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REMARKS ON RINGS OF QUOTIENTS OF RINGS OF FUNCTIONS

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This paper was inspired by [4]. The main result there suggests a close relationship between injective hulls of C^* algebras as studied in [2] and [3] and rational completions as studied in [1]. We shall prove an analogue of Theorem 1 in [3] for rational completions. The latter theorem states that the injective hull of the algebra $C(X)$ of all complex valued functions on the compact Hausdorff space X is the algebra $B(X)$ of all bounded Borel functions modulo sets of first category.

The situation in [1] differs from that of [3] in several respects. [1] deals with real rather than complex valued functions and with the category of rings rather than that of C^* algebras. Furthermore, rational completions are not necessarily the same as injective hulls although they agree in many cases.

We use the notation in [1]. In particular we use $C(X)$, $C^*(X)$, $Q(X)$, $Q^*(X)$, $\bar{Q}(X)$, and $\bar{Q}^*(X)$.

THEOREM. *Let X be a Baire space. Then $\bar{Q}(X)$ is the ring $B(X)$ of all Borel functions on X modulo sets of first category.*

Proof. By the principal representation theorem in [1], the elements of $Q(X)$ correspond to continuous functions f on dense open subsets U of X. By extending f arbitrarily on U', a nowhere dense set, we obtain a map from $Q(X)$ into $B(X)$. This map is clearly well defined and an algebraic homomorphism. Furthermore, the metric is preserved. (By [1, p. 28] metrics are meaningful even for unbounded functions.) This follows from the fact that nonempty open sets are of second category by the Baire property. Thus we obtain an embedding.

Now let f be a finite valued function in $B(X)$. Suppose $f=\sum_{i=i}^{n} c_i e_i$, where e_i is the characteristic function on E_i and $\{E_i\}$ is a partition of X by borel sets. By a result of Birkhoff (also used in [3]), E_i differs from a regular open set F_i by a set of first category. Then the F_i's are disjoint with union dense in X. Let f_i be the characteristic function on F_i. Then $\sum_{i=i}^{n} c_i f_i \in Q(X)$ and *im* $\sum_{i=i}^{n} c_i f_i$ differs from $\sum_{i=i}^{n} c_i e_i$ by a set of first category. This shows that the image of $Q(X)$ is dense in $B(X)$. Since $B(X)$ is a complete space, it follows that $\bar{Q}(X)=B(X)$. Q.E.D.

It can be shown by exactly the same method that $\bar{Q}^*(X)=B^*(X)$ where $B^*(X)$ is the subring of $B(X)$ of bounded functions. Otherwise, this can be obtained as a corollary of the above result by applying a standard trick to show that the

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isomorphism of $\bar{Q}(X)$ and $B(X)$ necessarily maps bounded functions into bounded functions.

It almost seems that one can prove that $Q(X)=B(X)$ by using the reasoning in [2] and [3]. Although many steps seem to work, e.g. the singular ideal of $C(X)$ is zero, the result is necessarily false since in general $\bar{Q}(X)\neq Q(X)$ (see [1, p. 34]).

As a final remark, consider the last theorem in [1] which states that the maximal ideal space of $Q(X)$ is the stone space of the completion of the free Boolean algebra with countably infinitely many generators for any space X without isolated points and with a countable base. Since completion corresponds to injective hull in the category of Boolean algebras, the above space is the projective cover of the cantor set in the category of compact totally disconnected spaces which is the same as the projective cover in the category of compact Hausdorff spaces. It thus seems natural to use the Cantor set itself as a canonical such X. Hence by the above theorem $\bar{Q}(X)$ is the ring of all Borel function on the Cantor set modulo sets of first category. Finally, we mention the corollary that $B(X)$ is the same for all spaces X without isolated points and having a countable base. The latter result is of interest since it makes no explicit mention of rings of quotients.

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