1
Introduction and Overview

1.1 What Will Be Covered

In this book, we are interested in building blocks of matter made out of atoms. Atoms are themselves made of nuclei and electrons. The characteristic microscopic length scale of such matter is the average distance between the nucleus and the electron in an hydrogen atom, that is, the Bohr radius. On this length scale, nuclei and electrons are point particles. On the one hand, nuclei and electrons have intrinsic properties such as mass, electric charge, linear momentum, angular momentum, and energy that are shared with point particles in classical mechanics. On the other hand, nuclei and electrons have wave-like attributes, discrete quantum numbers, and internal degrees of freedom that can only be described within the realm of quantum mechanics. For distances of the order of or larger than the Bohr radius, their dominant interactions are electromagnetic.

Because the matter of interest in this book is made of an astronomically large number of nuclei and electrons, it must be described within the framework of many-body physics. The ground state is a many-body ground state and so are the excitations built from it. Now, the quantum many-body ground state and its many-body excited states can be very different from their counterparts when all (electromagnetic) interactions between nuclei and electrons are turned off. The simplest treatment of interactions between nuclei and electrons is to assume that the nuclei are localized on the sites of a lattice (not necessarily periodic) and that their interactions with the electrons is approximated by a one-body classical potential for, otherwise, noninteracting electrons. Even in this limit, the many-body ground state for the electrons can acquire exotic properties in the thermodynamic limit. The main goal of this book is to study the conditions under which electronic many-body excitations carry fractions of the quantum numbers of an isolated electron. This exploration will deliver the main concept of this book, namely, that of *invertible topological phases of matter*.

The aim of equilibrium condensed matter physics is to identify what phases of matter are realized in a material. A phase of matter is characterized by a list of attributes. The electrical resistance decreases with decreasing temperature in a metal,

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a crystal supports sharp collective excitations in the form of phonons, a magnet supports sharp collective excitations in the form of magnons. A phase diagram is the outcome of the theoretical and experimental study of a material in thermodynamic equilibrium. The control parameters in a phase diagram are typically the temperature, chemical potential, pressure, and magnetic field. Regions of the phase diagram correspond to different phases of matter if they are separated by boundaries along which sharp transitions take place. When the parametric boundary between two phases of matter realizes a continuous phase transition, the parametric boundary is then characterized by attributes of a universal character, that is, they only depend on a countable set of dimensionless parameters. Some of the phases of matter can themselves be characterized by universal dimensionless parameters, say the quantized value of the Hall conductivity in the *quantum Hall effect*.

Many of the distinguishing attributes of phases of matter in thermodynamic equilibrium, at sufficiently low temperatures, and in the presence of translation symmetry are inherited from the presence or absence of a spectral gap between the ground and excited states. For example, there is no gap between the ground and excited states in a metal, whereas there is one in an insulator. In 1975, known examples of gapless phases of matter were as follows:

1. The Coulomb phase of relativistic U(1) gauge theories[1]
2. Fermi-liquid theory[2]
3. The non-Fermi-liquid fixed point induced by the current-current interaction between electrons[3,4]
4. Whenever there are Nambu-Goldstone bosons associated with the spontaneous breaking of a continuous symmetry provided space (spacetime) is larger than two[5]
5. Continuous (quantum) critical points
6. The algebraic phase when a Berezinskii–Kosterlitz–Thouless transition takes place (an example of a Luttinger liquid in two-dimensional spacetime).[6]

In 1975, known examples of (partially) gapped phases of matter were the following:

[1] The Coulomb phase of quantum electrodynamics in the electroweak standard model of high-energy physics is defined by demanding that the two-point function for the exponential of the U(1) field-strength tensor decays algebraically fast in four-dimensional Euclidean spacetime.

[2] L. D. Landau, Soviet Physics JETP-USSR **3**(6), 920–925 (1957); **5**(1), 101–108 (1957); **8**(1), 70–74 (1959). [Landau (1957a,b, 1959)].

[3] T. Holstein, R. E. Norton, and P. Pincus, Phys. Rev. B **8**(6), 2649–2656 (1973). [Holstein et al. (1973)].

[4] Although Fermi-liquid theory is robust to the long-range Coulomb interaction because of screening, it is not robust to current-current interactions for the latter are not screened because of gauge invariance. However, the smallness of the ratio between the Fermi velocity and the speed of light in metals renders this instability unobservable as this instability of Fermi-liquid theory is pre-empted by gaping instabilities for all practical purposes.

[5] Y. Nambu, Phys. Rev. **117**(3), 648–663 (1960); Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**(1), 345–358 (1961); J. Goldstone, Il Nuovo Cimento **19**(1), 154–164 (1961). [Nambu (1960); Nambu and Jona-Lasinio (1961); Goldstone (1961)].

[6] V. J. Emery, in *Highly Conducting One-Dimensional Solids,* pages 247–303, edited by J. T. Devreese, R. P. Evrard, V. E. van Doren, Springer, Boston, 1979. [Emery (1979)].

1. Bloch band insulators
2. Long-range ordered phases breaking spontaneously a discrete symmetry (the Ising model for localized spins, charge-density waves for interacting electrons, or bond-density waves arising from the electron-phonon coupling)
3. Charge Mott insulators such as the Hubbard model at half-filling for any bipartite lattice[6]
4. The confining phases in gauge theories[7]
5. Higgs phases of matter (nonrelativistic superconductors or the weak sector of the standard model of particle physics).[8]

After 1975, it was progressively understood that all gapped phases of matter are not equal in that they can be distinguished by dimensionless and quantized quantum numbers. One such family of gapped phases of matter is now called invertible topological phases of matter. They have a nondegenerate ground state that is separated from all excitations by a gap when space is any closed manifold and translation symmetry holds. They owe their topological attributes to the fact that they can also support (symmetry) protected gapless boundary states when open boundary conditions are selected.

Invertible topological phases of matter display excitations that carry fractional values of the quantum numbers carried by the elementary building blocks of matter in condensed matter physics, say the electric charge when the total electric charge is conserved. Although the emphasis of this book will be on the tenfold classification of strong topological insulators or superconductors, which are fermionic examples of invertible topological phases of matter, we shall also describe bosonic invertible topological phases of matter.

Conceptual discoveries are rarely punctual in time. Even if a paradigm changing concept can be attributed to one or two papers, these papers were not written out of thin air, they were motivated by earlier works. Bearing in mind this caveat, the concepts that will be explored in this book crystallized in the mid 1970s.[9,10] The 1970s saw a convergence of interest between theorists in high-energy and condensed matter physics, on the one hand, and mathematicians, on the other hand, that culminated with the realization that index theorems and quantum anomalies were related and had potentially observable consequences in physics. Predictions deriving from applications of the index theorem in one-dimensional space were made for polyacetylene. However, it is the experimental discovery of the integer[11]

[7] J. Schwinger, Phys. Rev. **128**(5), 2425–2429 (1962). [Schwinger (1962)].
[8] P. W. Anderson, Phys. Rev. **130**(1), 439–442 (1963); P. W. Higgs, Phys. Lett. **12**(2), 132–133 (1964); P. W. Higgs, Phys. Rev. Lett. **13**(16), 508–509 (1964). [Anderson (1963); Higgs (1964b,a)].
[9] R. Jackiw and C. Rebbi, Phys. Rev. D **13**(12), 3398–3409 (1976). [Jackiw and Rebbi (1976)].
[10] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42**(25), 1698–1701 (1979); Phys. Rev. B **22**(4), 2099–2111 (1980). [Su et al. (1979, 1980)].
[11] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**(6), 494–497 (1980). [Klitzing et al. (1980)].

and fractional[12] quantum Hall effects in 1980 and 1982, respectively, that opened the Pandora's box for those phenomena in condensed matter physics that require concepts from topology to be understood.

Paradoxically, after the nexus of common interest in the 1970s between condensed matter physics, high-energy physics, and algebraic topology, the discovery of high-temperature superconductivity on the one hand and progress in string theory on the other hand saw the three communities go their separate ways between 1990 and 2005. The discovery of graphene[13] in 2004 and the prediction by Kane and Mele[14] in 2005 that polyacetylene and the quantum Hall effect were not the only playgrounds for band topology in physics triggered a renewed convergence of interest between these three communities with topology as the focal point. The topological attributes of band insulators or band superconductors by which (symmetry-protected) topological boundary states can evade Anderson localization transcend condensed matter physics as they have classical counterparts in many different mediums supporting the propagation of waves that are of relevance to optics, mechanical engineering, or electrical engineering. Applied science has paid due notice to the possibility of improving the efficiency of data transmission by encoding information into delocalized (symmetry-protected) topological boundary states.

This book is organized into eight Chapters. Chapters 2–4 are dedicated to the phenomenon of charge fractionalization in polyacetylene. The material presented in Chapters 2 and 4 was understood over the decade from 1976 to 1986. Chapters 5 and 6 introduce the notion of invertible topological phases of matter, of which polyacetylene was the first example. The time period covered by Chapters 5 and 6 starts in 1964 and extends to today. Whereas polyacetylene is a fermionic example of an invertible topological phase of matter in one-dimensional space, Chapters 5 and 6 cover bosonic examples of symmetry-protected topological phases of matter in one-dimensional space and their relation to fermionic examples of invertible topological phases of matter. Finally, we return to fermionic invertible topological phases of matter in Chapters 7 and 8, for which we give an exhaustive classification in any dimension of space called the tenfold way. This classification presumes a combination of three discrete symmetries: time-reversal symmetry, charge-conjugation symmetry, or chiral symmetry that is realized by identical point particles obeying the Pauli exclusion principle (fermions). In particular, this classification does not presume any space-group symmetry.

[12] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**(22), 1559–1562 (1982). [Tsui et al. (1982)].

[13] K. S. Novoselov, A. K. Geim, S. V. Morozov, et al., Science **306**, 666—669 (2004). [Novoselov et al. (2004)].

[14] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**(14), 146802 (2005); Phys. Rev. Lett. **95**(22), 226801 (2005). [Kane and Mele (2005a,b)].

1.2 What Will Not Be Covered

Fermionic invertible topological phases of matter in two or higher dimensions of space and at zero temperature are realized by materials with a vanishing bulk thermal conductance, while their boundaries support a nonvanishing thermal conductance. From the point of view of the bulk, they realize an insulating state. From the point of view of their boundaries, they realize a thermal metal. What makes many fermionic invertible topological phases of matter remarkable is that their ability to transport energy along their boundaries survives the thought experiment by which the electron-electron interaction is adiabatically turned off, that is the existence of gapless (symmetry-protected) topological boundary states is not driven by electron-electron interactions in many fermionic invertible topological phases of matter. However, this need not necessarily be so.

For example, it is possible to construct a local electronic interaction such that the charge and thermal Hall conductance are independently quantized,[15] thereby breaking the Wiedemann-Franz law according to which the thermal and charge conductivity tensor must be proportional. As the Wiedemann-Franz law is a property of noninteracting electrons, it follows that the decoupling of the charge and thermal Hall conductance is a property driven by strong interactions. Although the terminology of fermionic invertible topological phase of matter still applies to this situation, we will not discuss such examples of physics driven by strong interactions in this book.

Another example is the fractional quantum Hall effect (FQHE), for which the value taken by the Hall conductance in units of e^2/h is quantized to a rational value that is not an integer. Hereto, electron-electron interactions are essential for the fractional quantum Hall effect. The manifestation of topology in the fractional quantum Hall effect is of a qualitatively different nature than that governing fermionic invertible topological phases of matter. The fractional quantum Hall effect is an example of a phase of matter displaying *topological order*.[16,17] Topological order is not possible in one-dimensional space when the quantum dynamics is a local one for the observable degrees of freedom. Topological order can only be realized when the dimensionality of space is larger than one.

The study of fractionalization that requires strong interactions will be treated in a separate book that covers bosonic and fermionic topological order. Another arena for fractionalization to be covered in such a book is that of fractons.[18,19,20,21]

[15] T. Neupert, C. Chamon, C. Mudry, and R. Thomale, Phys. Rev. B **90**(20), 205101 (2014). [Neupert et al. (2014)].

[16] X.-G. Wen, International Journal of Modern Physics B **04**(02), 239–271 (1990). [Wen (1990)].

[17] X.-G. Wen, *Quantum Field Theory of Many-Body Systems: From the Origin of Sound to an Origin of Light and Electrons,* Oxford University Press, New York 2007. [Wen (2007)].

[18] C. Chamon, Phys. Rev. Lett. **94**(4), 040402 (2005). [Chamon (2005)].

[19] S. Bravyi, B. Leemhuis, and B. M. Terhal, Ann. Phys. (N.Y.) **326**(4), 839–866 (2011). [Bravyi et al. (2011)].

[20] J. Haah, Phys. Rev. A **83**(4), 042330 (2011). [Haah (2011)].

[21] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **92**(23), 235136 (2015). [Vijay et al. (2015)].

1.3 Why One, Two, and More d-Dimensional Space

It is standard practice when teaching quantum mechanics to solve the Schrödinger equation when space is one-, two-, and so on dimensional. This allows to add stepwise the complexity brought about by angular degrees of freedom, say.

The same is true with classical and quantum statistical physics. Exact solutions of the Ising model for one- and two-dimensional lattices have played a very important role in the study of phase transitions, mean-field theory, the renormalization group, and so on.

In condensed matter physics, quantum confinement has allowed to realize experimentally models for which space is effectively zero-dimensional (quantum dots[22]), one-dimensional (carbon nanotubes[23]), or two-dimensional (graphene[13]). The same is possible by trapping cold atoms in an optical lattice. Conversely, by adiabatic tuning of some couplings entering a Hamiltonian (quantum pumping[24]), it is possible to study an effective Hamiltonian acting on a d greater than three-dimensional space.

The essence of what are invertible topological phases of matter is most easily explained in one-dimensional space. In fact, both fermionic and bosonic invertible topological phases of matter were first predicted theoretically for effectively one-dimensional space. In the case of bosonic invertible topological phases of matter, the first experimental discovery thereof was that of a one-dimensional spin-1 Heisenberg antiferromagnetic quantum magnet. In the case of a fermionic invertible topological phases of matter, the first experimental discovery thereof was that of a two-dimensional gas of electrons displaying the integer quantum Hall effect.

[22] S. M. Reimann and M. Manninen, Rev. Mod. Phys. **74**(4), 1283–1342 (2002); R. Hanson, L. P. Kouwenhoven, J. R. Petta, S. Tarucha, and L. M. K. Vandersypen, Rev. Mod. Phys. **79**(4), 1217–1265 (2007). [Reimann and Manninen (2002); Hanson et al. (2007)].

[23] M. S. Dresselhaus, G. Dresselhaus, and P. Avouris, *Carbon Nanotubes: Synthesis, Structure, Properties, and Applications*, Springer, Berlin Heidelberg, 2001. [Dresselhaus et al. (2001)].

[24] D. J. Thouless, Phys. Rev. B **27**(10), 6083–6087 (1983). [Thouless (1983)].