Bull. Aust. Math. Soc. **89** (2014), 70–78
doi:10.1017/S0004972713000609

PROPERTY L AND COMMUTING EXPONENTIALS IN DIMENSION AT MOST THREE

GERALD BOURGEOIS

(Received 25 October 2012; accepted 12 May 2013; first published online 28 June 2013)

Abstract

Let A, B be two square complex matrices of the same dimension $n \leq 3$. We show that the following conditions are equivalent. (i) There exists a finite subset $U \subset \mathbb{N}_{\geq 2}$ such that for every $t \in \mathbb{N} \setminus U$, $\exp(tA + B) = \exp(tA)\exp(B) = \exp(B)\exp(tA)$. (ii) The pair (A, B) has property L of Motzkin and Taussky and $\exp(A + B) = \exp(A)\exp(B) = \exp(B)\exp(A)$. We also characterise the pairs of real matrices (A, B) of dimension three, that satisfy the previous conditions.

2010 *Mathematics subject classification*: primary 15A16; secondary 15A22, 15A24.
Keywords and phrases: matrix exponential, matrix pencil, property L.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}^*$ the set of nonzero integers. For every $n \in \mathbb{N}$, I_n (0_n, respectively) denotes the identity matrix (the zero matrix, respectively) of dimension n. For $X \in \mathcal{M}_n(\mathbb{C})$, $s(X)$ denotes its spectrum, that is, the set of its eigenvalues. Two matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ are said to be simultaneously triangularisable (abbreviated to ST) if there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices.

It is well known that the map $\exp : \mathcal{M}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ is not a homomorphism. Thus it would be interesting to determine the matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ such that:

(i) $e^A e^B = e^B e^A = e^{A+B}$; or more simply
(ii) $e^A e^B = e^{A+B}$.

Unfortunately, the complete solution of (i) is known only for $n = 2$ and $n = 3$ (see [7]) and the complete solution of (ii) is known only for $n = 2$ (see [6]). In [2], the author dealt with square matrices $A, B \in \mathcal{M}_n(\mathbb{C})$, $n = 2$ or 3, satisfying the following more restrictive condition:

$$\text{for every } t \in \mathbb{N}, \quad \exp(tA + B) = \exp(tA)\exp(B) = \exp(B)\exp(tA). \tag{1.1}$$

The author concluded that these matrices are ST. It appears that the above conclusion is wrong in the case of dimension three. Indeed, Jean-Louis Tu communicated to the author the counterexample

$$A_0 = 2i\pi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_0 = 2i\pi \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & 0 \end{pmatrix}. \tag{1.2}$$

Clearly A_0, B_0 are not ST. However, it is easy to see that, for every $t \in \mathbb{C}$, the eigenvalues of $tA_0 + B_0$ are the entries of its diagonal. Moreover, for every $t \in \mathbb{N}$, the eigenvalues of $tA_0 + B_0$ belong to $2i\pi\mathbb{Z}$ and are distinct. Therefore, for every $t \in \mathbb{N}$,

$$\exp(A_0) = \exp(B_0) = \exp(tA_0 + B_0) = I_3.$$

In [8], Motzkin and Taussky introduced property L, as follows.

Definition 1.1. A pair $(A, B) \in \mathcal{M}_n(\mathbb{C})^2$ has property L if there exist orderings of the eigenvalues $(\lambda_j)_{j\leq n}, (\mu_j)_{j\leq n}$ of A, B such that for all $(x, y) \in \mathbb{C}^2$,

$$s(xA + yB) = (x\lambda_j + y\mu_j)_{j\leq n}.$$

Remark 1.2. If A, B are ST, then the pair (A, B) has property L. The converse is false in general, except when $n = 2$ (see [8]).

Verifying that (A, B) has property L can be done by a finite rational procedure. Let χ_U denote the characteristic polynomial of $U \in \mathcal{M}_n(\mathbb{C})$.

Proposition 1.3. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$. If there are orderings of the eigenvalues $(\lambda_j)_j, (\mu_j)_j$ of A, B and $(t_i)_{1\leq i\leq n-1} \in (\mathbb{C} \setminus \{0\})^{n-1}$ pairwise distinct, such that, for every $1 \leq i \leq n-1$, one has $s(t_iA + B) = (t_i\lambda_j + \mu_j)_j$, then (A, B) has property L.*

Proof. Clearly $\chi_{tA+B}(T) = T^n + \sum_{k=1}^n P_k(t)T^{n-k}$, where P_k is a polynomial of degree k. For instance, consider $P_n(t) = \alpha_n t^n + \cdots + \alpha_0$, where $\alpha_n = \pm\det(A)$, $\alpha_0 = \pm\det(B)$ are known. For every $1 \leq i \leq n-1$ we know $\sum_{j=1}^{n-1} \alpha_j t_i{}^j$. Solving a Vandermonde system, we obtain the $(\alpha_j)_{1\leq j\leq n-1}$. In the same way, we calculate the coefficients of the $(P_k)_{1\leq k\leq n-1}$ and χ_{tA+B} is determined. We conclude easily that, for every $t \in \mathbb{C}$, $s(tA + B) = (t\lambda_j + \mu_j)_j$ and, by a continuity argument, that (A, B) has property L. □

Recently, in [10, Proposition 4], de Seguins Pazzis proved the following result.

Proposition 1.4. *A pair $(A, B) \in \mathcal{M}_n(\mathbb{C})^2$ satisfying (1.1) has property L.*

In this paper, we are interested in the converse of Proposition 1.4. We can wonder whether the conditions $e^Ae^B = e^Be^A = e^{A+B}$ and (A, B) having property L imply (1.1). The answer is no. Indeed, the pair $(A_0, -2B_0)$ (see (1.2)) has property L and $\exp(A_0) = \exp(-2B_0) = I_3$. Moreover, one has $\exp(tA_0 - 2B_0) = I_3$ if and only if $t \in \mathbb{N} \setminus \{2, 3, 4\}$. Therefore, (1.1) does not hold for this pair. Thus, we weaken (1.1) and define the following condition:

$$\begin{cases} \text{there exists a finite subset } U \subset \mathbb{N}_{\geq 2} \text{ such that, for all } t \in \mathbb{N} \setminus U, \\ \exp(tA + B) = \exp(tA)\exp(B) = \exp(B)\exp(tA). \end{cases} \tag{1.3}$$

https://doi.org/10.1017/S0004972713000609 Published online by Cambridge University Press

We shall show that, in dimensions two and three, the pair of complex matrices (A, B) satisfies (1.3) if and only if $e^{A+B} = e^A e^B = e^B e^A$ and (A, B) has property L. Finally, we characterise the pairs of real matrices (A, B) of dimension three, that satisfy (1.3).

Studying expressions of the form $tA + B$ is useful as shown by the following result. Let A be an $n \times n$ matrix over $\mathbb{C}$. Knowing the characteristic polynomial of the matrix $tA + X$ for each complex t and each $n \times n$ matrix X allows us to deduce Jordan's form of A (see [1]).

2. Property L and condition (1.3)

The following generalisation of the example (1.2) provides a partial converse of Proposition 1.4.

Proposition 2.1. *Assume that $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathcal{M}_n(\mathbb{C})$ has n distinct eigenvalues in $2i\pi\mathbb{Z}$, that $B = [b_{jk}] \in \mathcal{M}_n(\mathbb{C})$ is diagonalisable (where, for every $j \le n$, $b_{jj} \in 2i\pi\mathbb{Z}$) and that the pair (A, B) has property L. Then the pair (A, B) satisfies (1.3).*

Proof. Note that $e^A = I_n$. According to [8, Theorem 1], for every $t \in \mathbb{C}$,

$$s(tA + B) = (t\lambda_j + b_{jj})_{j \le n}.$$

Thus $e^B = I_n$. Since for almost all $t \in \mathbb{N}$, $tA + B$ has n distinct eigenvalues in $2i\pi\mathbb{Z}$, $\exp(tA + B) = I_n$. □

Definition 2.2.

(1) The spectrum of $A \in \mathcal{M}_n(\mathbb{C})$ is said to be $2i\pi$ congruence-free (denoted by $2i\pi$ CF) if, for all $\lambda, \mu \in s(A)$, $\lambda - \mu \notin 2i\pi\mathbb{Z}^*$.

(2) Let $\log\colon \mathrm{GL}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ be the (noncontinuous) primary matrix function associated to the principal branch of the logarithm, defined for $z \in \mathbb{C}^*$ by $\mathrm{Im}(\log(z)) \in (-\pi, \pi]$ (see [3]). Thus, for every $X \in \mathrm{GL}_n(\mathbb{C})$, $s(\log(X)) \subset \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in (-\pi, \pi]\}$.

Lemma 2.3. *Let $A \in \mathcal{M}_n(\mathbb{C})$. There exists a unique pair $(\tilde{F}, \Delta) \in \mathcal{M}_n(\mathbb{C})^2$ such that*

$$A = \tilde{F} + \Delta, \quad e^{\tilde{F}} = e^A, \quad e^{\Delta} = I_n \quad \text{and, for all } \lambda \in s(\tilde{F}), \mathrm{Im}(\lambda) \in (-\pi, \pi].$$

Moreover, both $\tilde{F}$ and Δ are polynomials in A.

Proof. Necessarily, $\tilde{F} = \log(e^A)$. Let $f\colon x \in U \to e^x \in \mathbb{C}$, where U is a complex domain containing $s(\tilde{F})$. Then f is a holomorphic function such that f' is not zero on U. Moreover, we can choose U such that f is one-to-one on U. According to [5, Theorem 2], $\tilde{F}$ is a polynomial in $e^{\tilde{F}} = e^A$. Therefore, $\tilde{F}$ is a polynomial in A. Let $\Delta = A - \tilde{F}$. Then $A\tilde{F} = \tilde{F}A$ and $e^{\Delta} = e^A e^{-\tilde{F}} = I_n$. □

Remark 2.4. Note that $s(\tilde{F})$ is $2i\pi$ CF, Δ is diagonalisable and $s(\Delta) \subset 2i\pi\mathbb{Z}$.

https://doi.org/10.1017/S0004972713000609 Published online by Cambridge University Press

In the following two results, we use the notation of Lemma 2.3.

Lemma 2.5. *Let* (A, B) *be a pair of* $n \times n$ *complex matrices such that* $e^{A+B} = e^A e^B = e^B e^A$ *and* $AB \neq BA$. *Then* $\log(e^A)$ *and* $\log(e^B)$ *cannot be cyclic matrices.*

Proof. *Step 1.* According to [9], $s(A)$, $s(B)$ are not $2i\pi$ CF. Moreover, the equality

$$e^{A+B}e^{-A} = e^{-A}e^{A+B} = e^B$$

implies that $s(A + B)$ is not $2i\pi$ CF. By Lemma 2.3, $A = \tilde{F} + \Delta$, $B = \tilde{G} + \Theta$, where $e^{\tilde{F}} = e^A$, $e^{\tilde{G}} = e^B$ and $e^\Delta = e^\Theta = I_3$. Thus $e^{\tilde{F}}e^{\tilde{G}} = e^{\tilde{G}}e^{\tilde{F}}$. According to [11, Proof of Theorem 1], $\tilde{F}\tilde{G} = \tilde{G}\tilde{F}$.

Step 2. Assume, for instance, that $\tilde{F}$ is a cyclic matrix. Then the commutant of $\tilde{F}$ is $\mathbb{C}[\tilde{F}]$. Thus $\tilde{G}\Delta = \Delta\tilde{G}$ and $\tilde{F} + \Delta + \Theta, \tilde{G}$ commute. From $e^{\tilde{F}+\tilde{G}} = e^{\tilde{F}+\Delta+\Theta+\tilde{G}}$, we deduce that $e^{\tilde{F}} = e^{\tilde{F}+\Delta+\Theta}$. According to [4, Theorem 4], $\tilde{F}(\Delta + \Theta) = (\Delta + \Theta)\tilde{F}$. Therefore, $\Theta \in \mathbb{C}[\tilde{F}]$ and $\Delta\Theta = \Theta\Delta$. This implies $AB = BA$, which is a contradiction. □

Remark 2.6. The next two results concern the equation

$$e^{A+B} = e^A e^B = e^B e^A$$

in dimension three. The first one can be derived from [7, Case (I), pages 165–166]. However, the proof, dated 1954, is difficult to read. Thus we give an alternative proof.

Proposition 2.7. *Let* (A, B) *be a pair of* 3×3 *complex matrices such that* $e^{A+B} = e^A e^B = e^B e^A$ *and* $AB \neq BA$. *If* $\mathbb{C}^3$ *is an indecomposable* $\langle A, B\rangle$ *module, then there exist* $\sigma \in \mathbb{C}$ *and two* 3×3 *complex matrices* Δ *and* F, *that are polynomials in* A, *such that* $A = \sigma I_3 + \Delta + F$ *and* $e^\Delta = I_3$, $F^2 = 0_3$. *In the same way, there are* $\tau \in \mathbb{C}$ *and two* 3×3 *complex matrices* Θ *and* G, *that are polynomials in* B, *such that* $B = \tau I_3 + \Theta + G$ *and* $e^\Theta = I_3$, $G^2 = 0_3$. *Moreover,* $FG = GF$.

Proof. We use the decompositions $A = \tilde{F} + \Delta$, $B = \tilde{G} + \Theta$. By Lemma 2.5, $\tilde{F}$ has an eigenvalue σ with multiplicity at least two and its minimal polynomial has degree at most two. By Step 1 of the proof of Lemma 2.5, it remains to show that $(\tilde{F} - \sigma I_3)^2 = 0_3$. We put $F = \tilde{F} - \sigma I_3$. Then $s(F) = \{0, 0, *\}$ and, up to similarity, F has one of the following three forms:

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \text{where } \lambda \neq 0,$$
$$F = 0_3,$$

or

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

https://doi.org/10.1017/S0004972713000609 Published online by Cambridge University Press

In the last two cases, we are done. Assume

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \text{where } \lambda \neq 0.$$

In the same way as for $\tilde{F}$, we can prove that there is $\tau \in \mathbb{C}$ such that $G = \tilde{G} - \tau I_3$ is similar to one of the previous three forms. Note that

$$e^{F+G} = e^F e^G = e^{A-\sigma I_3} e^{B-\tau I_3} = e^{A+B-(\sigma+\tau)I_3}.$$

Thus, if $\mathrm{Im}(s(F+G)) \subset (-\pi, \pi]$, then $F+G = \log(e^{A+B-(\sigma+\tau)I_3})$. Clearly $F+G$ also has an eigenvalue with multiplicity at least two and its minimal polynomial has degree at most two. Since F, G commute, we obtain for G three possible values.

Case 1: $G = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{smallmatrix}\right)$. Then $\mathbb{C}^3$ is a decomposable $\langle A, B\rangle$ module.

Case 2: $G = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Then $F+G = \log(e^{A+B-(\sigma+\tau)I_3})$ but its minimal polynomial has degree three, which is a contradiction.

Case 3: $G = \left(\begin{smallmatrix} \nu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, where $\nu \neq 0$. We have $F+G = \log(e^{A+B-(\sigma+\tau)I_3})$ and necessarily $\nu = \lambda$. Moreover, $s(F+G)$ is $2i\pi$ CF and $e^{F+G} = e^{F+G+\Delta+\Theta}$. According to [4, Theorem 4], $F+G$ and $\Delta+\Theta$ commute. The commutativity conditions $[F, \Delta] = 0$, $[G, \Theta] = 0$, $[F+G, \Delta+\Theta] = 0$ imply that Δ and Θ are diagonal matrices and that $AB = BA$. This is a contradiction. □

Definition 2.8. Using the notation of Proposition 2.7, we say that

$$\text{a pair } (A, B) \in \mathcal{M}_3(\mathbb{C})^2 \text{ has property } (*)$$

if the Jordan–Chevalley decompositions of A, B, $A+B$ are in the form

$$A = (\sigma I_3 + \Delta) + F, \tag{2.1}$$
$$B = (\tau I_3 + \Theta) + G, \tag{2.2}$$
$$A + B = ((\sigma+\tau)I_3 + \Delta + \Theta) + (F+G) \tag{2.3}$$

and satisfy

$$F^2 = G^2 = FG = GF = 0_3,$$
$$e^{\Delta} = e^{\Theta} = e^{\Delta+\Theta} = I_3$$

and

$$[F, \Theta] = [\Delta, G].$$

https://doi.org/10.1017/S0004972713000609 Published online by Cambridge University Press

PROPOSITION 2.9. *If* $(A, B) \in \mathcal{M}_3(\mathbb{C})^2$ *satisfies*

$$e^{A+B} = e^A e^B = e^B e^A, \quad AB \neq BA$$

and is such that $\mathbb{C}^3$ *is an indecomposable* $\langle A, B\rangle$ *module, then the pair* (A, B) *has property* $(*)$*. Conversely, if the pair* (A, B) *has property* $(*)$*, then* $e^{A+B} = e^A e^B = e^B e^A$.

PROOF. We use the notation and results of Proposition 2.7. Note that $\sigma I_3 + \Delta$ is diagonalisable, F is nilpotent and both are polynomials in A. Thus (2.1) and (2.2) are the Jordan–Chevalley decompositions of A, B. Moreover,

$$e^A = e^{\sigma}(I_3 + F),$$
$$e^B = e^{\tau}(I_3 + G),$$

and

$$e^{A+B} = e^{\sigma+\tau}(I_3 + F + G + FG),$$

with $FG = GF$. Thus $F + G + FG$ is nilpotent. According to the proof of Proposition 2.7, $A + B = (\omega I_3 + \Sigma) + O$ with $O\Sigma = \Sigma O$, $e^{\Sigma} = I_3$, $O^2 = 0_3$. We have $e^{A+B} = e^{\omega}(I_3 + O)$ and then $e^{\omega} = e^{\sigma+\tau}$, $O = F + G + FG$. Finally, $O^2 = 0_3$ implies that $FG = 0_3$ and (2.3) is the Jordan–Chevalley decomposition of $A + B$. Since $\Delta + \Theta$ and $F + G$ commute, $[F, \Theta] = [\Delta, G]$. Obviously, $e^{\Delta+\Theta} = I_3$. The last assertion is clear. □

We get the following result in dimension two.

THEOREM 2.10. *A pair* $(A, B) \in \mathcal{M}_2(\mathbb{C})^2$ *satisfies (1.3) if and only if* $e^{A+B} = e^A e^B = e^B e^A$ *and* (A, B) *has property L.*

PROOF. If (A, B) satisfies (1.3), then there exists $t_0 \in \mathbb{N}$ such that $e^{tA+B} = e^{tA} e^B = e^B e^{tA}$ holds for every $t \geq t_0$. According to Proposition 1.4, the pair $(t_0 A, B)$ has property L, as does (A, B). Assume now that $e^{A+B} = e^A e^B = e^B e^A$, (A, B) has property L and $AB \neq BA$. According to [9], $s(A)$ and $s(B)$ are not $2i\pi$ CF and, since $n = 2$, A, B are diagonalisable. A homothety can be added to A or B and we may assume

$$A = \begin{pmatrix} 2i\pi\lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad s(B) = \{2i\pi\mu, 0\}, \quad \text{where } \lambda, \mu \in \mathbb{Z}^*.$$

Again, since $n = 2$, A and B are ST, that is, they have a common eigenvector. Thus we may assume $B = \left(\begin{smallmatrix} 2i\pi\mu & 1 \\ 0 & 0 \end{smallmatrix}\right)$ (replacing, if necessary, λ with $-\lambda$ or μ with $-\mu$). Note that $e^A e^B = e^{A+B}$ if and only if $\lambda + \mu \neq 0$. If $t \in \mathbb{N}$,

$$e^{tA} e^B = e^B e^{tA} = e^{tA+B},$$

except possibly if $t = -\mu/\lambda$. □

https://doi.org/10.1017/S0004972713000609 Published online by Cambridge University Press

Remark 2.11. The pair

$$A = i\pi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \pi \begin{pmatrix} -11i & 6 \\ 16 & 11i \end{pmatrix}$$

satisfies the condition $e^{A+B} = e^A e^B = e^B e^A$ but does not have property L.

Our main result, in dimension three, is as follows.

Theorem 2.12. *A pair* $(A, B) \in \mathcal{M}_3(\mathbb{C})^2$ *satisfies* (1.3) *if and only if* $e^{A+B} = e^A e^B = e^B e^A$ *and* (A, B) *has property L.*

Proof. We first suppose that (A, B) satisfies (1.3). Using the same argument as in the proof of the necessary condition of Theorem 2.10, we can verify that $e^{A+B} = e^A e^B = e^B e^A$ and (A, B) has property L.

Assume now that the pair (A, B) has property L, $AB \neq BA$ and

$$e^{A+B} = e^A e^B = e^B e^A.$$

If $\mathbb{C}^3$ is a decomposable $\langle A, B\rangle$ module, we are finished, using Theorem 2.10. Now, suppose that $\mathbb{C}^3$ is an indecomposable $\langle A, B\rangle$ module.

Step 1. The pair (A, B) has property $(*)$. Using the notation of Proposition 2.9, we obtain, for every $t \in \mathbb{N}$,

$$\begin{aligned} e^{tA} &= e^{t\sigma}(I_3 + tF), \\ e^{tA}e^B &= e^B e^{tA} = e^{t\sigma+\tau}(I_3 + tF + G), \\ e^{tA+B} &= e^{t\sigma+\tau} e^{t\Delta+\Theta}(I_3 + tF + G). \end{aligned}$$

Thus $e^{tA+B} = e^{tA}e^B = e^B e^{tA}$ if and only if $e^{t\Delta+\Theta} = I_3$.

Step 2. The pair $(\Delta + F, \Theta + G)$ has property L. We consider the associated orderings $s(\Delta + F) = s(\Delta) = (\lambda_j)_{j\le 3}$ and $s(\Theta + G) = s(\Theta) = (\mu_j)_{j\le 3}$. If $t \in \mathbb{C}$, then $s(t(\Delta + F) + \Theta + G) = s((t\Delta + \Theta) + (tF + G)) = (t\lambda_j + \mu_j)_{j\le 3}$. Since $t\Delta + \Theta$ commutes with the nilpotent matrix $tF + G$, $s(t\Delta + \Theta) = (t\lambda_j + \mu_j)_{j\le 3}$ and the pair (Δ, Θ) has property L.

Step 3. Since $s(\Delta) \subset 2i\pi\mathbb{Z}$, $s(\Theta) \subset 2i\pi\mathbb{Z}$, if $t \in \mathbb{N}$, then $s(t\Delta + \Theta) \subset 2i\pi\mathbb{Z}$. Thus it remains to prove that, for almost all $t \in \mathbb{N}$, $t\Delta + \Theta$ is diagonalisable. If Δ and Θ commute, we are done.

We assume that Δ and Θ do not commute. Suppose that, for an infinite number of values of $t \in \mathbb{N}$, $t\Delta + \Theta$ is not diagonalisable. Then, for these values of t, $(t\lambda_j + \mu_j)_{j\le 3}$ contains at least two equal elements. Thus, for instance, for an infinite number of values of t, $t\lambda_1 + \mu_1 = t\lambda_2 + \mu_2$. This implies that $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$ and we may assume that these eigenvalues are 0. Therefore, the associated orderings are $s(\Delta) = \{0, 0, \lambda\}$, where $\lambda \in 2i\pi\mathbb{Z}^*$, and $s(\Theta) = \{0, 0, \mu\}$, where $\mu \in 2i\pi\mathbb{Z}^*$. We may assume that $\Delta = \mathrm{diag}(0, 0, \lambda)$. According to [8, Theorem 1],

$$\Theta = \begin{pmatrix} W & \begin{pmatrix} u \\ v \end{pmatrix} \\ \begin{pmatrix} p & q \end{pmatrix} & \mu \end{pmatrix},$$

https://doi.org/10.1017/S0004972713000609 Published online by Cambridge University Press

where W is a nilpotent 2×2 matrix and u, v, p, q are complex numbers. We know that Θ and $\Delta + \Theta$ are diagonalisable, that is, their rank is one and $\lambda + \mu \neq 0$. It remains to show that, for almost all $t \in \mathbb{N}$, $\mathrm{rank}(tA + B) = 1$ and $t\lambda + \mu \neq 0$.

Case 1. $W = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Therefore, $\mathrm{rank}(\Theta) = 1$ implies $p = v = 0, \mu = qu$. It follows that $\mathrm{rank}(\Delta + \Theta) = 1$ implies $\lambda = 0$, which is a contradiction.

Case 2. $W = 0_3$. Therefore, $\mathrm{rank}(\Theta) = \mathrm{rank}(\Delta + \Theta) = 1$ implies that

$$pu = pv = qu = qv = 0.$$

The previous condition implies that $\mathrm{rank}(t\Delta + \Theta) = 1$, except if $t = -\mu/\lambda$. □

Corollary 2.13. *Let A, B be square complex matrices of the same dimension at most three, such that (A, B) has property L and $e^{A+B} = e^A e^B = e^B e^A$. Then there exists $\alpha \in \mathbb{N}$ such that, for every integer $t \notin [-\alpha, \alpha]$, $e^{tA+B} = e^{tA} e^B = e^B e^{tA}$ and $e^{A+tB} = e^A e^{tB} = e^{tB} e^A$.*

Proof. Since A, B play the same role, it is sufficient to show the first part of the assertion. Note that $e^B = e^{-A} e^{A+B} = e^{A+B} e^{-A}$ and $(-A, A + B)$ has property L. Then for $t \in \mathbb{N}$ large enough, $e^{(1-t)A+B} = e^{(1-t)A} e^B = e^B e^{(1-t)A}$. □

3. The real case

If $n = 2$, we have the following result.

Proposition 3.1 [2, Theorem 1]. *Let $A, B \in \mathcal{M}_2(\mathbb{R})$ be such that there exists a finite subset $U \subset \mathbb{N}_{\geq 2}$ such that, for all $t \in \mathbb{N} \setminus U$,*

$$\exp(tA + B) = \exp(tA) \exp(B).$$

Then $AB = BA$.

However, if $n = 3$ there exist real pairs of matrices satisfying (1.3) that are not ST.

Proposition 3.2. *Let $A, B \in \mathcal{M}_3(\mathbb{R})$ be such that $\mathbb{C}^3$ is an indecomposable $\langle A, B\rangle$ module. Then the following two conditions are equivalent.*

(i) *The pair (A, B) satisfies (1.3) and $AB \neq BA$.*
(ii) *There exist $\sigma, \tau \in \mathbb{R}$ such that the pair $(A - \sigma I_3, B - \tau I_3)$ is simultaneously similar to the pair*

$$\left(\begin{pmatrix} 0 & -2\pi k & 0 \\ 2\pi k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\rho & -2\pi l + \theta & -\alpha \\ 2\pi l + \theta & \rho & \beta \\ 2\gamma & 2\delta & 0 \end{pmatrix} \right),$$

where $k, l \in \mathbb{Z}^*$ and $\alpha, \beta, \gamma, \delta, \rho, \theta$ are not all zero real numbers such that

$$\gamma\beta + \alpha\delta = 0, \quad \delta\rho\beta + \gamma\theta\beta + \alpha\gamma\rho - \alpha\delta\theta = 0, \quad \rho^2 + \theta^2 + 2(\beta\delta - \alpha\gamma) = 0.$$

PROOF. Let (A, B) be a real pair satisfying (1.3) and $AB \neq BA$. We use the notation of Proposition 2.9. We may assume $\sigma = \tau = 0$. Since $s(A) = s(\Delta)$ is not $2i\pi$ CF and $e^{\Delta} = I_3$, $s(A)$ is in the form $\{2i\pi k, -2i\pi k, 0\}$, where $k \in \mathbb{Z}^*$. Thus $F = 0$ and $A = \Delta$ is diagonalisable over $\mathbb{C}$. In the same way, $s(B) = \{2i\pi l, -2i\pi l, 0\}$, where $l \in \mathbb{Z}^*$. Note that (A, B) is simultaneously similar over $\mathbb{R}$ to (R, S), where

$$R = \begin{pmatrix} 0 & -2\pi k & 0 \\ 2\pi k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = [s_{i,j}].$$

According to Theorem 2.12, if $t \in \mathbb{R}$, then $s(tR + S) = \{2i\pi(tk + l), -2i\pi(tk + l), 0\}$ (replacing, if necessary, l with $-l$). This is equivalent to:

$$\text{for every } t \in \mathbb{R}, \quad \chi_{tR+S}(T) = T^3 + 4\pi^2(tk + l)^2 T.$$

We obtain an algebraic system in the unknowns $(s_{i,j})_{i,j}$. Solving this system, we obtain the required form for S. □

Acknowledgements

The author thanks D. Adam and R. Oyono for many valuable discussions; he thanks the referee for his careful reading of the paper.

References

[1] L. Baribeau and S. Roy, 'Caractérisation spectrale de la forme de Jordan', *Linear Algebra Appl.* **320** (2000), 183–191.

[2] G. Bourgeois, 'On commuting exponentials in low dimensions', *Linear Algebra Appl.* **423** (2007), 277–286.

[3] N. J. Higham, *Functions of Matrices: Theory and Computation* (SIAM, Philadelphia, PA, 2008).

[4] E. Hille, 'On roots and logarithms of elements of a complex Banach algebra', *Math. Ann.* **136** (1958), 46–57.

[5] R. Horn and G. Piepmeyer, 'Two applications of the theory of primary matrix functions', *Linear Algebra Appl.* **361** (2003), 99–106.

[6] K. Morinaga and T. Nôno, 'On the non-commutative solutions of the exponential equation $e^x e^y = e^{x+y}$', *J. Sci. Hiroshima Univ. Ser. A* **17** (1954), 345–358.

[7] K. Morinaga and T. Nôno, 'On the non-commutative solutions of the exponential equation $e^x e^y = e^{x+y}$, II', *J. Sci. Hiroshima Univ. Ser. A* **18** (1954), 137–178.

[8] T. S. Motzkin and O. Taussky, 'Pairs of matrices with property L', *Trans. Amer. Math. Soc.* **73** (1952), 108–114.

[9] Ch. Schmoeger, 'Remarks on commuting exponentials in Banach algebras. II', *Proc. Amer. Math. Soc.* **128**(11) (2000), 3405–3409.

[10] C. de Seguins Pazzis, 'On commuting matrices and exponentials', *Proc. Amer. Math. Soc.* **141** (2013), 763–774.

[11] E. M. E. Wermuth, 'Two remarks on matrix exponentials', *Linear Algebra Appl.* **117** (1989), 127–132.

GERALD BOURGEOIS, GAATI, Université de la Polynésie Française,
BP 6570, 98702 FAA'A, Tahiti, Polynésie Française
e-mail: bourgeois.gerald@gmail.com

https://doi.org/10.1017/S0004972713000609 Published online by Cambridge University Press