (6.3), we may as well compute the summands of

$$\text{the component in } \bigwedge^n V \text{ of } \sum_{I,J:|I|,|J| \text{ even}} (c_I c_J \cdot e \cdot e_I f e_J^*) \bullet 1$$

that do not contain a factor e. Terms with $n \in I$ do not contribute because then $ee_I = 0$. Terms with $n \notin I$ but $n \in J$ do not contribute because when e gets contracted with f_n a factor e in e_J^* survives, and when e does not get contracted with f_n, we use $ee_J^* = 0$. So we may restrict attention to the terms with $n \notin I \cup J$. Let I, J correspond to such a term; that is, $|I|, |J|$ are even and $n \notin I \cup J$. Write $I = \{i_1 < \ldots < i_k\}$ and $J = \{j_1 < \ldots < j_l\}$. Then

$$\begin{aligned}
(ee_I f e_J^*) \bullet 1 &= ((-1)^{n-1} e_I f_1 \cdots f_{n-1} e f_n e_J^*) \bullet 1 \\
&= ((-1)^{n-1} e_I f_1 \cdots f_{n-1} e) \bullet (f_n \wedge e_{j_l} \wedge \cdots \wedge e_{j_1}) \\
&= ((-1)^{n-1} e_I f_1 \cdots f_{n-1}) \bullet (e_{j_l} \wedge \cdots \wedge e_{j_1} + e \wedge f_n \wedge e_{j_l} \wedge \cdots \wedge e_{j_1}).
\end{aligned}$$

The second term in the last expression will contribute only terms with a factor e to the final result, and the former term contributes

$$\text{the component in } \bigwedge^{n-1} V_e \text{ of } (-1)^{n-1} (\overline{e_I}\,\overline{f}\,\overline{e_J}^*) \bullet 1.$$

Comparing this with (6.2), we see that the diagram commutes on terms in $\mathrm{Cl}^+(V) f$ up to the factor $(-1)^{n-1}$. A similar computation shows that it commutes on terms in $\mathrm{Cl}^-(V) f$ up to a factor factor $(-1)^n$.

We now consider the second diagram, where V is split as the orthogonal direct sum $V_e \oplus \langle e, h \rangle$ with $e = e_n, h = f_n$. Consider $a \in \mathrm{Cl}(\langle e_1, \ldots, e_{n-1} \rangle)$. By the same argument as above, it suffices to consider the case where all summands of a in the basis e_I have indices I with $|I|$ of the same parity, say even. Then $\widehat{\nu_2} \circ \tau_h$ in the diagram sends $a\overline{f}$ to the component in $\bigwedge^n V$ of $afa^* \bullet 1$. Since the summands e_I in a all have $n \notin I$, in $afa^* \bullet 1$ all summands have a factor f_n, and indeed,

$$(afa^*) \bullet 1 = f_n \wedge (a\overline{f}a^* \bullet 1)$$

(when all terms in a have $|I|$ odd, we get a minus sign). The component in $\bigwedge^n V$ of this expression is the same as the one obtained via $m_h \circ \widehat{\nu_2}$. □

6.5. *Proof of Theorem 6.1*

In this section, we use the Cartan map to prove Corollary 6.1, and finish the proof of Corollary 5.8 via a similar argument.

Proof of Theorem 6.1. For a quadratic space of dimension $2n$, denote by $\widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V) \subseteq \bigwedge^n V$ the isotropic Grassmann cone over the Plücker embedding. Given a maximal isotropic subspace $F \subseteq V$ with basis

$f_1, \ldots, f_n$ and $f := f_1 \cdots f_n$, let $\hat{\nu}_2 : \mathrm{Cl}^+(V)f \to \bigwedge^n V$ be the Cartan map defined in §6.3. For any isotropic $v \in V \setminus F$, the diagram

$$\begin{tikzcd} \bigwedge^n V \arrow[r, "c_v"] & \bigwedge^{n-1} V_v \\ \mathrm{Cl}(V)f \arrow[u, "\hat{\nu}_2"] \arrow[r, "\pi_v"] & \mathrm{Cl}(V_v)\overline{f} \arrow[u, "\hat{\nu}_2"'] \end{tikzcd}$$

commutes up to scalar factor at the bottom by Proposition 6.5, where $V_v := v^\perp/\langle v\rangle$ and where $\overline{f}$ is the image of a product of a basis of $v^\perp \cap F_n$.

The proof of [16, Corollary 4.2] shows that for $\omega \in \bigwedge^n V$, the following are equivalent:

1. $\omega \in \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V)$;
2. For every sequence of isotropic vectors $v_1 \in V$, $v_2 \in V_{v_1}$, $v_3 \in (V_{v_1})_{v_2}, \ldots, v_{n-4} \in (\cdots((V_{v_1})_{v_2})_{v_3}\cdots)_{v_{n-3}}$, it holds

$$C(\omega) \in \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(W),$$

where we abbreviate $W := (\cdots((V_{v_1})_{v_2})_{v_3}\cdots)_{v_{n-4}}$ and $C : \bigwedge^n V \to \bigwedge^4 W$ is the composition $C := c_{v_{n-4}} \circ \cdots \circ c_{v_1}$ of the contraction maps c_{v_i} introduced in Section 3.2.

By slight abuse of notation, we also write $v_1, \ldots, v_{n-4}$ for preimages of these vectors in V. These span an $(n-4)$-dimensional isotropic subspace U of V (provided that each v_i chosen above in the successive quotients is nonzero), and W equals $U^\perp/U$. For any fixed ω, the condition that $C(\omega)$ lies in $\widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(W)$ is a closed condition on U, and hence, it suffices to check that condition for U in a dense subset of the Grassmannian of isotropic $(n-4)$-dimensional subspaces of V. In particular, it suffices to check this when $U \cap F_n = \{0\}$.

Fix $n \geq 4$ and $x \in \mathrm{Cl}(V_n)f_1\cdots f_n$ such that $p(g\cdot x) = 0$ for all $g \in \mathrm{Spin}(V_n)$ and all $p \in I_4$. This means precisely that $\pi_{n,4}(g\cdot x) \in \widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V_4)$ for all $g \in \mathrm{Spin}(V_n)$. We need to show that $x \in \widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V_n)$. To this end, consider $\omega := \hat{\nu}_2(x) \in \bigwedge^n V_n$. It suffices to show that $\omega \in \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V_n)$. Indeed, this follows from the fact that $\hat{\nu}_2\big(\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(V)\big)$ is one of the two irreducible components of $\widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(V)$ (see Example 6.3) and because ν_2 is an injective morphism by Lemma 6.2. Let $v_1, v_2, \ldots, v_{n-4} \in V_n$ as above: linearly independent, and such that the span $U := \langle v_1, \ldots, v_{n-4}\rangle$ is an isotropic space that intersects F_n trivially. Let $C := c_{v_{n-4}} \circ \cdots \circ c_{v_1}$ be the composition of the associated contractions. We need to show that $C(\omega) \in \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(W)$, where $W := U^\perp/U$.

Now $\hat{\nu}_2\big(\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(W)\big) \subseteq \widehat{\mathrm{Gr}}^{\mathrm{Pl}}_{\mathrm{iso}}(W)$ by Example 6.3, and the diagram

$$\begin{tikzcd} \bigwedge^n V_n \arrow[r, "C"] & \bigwedge^4 W \\ \mathrm{Cl}(V_n)f \arrow[u, "\hat{\nu}_2"] \arrow[r, "\pi_{v_{n-4}}\circ\cdots\circ\pi_{v_1}"] & \mathrm{Cl}(W)\overline{f}, \arrow[u, "\hat{\nu}_2"'] \end{tikzcd}$$

where $\overline{f}$ is the image of the product of a basis of $U^\perp \cap F_n$, commutes up to a scalar factor in the bottom map due to Proposition 6.5. Hence, it suffices to check that $\pi_{v_{n-4}} \circ \cdots \circ \pi_{v_1}(x) \in \widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(W)$. Now there exists an element $g \in \mathrm{Spin}(V_n)$ that maps F_n into itself (not with the identity!) and sends v_i to e_{n+1-i} for $i = 1, \ldots, n-4$. This induces an isometry $W := U^\perp/U \to (U')^\perp/U' = V_4 = \langle e_1, \ldots, e_4, f_1, \ldots, f_4\rangle$, where $U' := \langle e_5, \ldots, e_n\rangle$. This in turn induces a linear isomorphism (unique up to a scalar) $\mathrm{Cl}(W)\cdot\overline{f} \to \mathrm{Cl}(V_4)\cdot f_1\cdots f_4$ (where f on the left is the product of a basis of $F_n \cap U^\perp$) that maps $\widehat{\mathrm{Gr}}^{+}_{\mathrm{iso}}(W)$ onto

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$\widehat{\mathrm{Gr}}^+_{\mathrm{iso}}(V_4)$. Since, by assumption, $\pi_{n,4}(g\cdot x) = \pi_{e_5}\circ\cdots\circ\pi_{e_n}(g\cdot x)$ lies in the latter isotropic Grassmann cone, $\pi_{v_{n-4}}\circ\cdots\circ\pi_{v_1}(x)$ lies in the former. □

Lemma 6.6. *Let $q \geq n \geq n_0$. Then for all g in some open dense subset of $\mathrm{Spin}(V_q)$, there exist $g' \in \mathrm{Spin}(V_n)$ and $g'' \in \mathrm{Spin}(V_{n_0})$ such that*

$$\pi_{q,n_0}\circ g\circ \tau_{n,q} = g''\circ \pi_{n,n_0}\circ g'$$

holds up to a scalar factor.

Proof. The proof is similar to that above; we just give a sketch. Using the Cartan map, which is equivariant for the relevant spin groups, this lemma follows from a similar statement for the corresponding (halfs of) exterior power representations. Specifically, define

$$\begin{aligned} E &:= \langle e_{n_0+1},\dots,e_q\rangle \subseteq V_q,\\ E' &:= \langle e_{n_0+1},\dots,e_n\rangle \subseteq V_n, \text{ and}\\ F &:= \langle f_{n+1},\dots,f_q\rangle \subseteq V_q. \end{aligned}$$

Then the desired identity is

$$c_E\circ g\circ m_F = g''\circ c_{E'}\circ g' \tag{6.4}$$

(up to a scalar), where

$$\begin{aligned} c_E &:= c_{e_{n_0+1}}\circ\cdots\circ c_{e_q} : \bigwedge^q V_q \to \bigwedge^{n_0} V_{n_0},\\ c_{E'} &:= c_{n_0+1}\circ\cdots\circ c_{e_n} : \bigwedge^n V_n \to \bigwedge^{n_0} V_{n_0}, \text{ and}\\ m_F &:= m_{f_q}\circ\cdots\circ m_{f_{n+1}} : \bigwedge^n V_n \to \bigwedge^q V_q \end{aligned}$$

and the c_{e_i} and m_{f_j} are as defined in §3.2 and §3.3, respectively. Furthermore, since the exterior powers are representations of the special orthogonal groups, we may take g, g', g'' to be in $\mathrm{SO}(V_q), \mathrm{SO}(V_n), \mathrm{SO}(V_{n_0})$, respectively.

We investigate the effect of the map on the left on (a pure tensor in $\bigwedge^n V_n$ corresponding to) a maximal (i.e., n-dimensional) isotropic subspace W of V_n. First, W is extended to $W' := W\oplus F$, then g is applied to W', and the final contraction map sends gW' to the image in V_q/E of $(gW')\cap E^\perp$.

Instead of intersecting gW' with $E^\perp$, we may intersect $W' = W\oplus F$ with $(E'')^\perp$, where $E'' := g^{-1}E$, followed by the isometry $\overline{g} : (E'')^\perp/E'' \to E^\perp/E$ induced by g. Accordingly, one can verify that the map on the left-hand side of (6.4) becomes (a scalar multiple of)

$$\overline{g}\circ c_{E''}\circ m_F,$$

where $c_{E''} : \bigwedge^q V_q \to \bigwedge^{n_0}((E'')^\perp/E'')$ is the composition of contractions with a basis of E'', and where we write $\overline{g}$ also for the map that $\overline{g}$ induces from $\bigwedge^{n_0}((E'')^\perp/E'')$ to $\bigwedge^{n_0}(E^\perp/E)$.

Now consider the space $E''\cap(V_n\oplus F)\subseteq V_q$. For g in an open dense subset of $\mathrm{SO}(V_q)$, this intersection has the expected dimension $(q-n_0)+(2n+q-n)-2q = n-n_0$, and for g in an open dense subset of $\mathrm{SO}(V_q)$, we also have $(E'')^\perp\cap F = \{0\}$ (because $(E'')^\perp$ has codimension $q-n_0$, which is at least the dimension $q-n$ of F). We restrict ourselves to such g. Then in particular, $E''\cap F = \{0\}$, and therefore, the projection $\widetilde{E}\subseteq V_n$ of $E''\cap(V_n\oplus F)$ along F has dimension $n-n_0$, as well. Note that $\widetilde{E}$ is isotropic because E'' is and because F is the radical of the bilinear form on $V_n\oplus F$.

Furthermore, the projection $V_n\oplus F\to V_n$ restricts to a linear isomorphism $(V_n\oplus F)\cap(E'')^\perp \to \widetilde{E}^\perp$, where the latter is the orthogonal complement of $\widetilde{E}$ inside V_n. This linear isomorphism induces an isometry

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$$h_1 : ((V_n \oplus F) \cap (E'')^\perp)/((V_n \oplus F) \cap E'') \to \widetilde{E}^\perp/\widetilde{E}$$

between spaces of dimension $2n_0$ equipped with a nondegenerate bilinear forms. However, the inclusion $V_n \oplus F \to V_q$ also induces an isometry

$$h_2 : ((V_n \oplus F) \cap (E'')^\perp)/((V_n \oplus F) \cap E'') \to (E'')^\perp/E''.$$

Now a computation shows that, up to a scalar, we have

$$c_{E''} \circ m_F = h_2 \circ h_1^{-1} \circ c_{\widetilde{E}},$$

where $c_{\widetilde{E}} : \bigwedge^n V_n \to \bigwedge^{n_0}(\widetilde{E}^\perp/\widetilde{E})$ is a composition of contractions with a basis of $\widetilde{E}$. Now choose $g' \in \mathrm{SO}(V_n)$ such that $g'\widetilde{E} = E'$, so that we have

$$c_{E'} \circ g' = \overline{g'} \circ c_{\widetilde{E}},$$

where $\overline{g'}$ is the isometry $\widetilde{E}^\perp/\widetilde{E} \to (E')^\perp/E'$ induced by g'. We then conclude that

$$c_E \circ g \circ m_F = \overline{g} \circ h_2 \circ h_1^{-1} \circ (\overline{g'})^{-1} \circ c_{E'} \circ g',$$

and hence, we are done if we set

$$g'' := \overline{g} \circ h_2 \circ h_1^{-1} \circ (\overline{g'})^{-1} \in \mathrm{SO}((E')^\perp/E') = \mathrm{SO}(V_{n_0}). \qquad \square$$

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