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ON PROPER HOLOMORPHIC MAPPINGS FROM DOMAINS WITH T-ACTION

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Abstract. We describe the branch locus of a proper holomorphic mapping between two smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$ under the assumption that the first domain admits a transversal holomorphic action of the unit circle. As an application we show that any proper holomorphic self-mapping of a smoothly bounded pseudoconvex complete circular domain of finite type in $\mathbb{C}^2$ is biholomorphic.

§1. Introduction

In the present paper we study proper holomorphic mappings between smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$. We begin with the description of the branch locus of a proper holomorphic mapping. Let Ω be a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^2$. It follows by [3, 7] that the automorphism group action $\mathrm{Aut}(\Omega) \times \Omega \longrightarrow \Omega$, $(f, z) \mapsto f(z)$ extends smoothly to $\overline{\Omega}$. Thus we can assume that $\mathrm{Aut}(\Omega)$ acts smoothly on $\overline{\Omega}$ and in particular on $\partial\Omega$. We say (see [3, 4, 27]) that a subgroup G of $\mathrm{Aut}(\Omega)$ acts transversally on $\partial\Omega$ if for every point $p \in \partial\Omega$ the image of the tangent mapping $(\Psi_p)^* : T_eG \longrightarrow T_p(\partial\Omega)$ associated to the mapping $\Psi_p : G \longrightarrow \partial\Omega$, $f \mapsto f(p)$ is not contained in the holomorphic tangent space $H_p(\partial\Omega)$. We will denote by $\mathbf{T}$ the Lie group of the unite circle. If $\mathbf{T}$ is a subgroup of $\mathrm{Aut}(\Omega)$ and acts transversally on $\partial\Omega$, we will simply say that Ω admits a transversal $\mathbf{T}$-action.

Let $f : \Omega \longrightarrow D$ be a proper holomorphic mapping between two domains Ω and D. We will denote by $J_f(z)$ the Jacobian determinant of f and by $V_f = \{z \in \Omega : J_f(z) = 0\}$ the branch locus of f.

Our first main result is the following

Theorem 1.1. *Let $f : \Omega \longrightarrow D$ be a proper holomorphic mapping between two smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^2$.*

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Suppose that Ω admits a transversal $\mathbf{T}$-action. Then for any irreducible component V of the branch locus V_f the following holds:

(i) $(V, \partial V)$ *is a smooth manifold with boundary in a neighborhood of every point in* $\overline{V} \cap \partial\Omega$.

(ii) $\partial V := \overline{V} \backslash V$ *is a finite disjoint union of* $\mathbf{T}$*-orbits.*

It is well-known [1] that any proper holomorphic self-mapping of the unit ball is biholomorphic. This important result was extended to larger classes of domains by several authors [6, 8, 10, 13, 22, 26]. One of the natural conjectures here is to show that any proper holomorphic self-mapping of a pseudoconvex domain with smooth finite type boundary is biholomorphic (this question still remains open even in dimension 2). In our paper we establish a result confirming this general conjecture.

THEOREM 1.2. *Let Ω be a smoothly bounded pseudoconvex complete circular domain of finite type in $\mathbb{C}^2$. Then every proper holomorphic self-mapping of Ω is a biholomorphism.*

In contrast with the $\mathbf{T}^2$ action case a regularity of the boundary is essential here. Indeed, a basin of attraction of a polynomial homogeneous complex dynamic system in $\mathbb{C}^2$ is a complete circular domain [23]; this gives a large class of examples of circular domains with proper holomorphic self-mappings which are not automorphisms. For instance, there exists a complete circular domain D in $\mathbb{C}^2$ with real analytic strictly pseudoconvex boundary outside of the union of three circles (where the boundary is not smooth) such that there is a proper holomorphic self-mapping of D which is not biholomorphic [14] .

The basic idea in proof of Theorem 1.1 given in Section 2 and 3 is a special version of the scaling method developed in [17, 18]. One can consider this method as a quantitative version of deformation of a complex structure which reduces the determination of the branch locus to a very special class of domains with algebraic boundaries. Theorem 1.2 then follows by elementary complex dynamics arguments in Section 4.

§2. Branching of holomorphic mappings between algebraic domains

This section is devoted to the study of holomorphic mappings between algebraic domains in $\mathbb{C}^2$. The general situation will be reduced to this case in the next section.

We recall certain general facts about boundary behavior of proper holomorphic mappings. Let $f : D_1 \longrightarrow D_2$ be a proper holomorphic mapping between two pseudoconvex smoothly bounded domains in $\mathbb{C}^2$. We suppose that f is smooth up to the boundary. Let r_j be the defining function of D_j. Following [6, 8], we consider the Levi-determinant of D_j defined as follows:

$$\Lambda_{\partial D_j}(p) = -\det \begin{pmatrix} 0 & \dfrac{\partial r_j}{\partial z} & \dfrac{\partial r_j}{\partial w} \\ \dfrac{\partial r_j}{\partial \overline{z}} & \dfrac{\partial^2 r_j}{\partial z \partial \overline{z}} & \dfrac{\partial^2 r_j}{\partial z \partial \overline{w}} \\ \dfrac{\partial r_j}{\partial \overline{w}} & \dfrac{\partial^2 r_j}{\partial w \partial \overline{z}} & \dfrac{\partial^2 r_j}{\partial w \partial \overline{w}} \end{pmatrix}.$$

Obviously $\Lambda_{\partial D_2}(f(p))|J_f(z)|^2 = \Lambda_{\partial D_1}(p)$ for any $p \in \partial D_1$.

For any boundary point $p \in \partial D_j$ we consider also the order of vanishing of Λ_{D_j} at p denoted by $\tau_{\partial D_j}(p)$, which is defined as follows: we choose smooth real coordinates $x = (x_1, x_2, x_3)$ on ∂D_j such that p corresponds to $x = 0$, and the formal power series $\Lambda_{D_j}(x) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} a_\alpha x^\alpha$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We set $\tau_{\partial D_j}(p) = \min\{|\alpha| : a_\alpha \neq 0\}$ (of course, this definition does not depend on the choice of coordinates). The following properties of τ are well known (see [6, 10]):

(1) $\tau_{\partial D_j}(p)$ is an upper-semicontinuous function on ∂D_j.

(2) $\tau_{\partial D_2}(f(p)) \leq \tau_{\partial D_1}(p)$ and the equality holds if and only if $\overline{V_f}$ does not contain p i.e. f is a diffeomorphism on the boundary near p .

The main purpose of this section is to study the structure of the branch locus of a proper holomorphic mapping f between rigid algebraic domains $\Omega = \{(z, w) \in \mathbb{C}^2 : \rho(z, w) = \operatorname{Im} w + P(z) < 0\}$ and $D = \{(z, w) \in \mathbb{C}^2 : \phi(z, w) = \operatorname{Im} w + Q(z) < 0\}$, where P, Q are non identically zero subharmonic polynomials without purely harmonic terms.

The set of weakly pseudoconvex points of $\partial\Omega$ will be denoted by $w(\partial\Omega)$. One has $w(\partial\Omega) = \{z \in \mathbb{C} : (\Delta P)(z) = 0\} \times \mathbb{R}$.

Let us consider the set $\Sigma_\Omega \in \mathbb{C}$ of singular points of the set $S = \{z \in \mathbb{C} : (\Delta P)(z) = 0\}$, i.e. the set of points in $\mathbb{C}$ such that S is not a smooth curve in any neighborhood of such a point. Note that Σ_Ω is finite (as an algebraic set of dimension 0).

PROPOSITION 2.1. *Let U be a neighborhood of the origin and $f : \Omega \cap U \longrightarrow D$ be a holomorphic mapping verifying the following property: for every point $p \in U \cap \partial\Omega$ and any sequence of points $(p^j)_j$ in Ω converging* *to p the sequence of images $(f(p^j))_j$ has no cluster points in D. Then for every open $U' \subset U$ the branch locus $U' \cap V_f$ in U' is contained in the union of complex lines $\cup_{z_j \in \Sigma_\Omega} \{(z_j, w) \in \mathbb{C}^2 : \operatorname{Im} w < -P(z_j)\}$.*

The following corollary considers the case useful for the proof of our main results:

COROLLARY 2.2. *If P is homogeneous, then V_f is contained in the half-plane $\{(0, w) : \operatorname{Im} w < 0\}$.*

Proof of Proposition 2.1.

We have $S = S_1 \cup \ldots \cup S_k \cup \Sigma_\Omega$, where S_j are smooth connected real algebraic curves. We note that the order of vanishing of the laplacian ΔP is constant on every S_j by the connectivity.

Hence, we have $w(\partial\Omega) = \cup_{j=1}^N (S_j \times \mathbb{R}) \cup (\Sigma_\Omega \times \mathbb{R})$ and $\tau(p)$ is constant on every totally real manifold $S_j \times \mathbb{R}$.

Let us recall also the following property of the mapping f established in Proposition 2.2 and Lemma 4.1 of [17]: if p is a boundary point of Ω and there exists a sequence of points $(p^j)_j$ in Ω converging to p such that the sequence of images $(f(p^j))_j$ converges to a finite boundary point of ∂D, then f extends continuously up to the boundary in a neighborhood of p; then it follows by Lemma 6.2 of [18] that f extends to $\mathbb{C}^2$ as an algebraic mapping i.e. its graph is contained in a complex algebraic 2-dimensional variety X in $\mathbb{C}^2 \times \mathbb{C}^2$. Moreover, it follows by Proposition 6.3 of [18] and [11] that f is smooth up to the boundary in a neighborhood of p and then by Lemma 2.1 of [18] the Jacobian determinant J_f of f does not vanishes identically. Thus, $U \cap \partial\Omega$ is a disjoint union of two subsets: the subset A of points where f extends smoothly up to the boundary and the subset B of points $b \in U \cap \partial\Omega$ such that $\lim_{(z,w) \longrightarrow b} |f((z, w))| = \infty$. It was shown in Lemma 4.1 of [17] that A is an (non-empty) open dense subset of $U \cap \partial\Omega$. We will call B the "pull-back of infinity" and will denote by $f^{-1}(\infty)$.

We denote by $\mathbb{C}P^2$ the complex 2-dimensional projective space and by $\hat{X}$ the projective closure of X in $\mathbb{C}P^2 \times \mathbb{C}P^2$ which is an irreducible complex 2-dimensional projective variety. Let π_Ω (resp. π_D) be the natural projection of $\hat{X}$ to the copy of $\mathbb{C}P^2$ containing Ω (resp. D) (or more precisely its image under the canonical embedding $i : \mathbb{C}^2 \hookrightarrow \mathbb{C}P^2$). Since J_f does not vanishes identically, the composition $\pi_D \circ \pi_\Omega^{-1} : \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2$ is a proper holomorphic correspondence (see [24]) and in particular, surjective. Then

$\pi_\Omega \circ \pi_D^{-1}(\mathbb{C}P^2 \backslash i(\mathbb{C}^2))$ is an complex algebraic curve γ in $\mathbb{C}P^2$ containing $f^{-1}(\infty)$. Since $\partial\Omega$ is of finite type, the intersection $\gamma' = \gamma \cap (U \cap \partial\Omega)$ is at most a real algebraic curve.

LEMMA 2.3. *The closure $\overline{V_f}$ does not intersect the set of strictly pseudoconvex points in $\partial\Omega$.*

Proof. Let p be a strictly pseudoconvex point in $\overline{V_f} \cap U \cap \partial\Omega$ and W be its neighborhood which does not intersect $w(\partial\Omega)$. If there exists a point $p' \in \overline{V_f} \cap W$ which is not in $f^{-1}(\infty)$, then it follows by [17] that $f(p')$ is a strictly pseudoconvex point in ∂D and f extends to a biholomorphism in a neighborhood of p': a contradiction.

Thus, it suffices to establish the following

CLAIM. *For any open subset W of $\partial\Omega \cap U$ the intersection $\overline{V_f} \cap W \cap \partial\Omega$ is not contained in $f^{-1}(\infty)$.*

For the proof assume by contradiction that $\overline{V_f} \cap W \cap \partial\Omega$ is contained in $f^{-1}(\infty)$. Since f extends to an algebraic mapping, V_f is a piece of an complex algebraic subset in $\mathbb{C}^2$ and the set of its non-regular points Y is finite. It follows by the maximum principle applied to the restriction $\rho|V_f$ that the intersection $\overline{V_f} \cap \partial\Omega$ cannot contain only point from Y. Thus, one can assume that there exists a point p' in $\overline{V_f} \cap W \cap \partial\Omega$ such that V_f extends to a neighborhood of p' as a smooth complex manifold $\tilde{V}_f$. Now the well-known argument of [5] using the Hopf lemma shows that (moving slightly p') one can assume that $\tilde{V}_f$ intersects $\partial\Omega$ transversally at p'.

Let $g(z,w)$ is a holomorphic function on D , $|g(z,w)| < 1$ on D and $g(z,w) \longrightarrow 1$ as $|(z,w)| \longrightarrow \infty$ (see [17]). Let us consider the composition $g \circ f$. Since $\overline{V_f} \cap \partial\Omega$ is contained in $f^{-1}(\infty)$, $(g \circ f)(z,w) \longrightarrow 1$ as (z,w) tends to $\overline{V_f} \cap \partial\Omega$; then it follows by the boundary uniqueness theorem that $(g \circ f)|V_f$ is equal to 1 identically : a contradiction, and we get the claim.

Q.E.D.

Thus, we have the inclusion $\overline{V_f} \cap U \cap \partial\Omega \subset w(\partial\Omega)$.

LEMMA 2.4. *The intersection $\overline{V}_f \cap (S_j \times \mathbb{R})$ is empty for every j.*

Proof. Suppose by contradiction that $\overline{V_f} \cap (S_j \times \mathbb{R})$ contains a point p for some j. Then $\overline{V_f} \cap \partial\Omega$ is contained in $(S_j \times \mathbb{R})$ near p ; it follows by the Claim that there exists a point q in $\overline{V} \cap S_j \times \mathbb{R}$ such that $f(q)$ is

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finite and f extends smoothly up to the boundary near q . Since J_f does not vanish identically on $S_j \times \mathbb{R}$ by the boundary uniqueness theorem, there exists a sequence (q^k) in S_j converging to q such that $J_f(q^k) \neq 0$. Therefore, $\tau_{\partial\Omega}(q^k) = \tau_{\partial D}(f(q^k))$. On the other hand, $\tau_{\partial D}(f(q)) < \tau_{\partial\Omega}(q)$. Since $\tau_{\partial\Omega}$ is constant on S_j, we have $\tau_{\partial D}(f(q)) < \tau_{\partial\Omega}(q^k) = \tau_{\partial D}(f(q^k))$. This is a contradiction since the function $\tau_{\partial D}$ is upper semicontinuous.

Q.E.D.

Now the desired proposition follows by the uniqueness theorem.

§3. Scaling and branching

We begin with the following local version of Theorem 1.1:

Proposition 3.1. *Let $D_1 = \{(z,w) \in W : \operatorname{Im} w + P_{2m} + \varphi(z) < 0\}$ be a smooth pseudoconvex finite type domain in a neighborhood W of the origin, $\varphi(z) = o(|z|^{2m})$, P_{2m} is a non-zero subharmonic polynomial without purely harmonic terms and D_2 be a smoothly bounded pseudoconvex finite type domain. Let $f : D_1 \longrightarrow D_2$ be a holomorphic mapping smooth up to the boundary, $f(0) = 0$ and V is an irreducible component of the branch locus V_f such that $0 \in \overline{V}$. Then $V = D_1 \cap \{(z,w) \in W : z = 0\}$.*

In what follows we denote by Γ_j the boundary of D_j near the origin. Recall that by the well-known argument [5] the set $E \subset \overline{V} \cap \Gamma_1$ of points where V is a C^∞ smooth manifold with boundary transversal to Γ_1 is open dense in $\overline{V} \cap \Gamma_1$ ([5] considers the strictly pseudoconvex case but the argument easily can be adapted for our case in view of the existence of holomorphic peak functions [9] and plurisubharmonic exhaustion functions [19], see also [6]).

First, we assume that $0 \in E$, i.e. V is a C^∞ smooth variety with boundary near the origin and transversal to Γ_1; the general case will be reduced to this one. We proceed the proof by contradiction. Assume that the statement is false. Since V is an irreducible complex variety and a smooth manifold with boundary near 0, there exists a sequence $\zeta^\nu = (a^\nu, b^\nu)$ in $\overline{V} \cap \Gamma_1$ converging to 0 and such that $a^\nu \neq 0$ for any ν. Since V is transversal to the boundary at 0, it follows by the implicit function theorem that there exists a smoothly bounded domain X in $\mathbb{C}$, $0 \in \partial X$ and a neighborhood U of the origin in $\mathbb{C}^2$ such that

$$V \cap U = \{(z,w) \in U : z = h(w), w \in X\}, \tag{1}$$

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where h is a holomorphic function on X smooth up to the boundary. We extend h smoothly past the boundary and assume that it is defined in a neighborhood Y of 0.

Let $b^\nu = \alpha^\nu + i\beta^\nu$. Let us consider the translations $T^\nu : (z, w) \longrightarrow (z, w + \alpha^\nu)$. Then the sequence (T^ν) converges uniformly to the identity on any compact subset of $\mathbb{C}^2$ and $\eta^\nu = T^{\nu^{-1}}(\zeta^\nu) = (a^\nu, i\beta^\nu)$. Set $f^\nu = f \circ T^\nu$ and $W^\nu = T^{\nu^{-1}}(W)$. Then the sequence $f^\nu : D_1 \cap W^\nu \longrightarrow D_2$ is a sequence of proper holomorphic mappings ; the sequence (W^ν) converges in the Hausdorff distance sense to W and (f^ν) converges to f uniformly on any compact subset of $\overline{D}_1 \cap W$. We denote by V^ν the pullback $T^{\nu^{-1}}(V)$ which is contained in the branch locus of f^ν. We have $\eta^\nu \in V^\nu \cap \Gamma_1$.

LEMMA 3.2. *There exists a constant $C > 0$ such that for any ν and any $(z, w) \in D_1 \cap W^\nu$ one has*

$$C^{-1}\mathrm{dist}((z, w), \Gamma_1) \le \mathrm{dist}(f^\nu((z, w)), \Gamma_2) \le C\mathrm{dist}((z, w), \Gamma_1).$$

For the proof we observe that the statement holds for f by the Hopf Lemma (see [2]) and the linear mappings T^ν preserve the distance to Γ_1 with a uniform constant independent of ν.

We note that $V^\nu \cap W^\nu$ is defined by $\{(z, w) \in W^\nu : z = h^\nu(w) = h(w - \alpha^\nu)\}$, where h^ν is holomorphic on $X^\nu = X - \alpha^\nu$ and smooth up to the boundary ; evidently, h^ν converges together with all derivatives uniformly to h on any compact of X.

We set $\delta_\nu = |a^\nu|^{2m}$, $p^\nu = (0, -\delta_\nu i)$. Considering the Taylor expansion of h^ν near η^ν we get $z - a^\nu = \lambda_\nu(w - i\beta^\nu) + \psi^\nu(w)$, where ψ^ν is smooth function in a fixed neighborhood of 0 and there exists a constant $M > 0$ such that $|\psi^\nu(w)| \le M|w - i\beta^\nu|^2$ for any ν. Note also that the sequence (λ_ν) is bounded by (1).

Fix $\alpha < 0$. Let us slice V^ν by a complex line $w = i\alpha\delta_\nu + i\beta^\nu$. For ν large enough the intersection point is (by uniformity of neighborhoods) $(z^\nu, w^\nu) = (a^\nu + i\alpha\delta_\nu\lambda_\nu + o(\delta_\nu), i\alpha\delta_\nu + i\beta^\nu)$.

Set $r_1(z, w) = \mathrm{Im}\, w + P_{2m}(z) + \varphi(z)$. Since $r_1(a^\nu, b^\nu) = 0$, we have $r_1(z^\nu, w^\nu) = \delta_\nu\alpha + o(\delta_\nu) < 0$ for ν large enough. Hence, (z^ν, w^ν) is in $D_1 \cap W^\nu$ for ν large enough.

Now we can apply a version of the scaling construction developed in [17, 18]. We need the following well-known statement basic for analysis on pseudoconvex domains of finite type (see [16, 20]).

Let Ω be a domain in $\mathbb{C}^2$ with C^∞ smooth boundary near a point $p \in \partial\Omega$ of finite type $2k$. Then there exists a neighborhood U of p with the following properties:

(a) there exists a local biholomorphic change of coordinates such that in the new coordinates we have

$$\Omega \cap U = \{(z, w) \in \mathbb{C}^2 | r(z) = \operatorname{Im} w + \theta(z, \, Rew) < 0\},$$

where $\theta \in C^\infty$ and vanishes at the origin with the order of (at least) 2;

(b) there exists a mapping $\Phi : \mathbb{C}^2 \times U \to \mathbb{C}^2$ of class C^∞ such that

(b1) $\Phi(\bullet, \xi)$ is a polynomial and $\Phi(\xi, \xi) = 0$;

(b2) there exists a neighborhood V of p and $V_\xi \ni \xi$ such that $V \subset V_\xi \subset U$, $\Phi(\bullet, \xi)$ is a biholomorphism from V_ξ onto the unit ball $\mathbb{B} \subset \mathbb{C}^2$; the mapping $(t, \xi) \mapsto \Phi(\bullet, \xi)^{-1}(0, it)$ is a diffeomorphism between $(-1, 1) \times (\partial\Omega \cap V)$ and V (this implies by continuity that there exists an open cone C_0 with vertex at the origin in the direction of $\operatorname{Im} w$ and an open cone C_ξ with vertex on ξ in the direction of the inward normal at ξ and of the vertex angle independent of ξ such that $\Phi((C_\xi), \xi) \subset C_0$).

(b3) one has

$$r \circ \Phi(\bullet, \xi)^{-1} - r(\xi) = \operatorname{Im} w + \sum_{\ell=2}^{2k} P_\ell(z, \xi) + (\operatorname{Re} w) \sum_{\ell=1}^{k} Q_\ell(z, \xi)$$
$$+ \sigma_{2k+1}(z, \xi) + \sigma_2(\operatorname{Re} w, \xi) + (\operatorname{Im} z_2)\sigma_{k+1}(z, \xi)$$

on $V \times \mathbb{B}$; here P_ℓ and Q_ℓ are homogeneous polynomials in z and $\bar{z}$ of degree ℓ without purely harmonic terms; $\sigma_i(v, \xi)$ vanishes of order i in v;

(c) one has $\inf_\xi \sup_\ell \|P_\ell(\bullet, \xi)\| > 0$, where $\|\|$ is the norm of homogeneous polynomials.

For $\varepsilon > 0$ we set $\tau(\xi, \varepsilon) = \min_{\ell=2,\cdots,2k} \left(\frac{\varepsilon}{\|P_\ell(\bullet,\xi)\|}\right)^{1/\ell}$.

We suppose also that Γ_2 is of type $2k$ near the origin.

Set $q^\nu = f^\nu(p^\nu)$. We denote by ω^ν the point of Γ_2 closest to q^ν ; set also $\gamma_\nu = \operatorname{dist}(q^\nu, \Gamma_2) = |q^\nu - \omega^\nu|$. Let g^ν denote the polynomial biholomorphism $\Phi(\bullet, \omega^\nu)$ corresponding to Γ_2. Without loss of generality one can assume that $g^\nu \to \operatorname{id}$ uniformly on compact subsets of $\mathbb{C}^2$ as $\nu \to \infty$.

Let us consider the holomorphic mappings $\tilde{f}^\nu = g^\nu \circ f : D_1 \to g^\nu(D_2)$ and the following dilations of coordinates : $A^\nu : (z, w) \mapsto (\delta_\nu^{-1/2m} z, \, \delta_\nu^{-1} w)$ and $B^\nu : (z, w) \mapsto (\tau(\omega^\nu, \gamma_\nu)^{-1} z, \, \gamma_\nu^{-1} w)$. We set $D_1^\nu = A^\nu(D_1)$, $D_2^\nu =$

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$B^\nu \circ g^\nu(D_2)$ and consider the mappings

$$F^\nu = B^\nu \circ g^\nu \circ f^\nu \circ (A^\nu)^{-1} = B^\nu \circ \tilde{f}^\nu \circ (A^\nu)^{-1} \ : D_1^\nu \to D_2^\nu.$$

Let also $r_1^\nu(z,w) = \delta_\nu^{-1} r_1 \circ (A^\nu)^{-1} = \delta_\nu^{-1} r_1(\delta_\nu^{1/2m} z,\, \delta_\nu w)$ and $r_2^\nu = \gamma_\nu^{-1} r_2 \circ (g^\nu)^{-1} \circ (B^\nu)^{-1} = \gamma_\nu^{-1} r_2^\nu(\tau(\omega^\nu,\, \gamma_\nu)z,\, \gamma_\nu w)$.

Since Γ_2 is of type $2k$ at the origin, one has

$$r_2^\nu = \operatorname{Im} w + \gamma_\nu^{-1} \left(\sum_{\ell=2}^{2k} (\tau(\omega^\nu, \gamma_\nu))^\ell P_{2,\ell}(\omega^\nu,\, z) \right) + R^\nu,$$

where the sequence $(R^\nu)_\nu$ converges to zero uniformly on compact subsets of $\mathbb{C}^2$ as $\nu \to \infty$ (see [17, 18]).

Passing to the subsequence, one can assume that the polynomials $\gamma_\nu^{-1} \Sigma_{\ell=2}^{2k} P_{2,\ell}(\omega^\nu,\, \tau^\ell(\omega^\nu, \gamma_\nu) z)$ converge uniformly on compact subsets of $\mathbb{C}$ to a nonzero real polynomial Q of degree $\leq 2k$ that contains no purely harmonic terms. Let us consider the domain $\Omega_2 = \{w \in \mathbb{C}^2 | \psi(z,w) = \operatorname{Im} w + Q(z) < 0\}$.

The sequence (r_2^ν) converges to the function ψ uniformly together with all derivatives of any order ; hence Ω_2 is pseudoconvex as a smooth limit of pseudoconvex domains. In particular, Q is a subharmonic polynomial on $\mathbb{C}$.

Similarly, we have that the sequence (r_1^ν) converges uniformly on compact subsets of $\mathbb{C}^2$ to the function $\phi = \operatorname{Im} w + P_{2m}(z)$ (in what follows we write simply P). It is worth to note that P is a homogeneous polynomial. We define the domain $\Omega_1 = \{(z,w) \in \mathbb{C}^2 | \phi(z) < 0\}$.

Now quite similarly to [17], it follows by [12] that there exists a subsequence of $(F^\nu)_\nu$ uniformly converging on compact subsets of Ω_1.Thus, without loss of generality one can assume that $(F^\nu)_\nu$ converges uniformly on compact subsets of Ω_1 to a holomorphic mapping F. This was shown in [17, 18] that F takes its values in Ω_2 and , moreover, one has $\psi(F(z,w)) \leq C(R)\phi(z,w)$ for any $R > 0$ and $z \in \Omega_1 \cap R\mathbb{B}$ (here and below $\mathbb{B}$ denotes the unit ball of $\mathbb{C}^2$).

We have $A^\nu(p^\nu) = (0, -i)$, $A^\nu(\eta^\nu) = (e^{i\theta_\nu}, i\beta^\nu \delta_\nu^{-1})$. But $\beta^\nu = P_{2m}(a^\nu) + o(|a^\nu|^{2m})$ and by the choice of δ_ν the sequence $\beta^\nu \delta_\nu^{-1}$ tends to a finite point τ. Therefore,

$$A^\nu(z^\nu, w^\nu) = (e^{i\theta_\nu} + i\alpha\delta_\nu^{1-(1/2m)} + o(\delta_\nu^{1-(1/2m)}), i\alpha + i\beta^\nu/\delta_\nu)$$

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This sequence tends to the point $q = (e^{i\theta}, i\alpha + i\tau)$. Since $(e^{i\theta}, i\tau)$ is in the boundary of the limit model domain Ω_1, the point q is in Ω_1.

Since the limit mapping $F : \Omega_1 \longrightarrow \Omega_2$ satisfies conditions of Proposition 2.1 and P_{2m} is a homogeneous polynomial, by the previous section we obtain that $V_F = \{z = 0\}$. But by the construction q is in V_F : a contradiction. This proves the proposition 3.1 in the case where V is a smooth variety with boundary.

Suppose now that 0 is a point of $(\overline{V} \cap \Gamma_1) \setminus E$. Then we take a regular point $(a, b) \in E$ of type $2s$ and consider the polynomial change of variables $T : (z', w') \longrightarrow (z, w) = (z' + a, w' + b + Q(z'))$, where the polynomial Q is chosen such that $r_1 \circ T = \operatorname{Im} w' + R_{2s}(z') + \varphi'(z')$, where R_{2s} is a homogeneous subharmonic polynomial of degree $2s$ without purely harmonic terms and $\varphi'(z') = o(|z'|^{2s})$. Then 0 is the regular point for the branch locus of the mapping $f \circ T$ and as it was just shown $J_{f \circ T} = \{z' = 0\}$ near the origin. This implies that (in the old coordinates) V coincides with $\{(z, w) : z = a\}$ near (a, b). Since V is irreducible, V coincides with this line everywhere in U, and hence necessarily $a = 0$. This completes the proof of Proposition 3.1.

Q.E.D.

We can prove now our first main result.

Proof of Theorem 1.1.

By [4], f is smooth up to the boundary. Let V be an irreducible component of V_f and $p \in \partial V$ be a boundary point of V. It follows by [27] that there exists a neighborhood W of p in $\mathbb{C}^2$, a neighborhood U of 0 and a biholomorphic mapping $H : \Omega \cap W \longrightarrow \Omega' \cap U$ smooth up to the boundary $\partial\Omega$, $H(p) = 0$ such that $\Omega' \cap U$ is rigid; $\mathbf{T}$ acts on $\Omega' \cap U$ by translations $(z, w) \mapsto (z, w + t)$, $t \in \mathbb{R}$. In view of [2] we can assume that the mapping $f \circ H^{-1} : \Omega' \cap U \longrightarrow f(\Omega \cap V)$ is proper. Then it follows by Proposition 3.1 that $H(V \cap W) = \{(z, w) \in U : z = 0\}$. But then $V \cap W = H^{-1}(\{z = 0\})$ is a smooth manifold with boundary near p. This proves part (i) of Theorem 1.1.

Since the circle $\mathbf{T}$ acts (locally) by translation on Ω', we get that $(\partial V) \cap W$ coincides with the orbit $\mathbf{T}(p) \cap W$. By compactness of ∂V there exists a finite number of neighborhoods $W(p_j)$, $j = 1, \ldots, N$, $p_j \in \partial V$ such that $\partial V \subset \cup_1^N W(p_j)$ and for every j the intersection $(\partial V) \cap W(p_j)$ is equal to the orbit $\mathbf{T}(p_j) \cap W(p_j)$. Hence, ∂V is contained in a finite union of disjoint orbits.

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In order to show the inverse inclusion, we note that for any point p in $\partial\Omega$ its orbit $\mathbf{T}(p)$ is a smooth connected compact curve; since after an one-sided biholomorphic change of coordinates the $\mathbf{T}$ action is given by translations and therefore this curve can be transformed to a real line, we get that for any point a in $\mathbf{T}(p)$ there exists a neighborhood U such that $\mathbf{T}(p) \cap U$ is the boundary of a complex 1-dimensional manifold in $\Omega \cap U$. Since $\mathbf{T}(p)$ is compact, there exists a neighborhood W of $\mathbf{T}(p)$ such that $\mathbf{T}(p)$ is the boundary of a (connected) complex 1-dimensional manifold in $\Omega \cap W$ denoted by $\mathbf{T}(p)^{\mathbb{C}}$. If the intersection $\overline{V} \cap \mathbf{T}(p)$ contains a point a, then there exists a neighborhood U of a such that $V \cap U$ coincides with $\mathbf{T}(p)^{\mathbb{C}}$ on U. Since $\mathbf{T}(p)^{\mathbb{C}}$ is irreducible, it is contained in V by the uniqueness theorem. Hence , $\mathbf{T}(p)$ is contained in $\overline{V}$. This completes the proof of part (ii).

§4. Mappings from circular domains

Important special cases of domains with $\mathbf{T}$-action arise when the action is linear; classical examples are provided by circular domains. This section is devoted to the proof of our second main result Theorem 1.2. In what follows by a disc in $\mathbb{C}^2$ we mean a linear disc, i.e. the image of the unit disc in $\mathbb{C}$ under a linear mapping from $\mathbb{C}$ to $\mathbb{C}^2$; in particular, such a disc contains the origin.

LEMMA 4.1. *Let $f : \Omega \longrightarrow D$ be a proper holomorphic mapping between two smoothly bounded pseudoconvex finite type domains in $\mathbb{C}^2$. Suppose that Ω is a complete circular domain. Then the branch locus V_f is a finite union of discs.*

Proof. Since Ω is pseudoconvex, for every point $p \in \partial\Omega$ there exists a neighborhood U and a defining function ρ such that $-(-\rho)^\alpha$ is a plurisubharmonic function on $\Omega \cap U$ (shrinking U if necessarily, one can take $\alpha \in (0, 1)$ arbitrarily close to 1) [19]. Since Ω is a complete circular domain, every $\mathbf{T}$-orbit is a circle in the boundary which bounds a complex disc in Ω; it follows by the Hopf lemma that this disc is transversal to the boundary and hence the $\mathbf{T}$-action is transversal. Therefore, Theorem 1.1 implies that $\overline{V_f} \cap \partial\Omega$ is a finite union of circles. Then V_f coincides with the union of corresponding discs (say, by the maximum principle).

Q.E.D.

In the case of proper self-mappings the last proposition gives surprisingly strong corollaries which allow to prove our second main result.

Proof of Theorem 1.2.

Suppose that the branch locus V_f is not empty. The first step of the proof of Theorem 1.2 is the following

LEMMA 4.2. *The mapping f is polynomial homogeneous.*

Proof. First, we show that f is a polynomial mapping. We denote by $f^{(n)}$ the n-th iteration of f. It follows by Lemma 4.1 that for every n the branch locus $V_{f^{(n)}}$ is a finite union of discs .

We claim that there exists a sequence (L_n) of discs such that $L_n \subset V_{f^{(n)}}, L_{n+1} \subset f^{-1}(L_n)$.

We will construct the family (L_n) by induction. For every n we have $V_{f^{(n+1)}} = V_f \cup f^{-1}(V_{f^{(n)}})$. Fix any disc L_1 in V_f. Then $f^{-1}(L_1)$ is contained in $V_{f^{(2)}}$ and contains the disc L_2. Assume that the discs $L_1, \ldots, L_n$ are defined. Then $f^{-1}(L_n)$ is contained in $f^{-1}(V_{f^{(n)}}) \subset V_{f^{n+1}}$. So there exists a disc L_{n+1} such that $L_{n+1} \subset f^{-1}(L_n)$. Note that since every restriction $f : L_n \longrightarrow L_{n-1}$ is proper and $f(L_n) \subset L_{n-1}$, we have $f(L_n) = L_{n-1}$. We note that the discs (L_n) are distinct. Indeed, suppose by contradiction that m is the first integer such that there exists p with $L_m = L_{m+p}$. If $m \geq 2$, we have $f(L_m) = f(L_{m+p})$ and so $L_{m-1} = L_{m+p-1}$. This contradicts the definition of m. So $m = 1$.

Let $\tau_{\partial\Omega}(p)$ be the order of vanishing of the Levi determinant introduced in section 2. Since τ is invariant with respect to the $\mathbf{T}$-action, τ is constant on every ∂L_n. We denote it by τ_n. Since L_{n+1} is contained in $f^{-1}(L_n)$, the sequence $(\tau_n)_n$ is increasing . The domain Ω is of finite type, therefore the sequence (τ_n) is bounded , so it is a constant for n sufficiently large. Let n_0 be the first integer such that $\tau_n = \tau_{n_0}$ if $n \geq n_0$. Given $n \geq n_0$, for $z \in \partial L_{n+1}$ we have $\tau_{n+1} = \tau_{\partial\Omega}(z) \geq \tau_{\partial\Omega}(f(z)) = \tau_n = \tau_{n+1}$. Hence f is locally biholomorphic at z and z is not in V_f ; but then z cannot be in L_1. Thus $L_2 \neq L_n$ for any n large enough. This is a contradiction. So the discs L_n are different. In particular, since $f(L_{n+1}) = L_n$ is proper for every n we obtain that $f(0) = 0$. There exists a positive integer n_0 such that for every $n \geq n_0$ the sequence $(\tau_n)_n$ is constant and $L_n \cap V_f = \{0\}$ (since V_f is a finite union of discs). Consider a sequence of points $(p_n)_n$ in $\partial\Omega$ such that p_n is in L_n for every n. The restriction $f : L_{n+1} \rightarrow L_n$ is proper, $f(0) = 0$ and, since f is locally biholomorphic at $L_{n+1}\backslash\{0\}$, the

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derivative of the restriction $f_{|_{L_{n+1}}}$ cannot vanish at a point different of the origin. Then we have $f(\lambda p_{n+1}) = c_n \lambda^{k_n} p_n$ where $\lambda \in \mathbb{C} : |\lambda| < 1$, c_n is a constant, an integer k_n is smaller than the order of vanishing $or(f)$ of f at the origin. Fix $r > 0$ such that the ball $r\mathbb{B}$ is contained in Ω and consider the decomposition of f in homogeneous polynomials $f = \Sigma f_k$ on $r\mathbb{B}$ with $f_k(\lambda z) = \lambda^k f_k(z)$ for any z. Since $f(\lambda p_{n+1}) = c_n \lambda^{k_n} p_n$, we have $\sum \lambda^k f_k(p_{n+1}) = c_n \lambda^{k_n} p_n$ for every λ in a neighborhood of the origin in $\mathbb{C}$ independent of n; therefore $f_k(p_{n+1}) = 0$ for $k \geq or(f) + 1$ for any $n \geq n_0$. Thus, f_k vanishes on every L_n, $n \geq n_0$ for $k \geq or(f) + 1$. Since the lines L_n are different, every homogeneous polynomial f_k is zero. Thus, f is a polynomial of degree $\leq or(f)$.

Finally, let us show that f is homogeneous. For $k \leq or(f)$, let N_k denote the set of positive integer n such that $f(\lambda p_{n+1}) = c_n \lambda^k p_n$. There exists k_0 such that N_{k_0} is infinite. For every j different from k_0 we have $f_j(\lambda p_{n+1}) = 0$ for any $n \in \mathbb{N}_{k_0}$ and since N_{k_0} is infinite, we obtain that $f_j = 0$; hence $f = f_{k_0}$ and f is homogeneous.

Q.E.D.

The second basic step in our proof of Theorem 1.2 is an application of complex dynamics arguments. We refer the reader to [15, 23, 21] for standard definitions.

Since $f : \Omega \longrightarrow \Omega$ is proper and homogeneous, it follows that f is nondegenerate, i.e. $f^{-1}(0) = 0$. Let Ω_f denote the basin of attraction for f. Note that this is a complete circular domain (see also [23]).

LEMMA 4.3. *One has* $\Omega = \Omega_f$.

Proof. One can assume that $(f^{(k)})$ converges to F on Ω. For every $\lambda \in \mathbb{C}$, $|\lambda| < 1$ one has $f^{(k)}(\lambda z) \longrightarrow F(\lambda z)$ as $k \longrightarrow \infty$. But $f^{(k)}$ is homogeneous of degree d^k and $f^{(k)}(\lambda z) = \lambda^{d^k} f^{(k)}(z)$ which converges to 0. Hence, $F = 0$ on Ω and $\Omega \subset \Omega_f$. But f is proper and $f(\partial\Omega) \subset \partial\Omega$. Hence, Ω_f is contained in Ω.

Q.E.D.

In order to prove Theorem 1.2 , it suffices to prove that V_f is empty ([26]). Suppose by contradiction that it is not so. Then as it was just shown, f is a homogeneous polynomial, which is not linear.

We denote by $\pi : \mathbb{C}^2 \backslash \{0\} \longrightarrow \mathbb{C}P$ the canonical projection. Since f is nondegenerate, it takes lines to lines in $\mathbb{C}^2$ and naturally induces a rational mapping $\varphi : \mathbb{C}P \longrightarrow \mathbb{C}P$ on the projective space. We claim that its Julia set J_φ does not coincide with $\mathbb{C}P$. For the proof we apply an argument of [14]. Suppose by contradiction it does. This is known (see [15], pp.56-58) that in this case for every point $a \in J_\varphi$ there exists a neighborhood U and a positive integer n such that $\cup_{k=1}^n \varphi^{(k)}(U)$ covers $\mathbb{C}P$. Take a such that $\pi^{-1}(a)$ contains a strictly pseudoconvex point p in $\partial\Omega$. Then there exists a neighborhood W of p in $\mathbb{C}^2$ such that $\cup_{k=1}^n f^{(k)}(W)$ covers $\partial\Omega$. Since f takes any strictly pseudoconvex point to a strictly pseudoconvex one, we get that Ω is strictly pseudoconvex and by [26] V_f is empty: a contradiction.

Thus, J_φ is different from $\mathbb{C}P$. But then by the classical results J_φ is a closed subset of $\mathbb{C}P$ with empty interior ([15], Theorem 1.9). Therefore $\partial\Omega \backslash \pi^{-1}(J_\varphi)$ is a nonempty open subset of $\partial\Omega$ which in view of [23], Proposition 7.1, is foliated by Riemann surfaces: this is impossible since Ω is of finite type. This completes the proof of the theorem.

In conclusion we note that if Ω is a circular , but not complete circular domain, then the circled action in general is not transversal as shows the domain $D = (|z|^2 - 1)^2 + (|w|^2 - 1)^2 < \varepsilon$ for $\varepsilon > 0$ small enough. But if the action is transversal, the former proof is still valid with slight modifications (one has consider proper holomorphic mappings of annuli in linear sections).

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