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Γ-convergence and stochastic homogenization of degenerate integral functionals in weighted Sobolev spaces

Chiara D'Onofrio and Caterina Ida Zeppieri
Angewandte Mathematik, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany (chiara.donofrio@uni-muenster.de, caterina.zeppieri@uni-muenster.de)

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We study the Γ-convergence of nonconvex vectorial integral functionals whose integrands satisfy possibly degenerate growth and coercivity conditions. The latter involve suitable scale-dependent weight functions. We prove that under appropriate uniform integrability conditions on the weight functions, which shall belong to a Muckenhoupt class, the corresponding functionals Γ-converge, up to subsequences, to a degenerate integral functional defined on a limit weighted Sobolev space. The general analysis is then applied to the case of random stationary integrands and weights to prove a stochastic homogenization result for the corresponding functionals.

Keywords: Γ-convergence; degenerate growth conditions; Muckenhoupt weights; weighted Sobolev spaces; stochastic homogenization

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1. Introduction

In this paper, we study the effective behaviour of scale-dependent integral functionals with possibly *degenerate* integrands. Functionals of this kind typically model the energy of a heterogeneous material whose physical properties (elastic, thermal, electrical, etc.) may both deteriorate and vary significantly from point to point, on a mesoscopic scale.

The energy functionals we consider are of the form

$$F_k(u) = \int_A f_k(x, \nabla u)\,\mathrm{d}x, \tag{1.1}$$

where $A \subset \mathbb{R}^n$ is an open, bounded, Lipschitz set, $k \in \mathbb{N}$ is a parameter related to some material property (e.g. the size of the microstructure) and $u\colon A \to \mathbb{R}^m$ represents a physical variable (e.g. the elastic deformation of the body).

The degeneracy of the integrands $f_k\colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0, +\infty)$ is expressed in terms of growth and coercivity conditions which can depend both on the parameter

k and on the spatial variable x. These are given by introducing weight functions $\lambda_k : \mathbb{R}^n \to [0, +\infty)$ which modulate the typical superlinear growth in the gradient variable. That is, for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m\times n}$, and $k \in \mathbb{N}$ the integrands f_k satisfy

$$\alpha\lambda_k(x)(|\xi|^p - 1) \leqslant f_k(x, \xi) \leqslant \beta\lambda_k(x)(|\xi|^p + 1), \tag{1.2}$$

where $p > 1$, and $0 < \alpha \leqslant \beta < +\infty$.

If the weight functions λ_k are bounded in L^∞ uniformly in k, then (1.2) reduces to the standard growth and coercivity of order $p > 1$. In this case, the limit behaviour of F_k is well understood and can be described using the language of Γ-convergence. Namely, if $k \to \infty$, the functionals F_k Γ-converge (up to subsequences), on $W^{1,p}(A; \mathbb{R}^m)$, to an integral functional of the form

$$F(u) = \int_A f_0(x, \nabla u)\,\mathrm{d}x, \quad u \in W^{1,p}(A; \mathbb{R}^m), \tag{1.3}$$

with f_0 satisfying the same (nondegenerate) growth conditions satisfied by f_k (see [**9**]). Moreover, if $\varepsilon_k \to 0^+$ and $f_k(x, \xi) = f(x/\varepsilon_k, \xi)$ for some nondegenerate f, then the limit integrand f_0 is x-independent and subsequence-independent both in the periodic [**6**, **30**] and in the stationary random case [**15**, **16**, **28**], and given by a so-called homogenization formula. As a result, in this case, the whole sequence (F_k) Γ-converges to F.

In this paper, we consider sequences of weight functions (λ_k) which are not bounded in general. Specifically, for every $k \in \mathbb{N}$ we assume that

$$\lambda_k, \lambda_k^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \tag{1.4}$$

moreover, we additionally require the existence of a constant $K \geqslant 1$ such that for every $k \in \mathbb{N}$ there holds

$$\left(\fint_Q \lambda_k\,\mathrm{d}x\right)\left(\fint_Q \lambda_k^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1} \leqslant K, \tag{1.5}$$

for every cube $Q \subset \mathbb{R}^n$. The uniform integrability condition (1.5) is known as Muckenhoupt condition and the functions satisfying it are referred to as Muckenhoupt $A_p(K)$-weights [**29**].

In this case, the growth conditions (1.2) satisfied by f_k naturally set the problem in the parameter-dependent weighted Sobolev space $W^{1,p}_{\lambda_k}(A; \mathbb{R}^m)$ where, for a given $A_p(K)$-weight λ we have

$$W^{1,p}_\lambda(A; \mathbb{R}^m) = \left\{u \in W^{1,1}(A; \mathbb{R}^m) : \int_A \lambda|u|^p\,\mathrm{d}x + \int_A \lambda|\nabla u|^p\,\mathrm{d}x < +\infty\right\}.$$

The limit behaviour of functionals F_k with integrands satisfying (1.2) was studied for the first time in [**10**], in the convex, scalar case and under the sole integrability condition (1.4). Assuming that λ_k converges weakly to some λ_∞ in L^1, in [**10**] the authors proved a Γ-convergence and integral representation result for the Γ-limit of F_k, on the space of Lipschitz functions. The latter, though, in general is smaller than the domain of the Γ-limit. Moreover, in the setting considered in

[10] the functionals F_k are not equi-coercive and therefore a convergence result for the associated minimization problems cannot be derived from the Γ-convergence analysis.

In order to extend the Γ-convergence result in [10] to the domain of the Γ-limit and to gain compactness, in [17] the Muckenhoupt condition (1.5) was also required together with the additional bound

$$c_1 \leqslant \fint_{Q_0} \lambda_k \, \mathrm{d}x \leqslant c_2, \tag{1.6}$$

where $0 < c_1 \leqslant c_2 < +\infty$ and $Q_0 \subset \mathbb{R}^n$ is a given cube. The Muckenhoupt condition (1.5) guarantees the continuous embedding of $W^{1,p}_{\lambda_k}(A)$ in the Sobolev space $W^{1,1+\delta}(A)$, for some $\delta > 0$. Then, a combination of (1.5) and (1.6) ensures that sequences with equi-bounded energy are bounded in $W^{1,1+\delta}(A)$, and hence pre-compact in $L^1(A)$ (whenever $A \subset Q_0$). Therefore, in the setting considered in [17] the equi-coerciveness of the functionals F_k can be recovered. Moreover, again thanks to (1.5)–(1.6) a lower bound on the Γ-limit can be established, which shows that its domain is the weighted Sobolev space $W^{1,p}_{\lambda_\infty}(A)$, where λ_∞ belongs to a Muckenhoupt class and is the weak L^1-limit of (a subsequence of) λ_k.

Besides the contributions [10, 17], Γ-convergence and relaxation results for functionals of type (1.1)–(1.2) defined on weighted Sobolev spaces were also established in [3, 11, 18, 19, 21, 22, 31] without departing, though, from the convex/monotone operator, scalar setting, with the only exception of [31]. More specifically, in [31] the authors proved a stochastic homogenization result for a sequence of discrete nonconvex, vectorial energy functionals with degenerate integrands. Under suitable assumptions on the random weights, which are weaker than (1.5) in the scalar case but not really comparable to (1.5) in the vectorial case, the authors showed that in the stationary ergodic case the energies homogenize to a nondegenerate deterministic integral functional. We observe that the case of homogenization is somehow special since in this case the limit functional is always nondegenerate and thus defined on the space $W^{1,p}$.

In the present paper, we extend the analysis in [17] to the nonconvex, vectorial setting, without assuming any periodicity or stationarity of the integrands f_k. Namely, we assume that f_k satisfies (1.2), together with some mild continuity condition in ξ (cf. (3.3)), and that the weight functions λ_k are as in (1.4)–(1.6). Under these assumptions we show the existence of a subsequence (k_h), a limit Muckenhoupt weight λ_∞, with $\lambda_{k_h} \rightharpoonup \lambda_\infty$ in $L^1(Q_0)$, and a *degenerate* integrand f_∞ satisfying

$$\alpha\lambda_\infty(x)\left(\frac{1}{K}|\xi|^p - 1\right) \leqslant f_\infty(x,\xi) \leqslant \beta\lambda_\infty(x)(|\xi|^p + 1), \tag{1.7}$$

a.e. in Q_0 and for every $\xi \in \mathbb{R}^{m\times n}$, such that the functionals F_{k_h} Γ-converge, with respect to the strong $L^1(A;\mathbb{R}^m)$-convergence, to the integral functional

$$F_\infty(u) = \int_A f_\infty(x, \nabla u)\, \mathrm{d}x, \quad u \in W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m).$$

We also show that the Γ-convergence holds true, with the same subsequence (k_h), for every open, bounded, Lipschitz set $A \subset \mathbb{R}^n$, with $A \subset\subset Q_0$. Moreover, we derive an

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asymptotic formula for the limit integrand f_∞ which can be expressed as a (double) limit of sequences of scaled minimization problems as follows:

$$f_\infty(x,\xi) := \limsup_{\rho\to 0^+} \lim_{h\to\infty} \frac{1}{\rho^n} \inf\left\{ \int_{Q_\rho(x)} f_{k_h}(y,\nabla u+\xi)\,\mathrm{d}y\colon u\in W^{1,p}_{0,\lambda_{k_h}}(Q_\rho(x);\mathbb{R}^m)\right\}, \tag{1.8}$$

where $Q_\rho(x)\subset\mathbb{R}^n$ denotes the cube centred in x with side-length $\rho>0$, and

$$W^{1,p}_{0,\lambda_{k_h}}(Q_\rho(x);\mathbb{R}^m) = W^{1,1}_0(Q_\rho(x);\mathbb{R}^m)\cap W^{1,p}_{\lambda_{k_h}}(Q_\rho(x);\mathbb{R}^m).$$

The proof of this result is carried out in a number of intermediate steps. Namely, we first prove the Γ-convergence and integral representation result on the space $W^{1,\infty}(A;\mathbb{R}^m)\subsetneq W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m)$. To do so, we use the localization method of Γ-convergence and adapt the approach in [**7**, **14**] to our setting to get an integral representation result for functionals with degenerate integrands. We remark here that the most delicate part in the implementation of the localization method is the proof of the subadditivity of the Γ-limit, which requires to combine a fundamental estimate for the functionals F_k together with an *ad hoc* vectorial truncation argument, in the same spirit as, e.g. [**8**, lemma 3.5]. We then extend the Γ-convergence and integral representation result to the limit weighted Sobolev space $W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m)$. The latter coincides with the domain of F_∞, thanks to (1.7); hence we get a complete description of the Γ-limit of F_{k_h}. The passage from $W^{1,\infty}(A;\mathbb{R}^m)$ to $W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m)$ is performed by resorting to classical approximation argument (see [**1**, theorem II.4]) which exploits the property of the maximal function in relation to the Muckenhoupt weights. More precisely, we can adapt [**17**, theorem 3.1] to the vectorial setting to show that in the liminf inequality, we can replace a sequence (u_k), with $u_k\to u$ in $L^1(A;\mathbb{R}^m)$ and equi-bounded $W^{1,p}_{\lambda_k}(A;\mathbb{R}^m)$-norm, with a sequence of Lipschitz functions converging to a $W^{1,\infty}(A;\mathbb{R}^m)$-function which differs from u on a set with vanishing measure. Eventually, the asymptotic formula for f_∞ is obtained by combining a convergence result for minimization problems with prescribed Dirichlet conditions together with a derivation formula for f_∞ which is obtained by extending to the weighted Sobolev setting the method developed in [**4**, **5**].

Finally, the general Γ-convergence analysis is complemented by an application to the case of stationary random weights and integrands, thus generalizing the classical stochastic homogenization result in [**15**, **16**, **28**] to the degenerate setting.

That is, we specialize our general result to the choice

$$\lambda_k(\omega,x)=\lambda\left(\omega,\frac{x}{\varepsilon_k}\right),\quad f_k(\omega,x,\xi)=f\left(\omega,\frac{x}{\varepsilon_k},\xi\right),$$

where ω belongs to the sample space of a given probability space $(\Omega,\,\mathcal{F},\,P)$, λ is a random Muckenhoupt weight (cf. assumption 8.5), and f is a *degenerate stationary random integrand* (cf. definition 8.7). Then, following the same approach as in [**16**], we combine the deterministic analysis and the subadditive ergodic theorem

[2, theorem 2.9] to show that, almost surely, the random functionals

$$F_k(\omega)(u) = \int_A f\left(\omega, \frac{x}{\varepsilon_k}, \nabla u\right) \mathrm{d}x, \quad u \in W^{1,p}_{\lambda_k}(A;\mathbb{R}^m)$$

Γ-converge to a *nondegenerate* (spatially) homogeneous random functional

$$F_{\text{hom}}(\omega)(u) = \int_A f_{\text{hom}}(\omega, \nabla u)\, \mathrm{d}x \quad u \in W^{1,p}(A;\mathbb{R}^m),$$

where f_{hom} satisfies standard growth conditions of order $p > 1$ with *random coefficients* (cf. (8.10)) and is given by the following asymptotic cell formula

$$f_{\text{hom}}(\omega, \xi) = \lim_{t\to\infty} \frac{1}{t^n} \inf\left\{\int_{Q_t(0)} f(\omega, x, \nabla u + \xi)\, \mathrm{d}x \colon u \in W^{1,p}_{0,\lambda}(Q_t(0);\mathbb{R}^m)\right\}. \tag{1.9}$$

If, moreover, λ and f are ergodic, we show that f_{hom} is deterministic and given by the expected value of the right hand side of (1.9). Furthermore, in the ergodic case f_{hom} satisfies the following *deterministic* growth and coercivity conditions of order $p > 1$:

$$\alpha \left(\int_\Omega \lambda(\omega, 0)^{-1/(p-1)}\, \mathrm{d}P\right)^{1-p} (|\xi|^p - 1) \leqslant f_{\text{hom}}(\xi) \leqslant \beta \left(\int_\Omega \lambda(\omega, 0)\, \mathrm{d}P\right) (|\xi|^p + 1),$$

for every $\xi \in \mathbb{R}^{m\times n}$.

Outline of the paper. The paper is organized as follows. In § 2 we recall the notions of Muckenhoupt classes and weights and of weighted Sobolev spaces. Moreover, we recall here some well-known related results which will be used throughout. In § 3 we introduce the functionals we study and state the main result of this paper, theorem 3.2. The proof of theorem 3.2 is then carried out in § 4–7. Namely, in § 4 we prove a Γ-convergence and integral representation result in the space $W^{1,\infty}$, theorem 4.1. In § 5 we establish theorem 5.2 which extends the results in theorem 4.1 to the weighted Sobolev space $W^{1,p}_{\lambda_\infty}$, also showing that the latter coincides with the domain of the Γ-limit. On account of theorem 5.2, in § 6 we prove that in this setting Γ-convergence is stable under the addition of Dirichlet boundary conditions and we derive a convergence result for the associated minimization problems. In § 7 we prove a derivation formula for the integrand of the Γ-limit, theorem 7.1 (see also corollary 7.2). Eventually, in § 8 we prove a stochastic homogenization result for stationary random weights and integrands, theorem 8.12.

2. Preliminaries

In this section, we collect some useful definitions and preliminary results which will be used throughout.

2.1. Muckenhoupt classes

We start by recalling the definition of the so-called Muckenhoupt classes. An introduction to the theory of Muckenhoupt classes can be found in [**24**].

Definition 2.1. *Let $p > 1$ and $K \geqslant 1$. The Muckenhoupt class $A_p(K)$ is defined as the collection of all nonnegative functions $\lambda\colon \mathbb{R}^n \to [0, +\infty)$, with λ, $\lambda^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, such that*

$$\left(\fint_Q \lambda \,\mathrm{d}x\right)\left(\fint_Q \lambda^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1} \leqslant K,$$

for every cube $Q \subset \mathbb{R}^n$ with faces parallel to the coordinate hyperplanes.

Moreover, we set $A_p := \bigcup_{K\geqslant 1} A_p(K)$.

The elements of the class A_p (resp. $A_p(K)$) are usually referred to as A_p-weights (resp. $A_p(K)$-weights). Simple examples of A_p-weights are radially symmetric functions of the type

$$\lambda(x) = |x|^\gamma \quad \text{for} \quad -n < \gamma < n(p-1).$$

Further examples can be found, e.g. in [**25**].

We recall the following 'reverse Hölder inequality' which holds for functions in A_p and whose proof can be found in [**13**, theorem IV].

Theorem 2.2. *Let $p > 1$ and $K \geqslant 1$. Then there exist an exponent $\sigma = \sigma(K, p, n) > 0$ and a constant $c = c(K, p, n) > 0$ such that*

$$\left(\fint_Q \lambda^{1+\sigma}\,\mathrm{d}x\right)^{1/(1+\sigma)} \leqslant c\left(\fint_Q \lambda\,\mathrm{d}x\right), \tag{2.1}$$

$$\left(\fint_Q \lambda^{-(1+\sigma)/(p-1)}\,\mathrm{d}x\right)^{1/(1+\sigma)} \leqslant c\left(\fint_Q \lambda^{-1/(p-1)}\,\mathrm{d}x\right), \tag{2.2}$$

for every cube Q and for every $\lambda \in A_p(K)$.

Remark 2.3. We observe that since $\lambda^{-1/(p-1)} \in A_{p'}(K)$ with $p' := p/(p-1)$, then inequality (2.2) can be obtained by applying (2.1) to the weight $\lambda^{-1/(p-1)}$.

In this paper, we will deal with sequences of $A_p(K)$-weights. The following result is a consequence of theorem 2.2 and its proof can be found in [**17**, proposition 4.1].

Proposition 2.4. *Let $K \geqslant 1$, $p > 1$, and let (λ_k) be a sequence of functions in $A_p(K)$. Let $Q_0 \subset \mathbb{R}^n$ be a cube and assume that there exist two constants c_1, c_2*

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with $0 < c_1 \leqslant c_2$ *such that*

$$c_1 \leqslant \fint_{Q_0} \lambda_k \,\mathrm{d}x \leqslant c_2, \tag{2.3}$$

for every $k \in \mathbb{N}$*. Then there exist a subsequence* $(\lambda_{k_h}) \subset (\lambda_k)$*, a constant* $c_3 = c_3(n)$*, depending only on* n*, and functions* λ_∞ *and* $\tilde{\lambda}_\infty$ *in* $A_p(c_3^p K)$ *such that*

$$\lambda_{k_h} \rightharpoonup \lambda_\infty \quad \textit{in} \quad L^{1+\sigma}(Q_0) \tag{2.4}$$

and

$$\lambda_{k_h}^{-1/(p-1)} \rightharpoonup \tilde{\lambda}_\infty^{-1/(p-1)} \quad \textit{in} \quad L^{1+\sigma}(Q_0), \tag{2.5}$$

for some $\sigma > 0$*. Moreover, there holds*

$$\tilde{\lambda}_\infty \leqslant \lambda_\infty \leqslant K\tilde{\lambda}_\infty, \tag{2.6}$$

a.e. in Q_0*.*

If (2.3) *is replaced by the stronger condition*

$$0 < \liminf_{k\to\infty} \fint_{Q} \lambda_k \,\mathrm{d}x, \quad \limsup_{k\to\infty} \fint_{Q} \lambda_k \,\mathrm{d}x < +\infty \textit{ for every cube } Q \subset \mathbb{R}^n,$$

then (2.4) *and* (2.5) *holds true for every cube* $Q \subset \mathbb{R}^n$*,* (2.6) *holds a.e. in* $\mathbb{R}^n$*, and* $\lambda_\infty \in A_p(K)$*.*

The equi-integrability estimate below is another immediate consequence of theorem 2.2.

Proposition 2.5. *Let* $p > 1$*,* $K \geqslant 1$*, and let* (λ_k) *be a sequence of functions in* $A_p(K)$ *satisfying* (2.3)*. Then there exist* $\sigma = \sigma(K, p, n) > 0$ *and* $c = c(K, p, n) > 0$ *such that*

$$\int_E \lambda_k \,\mathrm{d}x \leqslant c\, c_2 |Q_0| \left(\frac{|E|}{|Q_0|}\right)^{\sigma/(1+\sigma)},$$

for every measurable set $E \subset Q_0$ *and every* $k \in \mathbb{N}$*.*

Proof. Let $\sigma > 0$ and $c > 0$ be the constants given by theorem 2.2. By (2.1) and (2.3) we get

$$\left(\fint_{Q_0} \lambda_k^{1+\sigma} \,\mathrm{d}x\right)^{1/(1+\sigma)} \leqslant c \left(\fint_{Q_0} \lambda_k \,\mathrm{d}x\right) \leqslant c\, c_2,$$

for every $k \in \mathbb{N}$. Therefore, the Hölder inequality easily gives

$$\begin{aligned}\int_E \lambda_k \,\mathrm{d}x &\leqslant |E|^{\sigma/(1+\sigma)} |Q_0|^{1/(1+\sigma)} \left(\fint_{Q_0} \lambda_k^{1+\sigma} \,\mathrm{d}x\right)^{1/(1+\sigma)} \\ &\leqslant c\, c_2 |E|^{\sigma/(1+\sigma)} |Q_0|^{1/(1+\sigma)} = c\, c_2 |Q_0| \left(\frac{|E|}{|Q_0|}\right)^{\sigma/(1+\sigma)}.\end{aligned}$$

□

2.2. Weighted Sobolev spaces

In this short subsection, we recall the definition and the basic properties of weighted Sobolev spaces. For a comprehensive treatment of this subject we refer the reader to the monographs [**25**, **34**]. For further relevant results concerning weighted Sobolev spaces, we will provide a precise reference to the literature whenever these results are used in the paper.

Let $p > 1$, let $\lambda \in A_p$. In all that follows $A \subset \mathbb{R}^n$ denotes an open and bounded set with Lipschitz boundary. Let $m \in \mathbb{N}$, $m \geqslant 1$; we define the weighted Lebesgue space

$$L^p_\lambda(A;\mathbb{R}^m) := \left\{ u \in L^1(A;\mathbb{R}^m) \colon \int_A \lambda |u|^p \,\mathrm{d}x < +\infty \right\};$$

we recall that $L^p_\lambda(A;\mathbb{R}^m)$ equipped with the norm

$$\|u\|_{L^p_\lambda(A;\mathbb{R}^m)} := \left(\int_A \lambda |u|^p \,\mathrm{d}x \right)^{1/p}$$

is a reflexive Banach space. Moreover, we define the weighted Sobolev space $W^{1,p}_\lambda(A;\mathbb{R}^m)$ as

$$W^{1,p}_\lambda(A;\mathbb{R}^m) := \left\{ u \in W^{1,1}(A;\mathbb{R}^m) \cap L^p_\lambda(A;\mathbb{R}^m) \colon \int_A \lambda |\nabla u|^p \,\mathrm{d}x < +\infty \right\},$$

the latter is also a reflexive Banach space when endowed with the norm

$$\|u\|_{W^{1,p}_\lambda(A;\mathbb{R}^m)} := \left(\int_A \lambda |u|^p \,\mathrm{d}x + \int_A \lambda |\nabla u|^p \,\mathrm{d}x \right)^{1/p}.$$

We recall that the embedding of $W^{1,p}_\lambda(A;\mathbb{R}^m)$ in $L^p_\lambda(A;\mathbb{R}^m)$ is compact (see, e.g. [**23**, lemma 1]). Furthermore, we have the following continuous embeddings:

$$\begin{aligned} &L^\infty(A;\mathbb{R}^m) \hookrightarrow L^p_\lambda(A;\mathbb{R}^m) \hookrightarrow L^{1+\delta}(A;\mathbb{R}^m), \\ &W^{1,\infty}(A;\mathbb{R}^m) \hookrightarrow W^{1,p}_\lambda(A;\mathbb{R}^m) \hookrightarrow W^{1,1+\delta}(A;\mathbb{R}^m), \end{aligned}$$

for some $\delta > 0$.

Throughout the paper, we will also use the fact that $C^\infty(\overline{A};\mathbb{R}^m)$ is dense in $W^{1,p}_\lambda(A;\mathbb{R}^m)$ (see, e.g. [**34**, corollary 2.1.6]).

The following characterization of $W^{1,p}_\lambda(A;\mathbb{R}^m)$ will be useful for our purposes.

Proposition 2.6. *Let $p > 1$, $\lambda \in A_p$, and let $A \subset \mathbb{R}^n$ be open, bounded, and with Lipschitz boundary. Define*

$$\widehat{W}^{1,p}_\lambda(A;\mathbb{R}^m) := \left\{ u \in W^{1,1}(A;\mathbb{R}^m) \colon \int_A \lambda |\nabla u|^p \,\mathrm{d}x < +\infty \right\},$$

then $\widehat{W}^{1,p}_\lambda(A;\mathbb{R}^m) = W^{1,p}_\lambda(A;\mathbb{R}^m)$.

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Proof. The inclusion $W^{1,p}_\lambda(A;\mathbb{R}^m) \subset \widehat{W}^{1,p}_\lambda(A;\mathbb{R}^m)$ is obvious, therefore we only need to show that

$$\widehat{W}^{1,p}_\lambda(A;\mathbb{R}^m) \subset W^{1,p}_\lambda(A;\mathbb{R}^m). \tag{2.7}$$

To prove (2.7) we will establish the following Poincaré-type inequality: there exists $C > 0$ such that

$$\|u\|_{L^p_\lambda(A;\mathbb{R}^m)} \leqslant C\big(\|u\|_{L^1(A;\mathbb{R}^m)} + \|\nabla u\|_{L^p_\lambda(A;\mathbb{R}^{m\times n})}\big), \tag{2.8}$$

for every $u \in C^\infty(\overline{A};\mathbb{R}^m)$.

We will obtain (2.8) arguing by contradiction. Were (2.8) false, then for every $j \in \mathbb{N}$ there would exist $u_j \in C^\infty(\overline{A};\mathbb{R}^m)$ such that

$$\|u_j\|_{L^p_\lambda(A;\mathbb{R}^m)} > j\big(\|u_j\|_{L^1(A;\mathbb{R}^m)} + \|\nabla u_j\|_{L^p_\lambda(A;\mathbb{R}^{m\times n})}\big).$$

Define the renormalized functions $v_j \in C^\infty(\overline{A};\mathbb{R}^m)$ as

$$v_j := \frac{u_j}{\|u_j\|_{L^p_\lambda(A;\mathbb{R}^m)}}, \quad \text{for every } j \in \mathbb{N}.$$

Then,

$$\|v_j\|_{L^p_\lambda(A;\mathbb{R}^m)} = 1 \quad \text{and} \quad \|v_j\|_{L^1(A;\mathbb{R}^m)} + \|\nabla v_j\|_{L^p_\lambda(A;\mathbb{R}^{m\times n})} < \frac{1}{j}, \tag{2.9}$$

for every $j \in \mathbb{N}$. Hence, in particular, the sequence (v_j) is bounded in $W^{1,p}_\lambda(A;\mathbb{R}^m)$. Therefore, by the compact embedding of $W^{1,p}_\lambda(A;\mathbb{R}^m)$ in $L^p_\lambda(A;\mathbb{R}^m)$, up to subsequences (not relabelled), $v_j \to v$ in $L^p_\lambda(A;\mathbb{R}^m)$, for some $v \in L^p_\lambda(A;\mathbb{R}^m)$. Moreover, since the embedding of $L^p_\lambda(A;\mathbb{R}^m)$ in $L^1(A;\mathbb{R}^m)$ is continuous, we also have $v_j \to v$ in $L^1(A;\mathbb{R}^m)$. Therefore, (2.9) entails both $\|v\|_{L^p_\lambda(A;\mathbb{R}^m)} = 1$ and $v = 0$ a.e. in A and hence a contradiction.

Now let $u \in \widehat{W}^{1,p}_\lambda(A;\mathbb{R}^m)$; by [**11**, proposition 3.5] (see also [**12**, theorem 6.1]) there exists $(u_j) \subset C^\infty(\overline{A};\mathbb{R}^m)$ such that

$$\|u_j - u\|_{L^1(A;\mathbb{R}^m)} + \|\nabla u_j - \nabla u\|_{L^p_\lambda(A;\mathbb{R}^{m\times n})} \to 0 \quad \text{as } j \to \infty.$$

Moreover, in view of (2.8) the sequence (u_j) is bounded in $W^{1,p}_\lambda(A;\mathbb{R}^m)$, therefore again appealing to the compact embedding of $W^{1,p}_\lambda(A;\mathbb{R}^m)$ in $L^p_\lambda(A;\mathbb{R}^m)$ we deduce that $u_j \to u$ in $L^p_\lambda(A;\mathbb{R}^m)$ and hence $u \in W^{1,p}_\lambda(A;\mathbb{R}^m)$, as desired. ☐

REMARK 2.7. We note that by the density of $C^\infty(\overline{A};\mathbb{R}^m)$ in $W^{1,p}_\lambda(A;\mathbb{R}^m)$ inequality (2.8) actually holds in the whole space $W^{1,p}_\lambda(A;\mathbb{R}^m)$. That is, there exists a constant $C > 0$ such that

$$\|u\|_{L^p_\lambda(A;\mathbb{R}^m)} \leqslant C\big(\|u\|_{L^1(A;\mathbb{R}^m)} + \|\nabla u\|_{L^p_\lambda(A;\mathbb{R}^{m\times n})}\big), \tag{2.10}$$

for every $u \in W^{1,p}_\lambda(A;\mathbb{R}^m)$.

Finally, in this paper, we will also consider the space

$$W^{1,p}_{0,\lambda}(A;\mathbb{R}^m) := W^{1,1}_0(A;\mathbb{R}^m) \cap W^{1,p}_\lambda(A;\mathbb{R}^m).$$

We recall that $W^{1,p}_{0,\lambda}(A;\mathbb{R}^m)$ agrees with the closure of $C^\infty_0(A;\mathbb{R}^m)$ in $W^{1,p}_\lambda(A;\mathbb{R}^m)$ (see, e.g. [**32**, theorem 1.4] or [**17**, proposition 2.1]).

2.3. Maximal function and measure theory

In this subsection, we recall the definition of maximal function and some of its properties which are useful for our purposes. Moreover, for the readers' convenience we also recall some classical result in measure theory which we are going to employ in the paper.

For the theory of maximal functions we refer to [**33**].

Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the Hardy maximal function of u at x is defined as

$$(Mu)(x) := \sup_{r>0} \fint_{Q_r(x)} |u| \,\mathrm{d}y$$

where $Q_r(x)$ is the cube centred at x, with side length r and faces parallel to the coordinate planes. The following property will be useful for our purposes: there exists a constant $\tilde{c} = \tilde{c}(n) > 0$ depending only on n such that

$$|\{x \in \mathbb{R}^n : (Mu)(x) \geqslant l\}| \leqslant \frac{\tilde{c}(n)}{l} \|u\|_{L^1(\mathbb{R}^n)}, \tag{2.11}$$

for every $u \in L^1(\mathbb{R}^n)$ and every $l > 0$.

The following result is proven in [**29**, theorem 9].

THEOREM 2.8. *Let $p > 1$, $K \geqslant 1$, and let $\lambda \in A_p(K)$. Then there exists a constant $c_4 = c_4(K, p, n) > 0$ such that*

$$\int_{\mathbb{R}^n} \lambda |Mu|^p \,\mathrm{d}x \leqslant c_4 \int_{\mathbb{R}^n} \lambda |u|^p \,\mathrm{d}x,$$

for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We observe that theorem 2.8 in particular implies that if $u \in L^p_\lambda(\mathbb{R}^n)$ then $Mu \in L^p_\lambda(\mathbb{R}^n)$.

For the following lemma we refer to [**1**, lemma I.11].

LEMMA 2.9. *Let $u \in C^\infty_0(\mathbb{R}^n)$ and let $l > 0$. Set*

$$H^l := \{x \in \mathbb{R}^n : (M|\nabla u|)(x) < l\}.$$

Then u is Lipschitz continuous in H^l; i.e. there exists a constant $c_5 = c_5(n) > 0$ such that

$$|u(x) - u(y)| \leqslant c_5\, l |x - y|,$$

for every $x, y \in H^l$.

We recall the following result which can be found in [20].

LEMMA 2.10. *Let $G \subset \mathbb{R}^n$ be measurable with $|G| < +\infty$. Let (E_h) be a sequence of measurable subsets of G such that $|E_h| \geqslant \tau$ for every $h \in \mathbb{N}$ and for some $\tau > 0$. Then there exists a subsequence $(E_{h_j}) \subset (E_h)$ such that $\bigcap_{j\in\mathbb{N}} E_{h_j} \neq \emptyset$.*

Eventually, we state the following technical lemma whose proof can be found in [1, lemma I.7].

LEMMA 2.11. *Let (ϕ_h) be a bounded sequence in $L^1(\mathbb{R}^n)$. Then for every $\tau > 0$ there exist a measurable set E_τ with $|E_\tau| < \tau$, $\delta_\tau > 0$, and a sequence (h_j^τ) such that for every $j \in \mathbb{N}$*

$$\int_B |\phi_{h_j^\tau}|\,\mathrm{d}x < \tau,$$

for every measurable set B such that $B \cap E_\tau = \emptyset$ and $|B| < \delta_\tau$.

3. Setting of the problem and statement of the main result

In this section, we introduce the functionals we are going to study and state the main result of the paper.

ASSUMPTION 3.1 (Admissible weights). Let $p > 1$, $K \geqslant 1$, and let $A_p(K)$ denote the Muckenhoupt class as in definition 2.1. A sequence of measurable weight functions $\lambda_k\colon \mathbb{R}^n \to [0, +\infty)$ is admissible if:

- $\lambda_k \in A_p(K)$, for every $k \in \mathbb{N}$;
- there exists a cube $Q_0 \subset \mathbb{R}^n$ such that for every $k \in \mathbb{N}$ there holds

$$c_1 \leqslant \fint_{Q_0} \lambda_k\,\mathrm{d}x \leqslant c_2, \tag{3.1}$$

  for some constants $0 < c_1 \leqslant c_2 < +\infty$.

Let (λ_k) be a sequence of weights satisfying assumption 3.1; in this paper, we consider Borel integrands $f_k : \mathbb{R}^n \times \mathbb{R}^{m\times n} \to [0, +\infty)$ satisfying the two following conditions:

(1) (degenerate growth conditions) there exist two constants $0 < \alpha \leqslant \beta < +\infty$ such that for almost every $x \in \mathbb{R}^n$

$$\alpha\lambda_k(x)(|\xi|^p - 1) \leqslant f_k(x,\xi) \leqslant \beta\lambda_k(x)(|\xi|^p + 1), \tag{3.2}$$

for every $\xi \in \mathbb{R}^{m\times n}$ and every $k \in \mathbb{N}$;

(2) (continuity in ξ) there exists $L > 0$ such that for almost every $x \in \mathbb{R}^n$

$$|f_k(x,\xi_1) - f_k(x,\xi_2)| \leqslant L\lambda_k(x)(|\xi_1|^{p-1} + |\xi_2|^{p-1} + 1)|\xi_1 - \xi_2|, \tag{3.3}$$

for every ξ_1, $\xi_2 \in \mathbb{R}^{m\times n}$, and every $k \in \mathbb{N}$.

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Let $\mathcal{A}(Q_0)$ denote the collection of all open subsets of Q_0 with Lipschitz boundary. We consider the sequence of localized integral functionals $F_k : W^{1,1}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0, +\infty)$ defined as

$$F_k(u, A) := \begin{cases} \displaystyle\int_A f_k(x, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}_{\lambda_k}(A;\mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.4}$$

We endow $W^{1,1}(Q_0;\mathbb{R}^m)$ with the strong $L^1(Q_0;\mathbb{R}^m)$-topology. If not otherwise specified, throughout the paper the Γ-limits will all be computed with respect to this topology.

The following theorem is the main result of this paper.

THEOREM 3.2. *Let F_k be the functionals defined in (3.4). Then there exists a subsequence (F_{k_h}) such that for every $A \in \mathcal{A}(Q_0)$, $A \subset\subset Q_0$, the functionals $F_{k_h}(\cdot, A)$ Γ-converge to the functional $F_\infty(\cdot, A)$ with $F_\infty : W^{1,1}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0, +\infty]$ given by*

$$F_\infty(u, A) := \begin{cases} \displaystyle\int_A f_\infty(x, \nabla u)\,\mathrm{d}x & \textit{if } \ u \in W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m), \\ +\infty & \textit{otherwise,} \end{cases}$$

where, for some $c_3 = c_3(n) > 0$, λ_∞ belongs to $A_p(c_3^p K)$ and satisfies

$$\lambda_{k_h} \rightharpoonup \lambda_\infty \quad \textit{weakly in } L^1(Q_0).$$

The integrand $f_\infty \colon Q_0 \times \mathbb{R}^{m\times n} \to [0, +\infty)$ is a Borel function and for a.e. $x \in Q_0$ and every $\xi \in \mathbb{R}^{m\times n}$ is given by the following asymptotic formula

$$f_\infty(x, \xi) := \limsup_{\rho\to 0^+} \lim_{h\to\infty} \frac{m_{F_{k_h}}(u_\xi, Q_\rho(x))}{\rho^n}, \tag{3.5}$$

where, for every $A \in \mathcal{A}(Q_0)$,

$$m_{F_{k_h}}(u_\xi, A) := \inf\left\{F_{k_h}(u, A)\colon v \in W^{1,p}_{0,\lambda_{k_h}}(A;\mathbb{R}^m) + u_\xi\right\},$$

with $u_\xi(x) := \xi x$.

Moreover, f_∞ satisfies the following properties for almost every $x \in Q_0$:

(i) *for every $\xi \in \mathbb{R}^{m\times n}$*

$$\alpha\lambda_\infty(x)\left(\frac{1}{K}|\xi|^p - 1\right) \leqslant f_\infty(x, \xi) \leqslant \beta\lambda_\infty(x)(|\xi|^p + 1); \tag{3.6}$$

(ii) *for every $\xi_1, \xi_2 \in \mathbb{R}^{m\times n}$*

$$|f_\infty(x, \xi_1) - f_\infty(x, \xi_2)| \leqslant L'\lambda_\infty(x)(|\xi_1|^{p-1} + |\xi_2|^{p-1} + 1)|\xi_1 - \xi_2|, \tag{3.7}$$

for some $L' > 0$.

REMARK 3.3. We observe that if we replace (3.1) with the following stronger condition:

$$0 < \liminf_{k\to\infty} \fint_Q \lambda_k \,\mathrm{d}x, \quad \limsup_{k\to\infty} \fint_Q \lambda_k \,\mathrm{d}x < +\infty \quad \text{for every cube } Q \subset \mathbb{R}^n, \tag{3.8}$$

then theorem 3.2 holds true without the restriction $A \subset\subset Q_0$. Specifically, if (3.8) holds, then if we define the functionals F_k on $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m) \times \mathcal{A}_0$, where $\mathcal{A}_0$ is the collection of open, bounded, and Lipschitz subsets of $\mathbb{R}^n$, thanks to a diagonal argument, it can be shown that the functionals $F_{k_h}(\cdot, A)$ Γ-converge with respect to the $L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$-convergence to $F_\infty(\cdot, A)$, for every $A \in \mathcal{A}_0$; moreover, $\lambda_\infty \in A_p(K)$ (cf. proposition 2.4) and f_∞ is defined through (3.5) for a.e. $x \in \mathbb{R}^n$.

We note that (3.8) holds true in the case of admissible periodic or stationary weights (cf. § 8 and remark 8.11).

The proof of theorem 3.2 will be broken up in several intermediate results and it will be carried out in § 4–7. Namely, in § 4 we prove that (up to subsequences) the functionals F_k Γ-converge to the integral functional F_∞ on the space $W^{1,\infty}(Q_0;\mathbb{R}^m)$. Moreover, in this section we also prove that the limit integrand f_∞ satisfies the desired growth conditions as well as the continuity property. By means of an approximation argument, in § 5 we extend the Γ-convergence result established in § 4 to the whole $W^{1,1}(Q_0;\mathbb{R}^m)$, also showing that the domain of F_∞ coincides with the 'limit' weighted Sobolev space $W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$. Eventually, by combining the analysis in § 6 and § 7, we derive the asymptotic formula (3.5) for f_∞.

4. Γ-convergence and integral representation in $W^{1,\infty}$

In this section, we show that on $W^{1,\infty}(Q_0;\mathbb{R}^m)$ the sequence (F_k) Γ-converges (up to subsequences) to a limit functional which can be represented in an integral form.

The following theorem is the main result of the present section.

THEOREM 4.1 (Γ-convergence in $W^{1,\infty}$). *Let F_k be the functionals defined in (3.4). Then there exists a subsequence (F_{k_h}) such that for every $A \in \mathcal{A}(Q_0)$ the functionals $F_{k_h}(\cdot, A)$ Γ-converge on $W^{1,\infty}(A;\mathbb{R}^m)$ to the functional $F(\cdot, A)$ with $F : W^{1,\infty}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0, +\infty)$ given by*

$$F(u, A) = \int_A f_\infty(x, \nabla u)\,\mathrm{d}x, \tag{4.1}$$

for some Borel function $f_\infty : Q_0 \times \mathbb{R}^{m\times n} \to [0, +\infty)$. Moreover, the function f_∞ satisfies the following properties for almost every $x \in Q_0$:

(i) *for every $\xi \in \mathbb{R}^{m\times n}$*

$$\alpha\lambda_\infty(x)\left(\frac{1}{K}|\xi|^p - 1\right) \leqslant f_\infty(x, \xi) \leqslant \beta\lambda_\infty(x)(|\xi|^p + 1), \tag{4.2}$$

where $\lambda_\infty \in A_p(c_3^p K)$, for some $c_3 = c_3(n) > 0$, and $\lambda_{k_h} \rightharpoonup \lambda_\infty$ in $L^1(Q_0)$;

(ii) *for every* $\xi_1,\, \xi_2 \in \mathbb{R}^{m\times n}$

$$|f_\infty(x,\xi_1) - f_\infty(x,\xi_2)| \leqslant L'\lambda_\infty(x)(|\xi_1|^{p-1} + |\xi_2|^{p-1} + 1)|\xi_1 - \xi_2|, \tag{4.3}$$

for some $L' > 0$.

The proof of theorem 4.1 will be achieved in a number of intermediate steps by means of the so-called localization method of Γ-convergence (see, e.g.[**7**, chapters 9–11] or [**14**, chapters 16–19]).

To this end, we consider the localized Γ-liminf and the Γ-limsup of F_k; i.e. we consider the functionals $F',\, F''\colon W^{1,1}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0, +\infty]$ defined as

$$F'(u,A) := \Gamma\text{-}\liminf_{k\to\infty} F_k(u,A), \tag{4.4}$$

$$F''(u,A) := \Gamma\text{-}\limsup_{k\to\infty} F_k(u,A), \tag{4.5}$$

for $u \in W^{1,1}(Q_0;\mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$. Then, the aim of this section is to show that, up to subsequences, for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$ we have

$$F'(u,A) = F''(u,A) = F(u,A),$$

where F is as in (4.1).

Remark 4.2. We observe that F' and F'' are lower semicontinuous with respect to the strong topology of $L^1(Q_0;\mathbb{R}^m)$ [**14**, proposition 6.8]. They also inherit some of the properties of the functionals F_k. Namely, as set functions they are both increasing [**14**, proposition 6.7], moreover, F' is superadditive on pairwise-disjoint sets [**14**, proposition 16.12]; while as functionals they are both local [**14**, proposition 16.15].

Thanks to assumption 3.1 we can invoke proposition 2.4 to deduce the existence of $\lambda_\infty \in A_p(c_3^p K)$ such that $\lambda_k \rightharpoonup \lambda_\infty$ in $L^{1+\sigma}(Q_0)$.

Then in the following lemma we show that the domain of F'' (and hence also the domain of F') contains the space $W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$.

Lemma 4.3. *Up to subsequences, there holds*

$$F''(u,A) \leqslant \beta \int_A \lambda_\infty(|\nabla u|^p + 1)\,\mathrm{d}x, \tag{4.6}$$

for every $u \in W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$ *and* $A \in \mathcal{A}(Q_0)$.

Proof. Let $(\lambda_{k_h}) \subset (\lambda_k)$ be the subsequence whose existence is established by proposition 2.4. Hence, in particular, $\lambda_{k_h} \rightharpoonup \lambda_\infty$ weakly in $L^1(Q_0)$ and $\lambda_\infty \in A_p(c_3^p K)$, for some $c_3 = c_3(n) > 0$.

Let $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and let $A \in \mathcal{A}(Q_0)$. Therefore, if F'' is as in (4.5) with k replaced by k_h, by (3.2) we readily get

$$
\begin{aligned}
F''(u,A) &\leqslant \limsup_{h\to\infty} F_{k_h}(u,A) \leqslant \lim_{h\to\infty} \beta \int_A \lambda_{h_k}(|\nabla u|^p + 1)\,\mathrm{d}x \\
&\leqslant \beta \int_A \lambda_\infty(|\nabla u|^p + 1)\,\mathrm{d}x, \qquad (4.7)
\end{aligned}
$$

hence (4.6) is proven for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$.

Now let $u \in W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$; then there exists $(u_j) \subset C^\infty(\overline{Q}_0;\mathbb{R}^m) \subset W^{1,\infty}(Q_0;\mathbb{R}^m)$ such that $u_j \to u$ in $W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$. Hence, thanks to (4.7), to the fact that $u_j \to u$ in $L^1(Q_0;\mathbb{R}^m)$, and to the lower semicontinuity of F'' with respect to the strong $L^1(Q_0;\mathbb{R}^m)$-convergence, we obtain

$$
\begin{aligned}
F''(u,A) &\leqslant \liminf_{j\to\infty} F''(u_j,A) \\
&\leqslant \lim_{j\to\infty} \beta \int_A \lambda_\infty(|\nabla u_j|^p + 1)\,\mathrm{d}x \;= \beta \int_A \lambda_\infty(|\nabla u|^p + 1)\,\mathrm{d}x
\end{aligned}
$$

and thus the claim. □

The following lemma shows that F_k (almost) decreases by smooth truncations.

Lemma 4.4. *Let F_k be the functionals defined in (3.4). Let $A \in \mathcal{A}(Q_0)$ and let $(u_k) \subset W^{1,1}(Q_0;\mathbb{R}^m)$ be such that*

$$
\sup_{k\in\mathbb{N}} \left(F_k(u_k,A) + \|u_k\|_{L^1(A;\mathbb{R}^m)}\right) < +\infty. \qquad (4.8)
$$

Then for every $\eta > 0$, $M > 0$ and for every $k \in \mathbb{N}$ there exists a Lipschitz function $\varphi_k : \mathbb{R}^m \to \mathbb{R}^m$ with Lipschitz constant less than or equal to 1 satisfying

$$
\varphi_k(y) = \begin{cases} y & \text{if } |y| \leqslant a_k, \\ 0 & \text{if } |y| > b_k, \end{cases}
$$

for suitable constants a_k, $b_k > 0$ with $M \leqslant a_k < b_k$, such that

$$
F_k(\varphi_k(u_k),A) \leqslant F_k(u_k,A) + \eta,
$$

for every $k \in \mathbb{N}$. Moreover, the function φ_k can be chosen in a finite family independent of k.

Proof. The proof of this lemma is classical and follows the line of, e.g. [**8**, lemma 3.5] with minor modifications. However, since we work in a different functional setting, we repeat the proof here for the readers' convenience.

Let $\eta > 0$ and $M > 0$ be fixed. Let (a_j) be a strictly increasing sequence of positive real numbers such that for every $j \in \mathbb{N}$ there exists a Lipschitz function

$\varphi_j : \mathbb{R}^m \to \mathbb{R}^m$ with Lipschitz constant less than or equal to 1 satisfying

$$\varphi_j(y) = \begin{cases} y & \text{if } |y| \leqslant a_j, \\ 0 & \text{if } |y| > a_{j+1}. \end{cases}$$

For every $k \in \mathbb{N}$ and every $j \in \mathbb{N}$ set $w_k^j := \varphi_j(u_k)$. We have

$$\begin{aligned} \int_A f_k(x, \nabla w_k^j)\,\mathrm{d}x &= \int_{A\cap\{|u_k|\leqslant a_j\}} f_k(x,\nabla u_k)\,\mathrm{d}x + \int_{A\cap\{|u_k|>a_{j+1}\}} f_k(x,0)\,\mathrm{d}x \\ &\quad + \int_{A\cap\{a_j<|u_k|\leqslant a_{j+1}\}} f_k(x,\nabla w_k^j)\,\mathrm{d}x \\ &\leqslant \int_A f_k(x,\nabla u_k)\,\mathrm{d}x + \beta \int_{A\cap\{|u_k|>a_{j+1}\}} \lambda_k\,\mathrm{d}x \\ &\quad + \beta \int_{A\cap\{a_j<|u_k|\leqslant a_{j+1}\}} \lambda_k(|\nabla u_k|^p + 1)\,\mathrm{d}x, \end{aligned}$$

where to establish the last inequality we have used the nonnegativity of f_k, (3.2), and the fact that φ_j has Lipschitz constant less than or equal to 1.

Let $N \in \mathbb{N}$ be arbitrary; we now want to estimate $1/N \sum_{j=1}^N F_k(w_k^j, A)$, for every $k \in \mathbb{N}$. To this end we start noticing that $(\{a_j < |u_k| \leqslant a_{j+1}\})_{j\in\mathbb{N}}$ is a family of pairwise-disjoint sets. Therefore, we get

$$\begin{aligned} \frac{1}{N}\sum_{j=1}^N F_k(w_k^j, A) &\leqslant F_k(u_k, A) \\ &\quad + \frac{\beta}{N}\sum_{j=1}^N \int_{A\cap\{|u_k|>a_{j+1}\}} \lambda_k\,\mathrm{d}x + \frac{\beta}{N}\int_A \lambda_k(|\nabla u_k|^p+1)\,\mathrm{d}x. \end{aligned} \tag{4.9}$$

In view of (3.1) and (4.8) we can find a constant $C > 0$ such that

$$\beta \int_A \lambda_k(|\nabla u_k|^p + 1)\,\mathrm{d}x \leqslant C, \tag{4.10}$$

for every $k \in \mathbb{N}$. Moreover, thanks to proposition 2.5 there exist $c, \sigma > 0$ such that

$$\int_{A\cap\{|u_k|>a_{j+1}\}} \lambda_k\,\mathrm{d}x \leqslant c\,c_2|Q_0| \left(\frac{|A\cap\{|u_k|>a_{j+1}\}|}{|Q_0|}\right)^{\sigma/(1+\sigma)}, \tag{4.11}$$

for every $k \in \mathbb{N}$ and every $j \in 1, \ldots, N$.

Therefore, we define the sequence (a_j) recursively by imposing the following condition on a_1:

$$|A\cap\{|u_k|>a_1\}| \leqslant \left(\frac{\eta}{2\beta c\,c_2}\right)^{(1+\sigma)/\sigma} |Q_0|^{-\sigma} \quad \text{for every } k\in\mathbb{N}, \quad a_1 \geqslant M, \tag{4.12}$$

which is clearly possible thanks to the boundedness of (u_k) in $L^1(A;\mathbb{R}^m)$. Eventually, by choosing $N \in \mathbb{N}$ in a way such that $C/N \leqslant \eta/2$, gathering (4.9)–(4.12) we

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obtain

$$\frac{1}{N}\sum_{j=1}^{N} F_k(w_k^j, A) \leqslant F_k(u_k, A) + \eta.$$

Therefore, for every $k \in \mathbb{N}$ we can find $j(k) \in \{1, \ldots, N\}$ such that

$$F_k(w_k^{j(k)}, A) \leqslant F_k(u_k, A) + \eta,$$

hence the proof is accomplished by setting $\varphi_k := \varphi_{j(k)}$. Finally, we note that N is independent of k. □

We now use lemma 4.4 to show that if $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ then for every $A \in \mathcal{A}(Q_0)$ the value of the Γ-limsup $F''(u, A)$ can be recovered along a sequence (w_k) which is bounded in $L^\infty(Q_0;\mathbb{R}^m)$ and such that $u_k \to u$ in $L^q(Q_0;\mathbb{R}^m)$, for every $1 \leqslant q < +\infty$.

Proposition 4.5. *Let F_k be the functionals defined in (3.4) and let $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$. Then there exists a sequence $(w_k) \subset W^{1,1}(Q_0;\mathbb{R}^m)$ satisfying the following properties*:

(i) $\sup_k \|u_k\|_{L^\infty(Q_0;\mathbb{R}^m)} < +\infty$;

(ii) $w_k \to u$ *in* $L^q(Q_0;\mathbb{R}^m)$ *for every* $1 \leqslant q < +\infty$;

(iii) $\limsup_{k\to\infty} F_k(w_k, A) = F''(u, A)$, *for every* $A \in \mathcal{A}(Q_0)$.

Proof. Let $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$; by [**14**, proposition 8.1] there exists $(u_k) \subset W^{1,1}(Q_0;\mathbb{R}^m)$ such that $u_k \to u$ in $L^1(Q_0;\mathbb{R}^m)$ and

$$\limsup_{k\to\infty} F_k(u_k, A) = F''(u, A) < +\infty, \tag{4.13}$$

where the last inequality follows by lemma 4.3.

Let $\eta > 0$ be fixed: by applying lemma 4.4 to the sequence (u_k) with $M := \|u\|_{L^\infty(Q_0;\mathbb{R}^m)}$ we obtain a sequence $(w_k) \subset W^{1,1}(Q_0;\mathbb{R}^m) \cap L^\infty(Q_0;\mathbb{R}^m)$ which is bounded in $L^\infty(Q_0;\mathbb{R}^m)$, such that $w_k \to u$ in $L^q(Q_0;\mathbb{R}^m)$ for every $1 \leqslant q < +\infty$ and

$$F_k(w_k, A) \leqslant F_k(u_k, A) + \eta, \tag{4.14}$$

for every $A \in \mathcal{A}(Q_0)$. Then, taking the lim sup as $k \to \infty$ in (4.14) and appealing to (4.13) we obtain

$$\limsup_{k\to\infty} F_k(w_k, A) \leqslant F''(u, A) + \eta.$$

Eventually, the claim follows by the definition of F'' and the arbitrariness of η. □

The following proposition shows that the functionals F_k satisfy the fundamental estimate, uniformly in k.

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Proposition 4.6 (Fundamental estimate). *Let F_k be the functionals defined in* (3.4) *and let $A \in \mathcal{A}(Q_0)$. For every $\eta > 0$ and for every A', A'', $B \in \mathcal{A}(Q_0)$ with $A' \subset\subset A'' \subset\subset A$ there exists a constant $M_\eta > 0$ with the following property: for every $k \in \mathbb{N}$ and for every u, $\tilde{u} \in W^{1,p}_{\lambda_k}(A;\mathbb{R}^m)$ there exists a function $\varphi \in C_0^\infty(A'')$ with $\varphi = 1$ in a neighbourhood of $\overline{A'}$ and $0 \leqslant \varphi \leqslant 1$ such that*

$$\begin{aligned}&F_k(\varphi u + (1-\varphi)\tilde{u}, A' \cup B)\\ &\quad\leqslant (1+\eta)\left(F_k(u, A'') + F_k(\tilde{u}, B)\right) + M_\eta\left(\int_S \lambda_k|u-\tilde{u}|^p\,\mathrm{d}x\right) + \eta,\end{aligned}$$

where $S := B \cap (A'' \setminus A')$.

Proof. Let $\eta > 0$, A, A', A'', B and S be as in the statement. We start observing that by (3.1) there exists a constant $C > 0$ such that

$$\int_S \lambda_k\,\mathrm{d}x \leqslant C \tag{4.15}$$

for every $k \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that

$$\frac{1}{N}\max\left\{\frac{3^{p-1}\beta}{\alpha}, 3^p\beta C\right\} \leqslant \eta. \tag{4.16}$$

Let $A_1, \ldots, A_{N+1}$ be $N+1$ open sets satisfying $A' \subset\subset A_1 \subset\subset \cdots \subset\subset A_{N+1} \subset\subset A''$, and for $i = 1, \ldots, N$ consider the function $\varphi_i \in C_0^\infty(A)$ such that $\operatorname{supp}\varphi_i \subset A_{i+1}$ and $\varphi_i = 1$ on a neighbourhood of $\overline{A_i}$. Finally, define

$$M_\eta := \frac{1}{N}3^{p-1}\max_{1\leqslant i\leqslant N}\|\nabla\varphi_i\|_\infty.$$

For every $k \in \mathbb{N}$ and for $i = 1, \ldots, N$ we have

$$\begin{aligned}&F_k(\varphi_i u + (1-\varphi_i)\tilde{u}, A' \cup B)\\ &\quad= F_k^*(u, (A'' \cup B) \cap \overline{A_i}) + F_k^*(\tilde{u}, B \setminus A_{i+1}) + F_k(\varphi_i u\\ &\qquad+ (1-\varphi_i)\tilde{u}, B \cap (A_{i+1} \setminus \overline{A_i}))\\ &\quad\leqslant F_k(u, A'') + F_k(\tilde{u}, B) + F_k(\varphi_i u + (1-\varphi_i)\tilde{u}, B \cap (A_{i+1} \setminus \overline{A_i})),\end{aligned} \tag{4.17}$$

where F_k^* denotes the extension of F_k to the Borel subsets of Q_0.

Denote by $I_{k,i}$ the last term in (4.17). For every $k \in \mathbb{N}$ and for $i = 1, \ldots, N$, using (3.2) we obtain

$$\begin{aligned}
I_{k,i} &\leqslant \beta \int_{S_i} \lambda_k |\nabla(\varphi_i u + (1-\varphi_i)\tilde{u})|^p \,\mathrm{d}x + \beta \int_{S_i} \lambda_k \,\mathrm{d}x \\
&\leqslant 3^{p-1}\beta \int_{S_i} \lambda_k (|\nabla\varphi_i|^p |u-\tilde{u}|^p + |\nabla u|^p + |\nabla \tilde{u}|^p)\,\mathrm{d}x + \beta \int_{S_i} \lambda_k \,\mathrm{d}x \\
&\leqslant 3^{p-1}\beta \int_{S_i} \lambda_k |\nabla u|^p \,\mathrm{d}x + 3^{p-1}\beta \int_{S_i} \lambda_k |\nabla\tilde{u}|^p \,\mathrm{d}x \ + N M_\eta \int_{S_i} \lambda_k |u-\tilde{u}|^p \,\mathrm{d}x \\
&\quad + \beta \int_{S_i} \lambda_k \,\mathrm{d}x \\
&\leqslant 3^{p-1}\beta \int_{S_i} \lambda_k (|\nabla u|^p - 1)\,\mathrm{d}x + 3^{p-1}\beta \int_{S_i} \lambda_k (|\nabla\tilde{u}|^p - 1)\,\mathrm{d}x \\
&\quad + N M_\eta \int_{S_i} \lambda_k |u-\tilde{u}|^p \,\mathrm{d}x + 3^p\beta \int_{S_i} \lambda_k \,\mathrm{d}x,
\end{aligned}$$

where $S_i := B \cap (A_{i+1} \setminus \overline{A_i})$. Therefore, by the growth condition from below (3.2) on f_k we get

$$I_{k,i} \leqslant \frac{3^{p-1}\beta}{\alpha}\,(F_k(u,S_i) + F_k(\tilde{u},S_i)) + N M_\eta \int_{S_i} \lambda_k |u-\tilde{u}|^p \,\mathrm{d}x + 3^p\beta \int_{S_i} \lambda_k(x)\,\mathrm{d}x,$$

for every $k \in \mathbb{N}$ and for $i = 1, \ldots, N$. Hence, there exists $i_0 \in \{1, \ldots, N\}$ such that

$$\begin{aligned}
I_{k,i_0} \leqslant \frac{1}{N}\sum_{i=1}^{N} I_{k,i} &\leqslant \frac{1}{N}\frac{3^{p-1}\beta}{\alpha}\,(F_k(u,S) + F_k(\tilde{u},S)) \\
&\quad + M_\eta \int_S \lambda_k |u-\tilde{u}|^p \,\mathrm{d}x + \frac{1}{N} 3^p\beta \int_S \lambda_k \,\mathrm{d}x
\end{aligned}$$

for every $k \in \mathbb{N}$; thus by (4.15) we get

$$I_{k,i_0} \leqslant \frac{1}{N}\frac{3^{p-1}\beta}{\alpha}\,(F_k(u,A'') + F_k(\tilde{u},B)) + M_\eta \int_S \lambda_k |u-\tilde{u}|^p \,\mathrm{d}x + \frac{1}{N} 3^p\beta C.$$

Eventually, in view of (4.16) and (4.17) the proof is accomplished choosing $\varphi := \varphi_{i_0}$. □

With the help of propositions 4.5 and 4.6 we can deduce the following result which will eventually lead to the inner regularity and subadditivity of the set function $F''(u,\cdot)$, for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$.

LEMMA 4.7. *Let F'' be as in (4.5). Let $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and let $A', A'', B \in \mathcal{A}(Q_0)$ be such that $A' \subset\subset A'' \subset\subset A$; then*

$$F''(u, A' \cup B) \leqslant F''(u,A'') + F''(u,B). \tag{4.18}$$

Proof. Let $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$; by proposition 4.5 there exist $(u_k) \subset W^{1,p}_{\lambda_k}(A'';\mathbb{R}^m)$ and $(\tilde u_k) \subset W^{1,p}_{\lambda_k}(B;\mathbb{R}^m)$ which are bounded in $L^\infty(Q_0;\mathbb{R}^m)$, converge to u in $L^q(Q_0;\mathbb{R}^m)$ for every $q \geqslant 1$, and satisfy

$$\limsup_{k\to\infty} F_k(u_k, A'') = F''(u, A'') \text{ and } \limsup_{k\to\infty} F_k(\tilde u_k, B) = F''(u, B). \tag{4.19}$$

Let $\eta > 0$ be fixed; then, in view of proposition 4.6 we can find a constant $M_\eta > 0$ and a sequence (φ_k) of cut-off functions between A' and A'' such that

$$\begin{aligned} &F_k(\varphi_k u_k + (1-\varphi_k)\tilde u_k, A' \cup B) \\ &\quad \leqslant (1+\eta)\left(F_k(u_k, A'') + F_k(\tilde u_k, B)\right) + M_\eta \int_S \lambda_k |u_k - \tilde u_k|^p \, \mathrm{d}x + \eta, \end{aligned}$$

where $S = B \cap (A'' \setminus A')$. Since the sequence $\varphi_k u_k + (1-\varphi_k)\tilde u_k$ converges to u in $L^1(Q_0;\mathbb{R}^m)$, by (4.19) we obtain

$$\begin{aligned} F''(u, A' \cup B) &\leqslant \limsup_{k\to\infty} F_k(\varphi_k u_k + (1-\varphi_k)\tilde u_k, A' \cup B) \\ &\leqslant (1+\eta)\left(F''(u, A'') + F''(u, B)\right) \\ &\quad + M_\eta \limsup_{k\to\infty} \int_S \lambda_k |u_k - \tilde u_k|^p \, \mathrm{d}x + \eta. \end{aligned}$$

Now let $\sigma > 0$ be the exponent as in theorem 2.2, using the Hölder inequality and recalling (3.1) we get

$$\begin{aligned} \int_S \lambda_k |u_k - \tilde u_k|^p \, \mathrm{d}x &\leqslant \left(\int_S \lambda_k^{1+\sigma} \, \mathrm{d}x\right)^{1/(1+\sigma)} \left(\int_S |u_k - \tilde u_k|^{p(1+\sigma)/\sigma} \, \mathrm{d}x\right)^{\sigma/(1+\sigma)} \\ &\leqslant c\, c_2 |Q_0|^{1/(1+\sigma)} \left(\int_{Q_0} |u_k - \tilde u_k|^{p(1+\sigma)/\sigma} \, \mathrm{d}x\right)^{\sigma/(1+\sigma)}. \end{aligned}$$

Therefore, since $\|u_k - \tilde u_k\|_{L^q(Q_0;\mathbb{R}^m)} \to 0$ for every $q \geqslant 1$, we immediately obtain

$$\limsup_{k\to\infty} \int_S \lambda_k |u_k - \tilde u_k|^p \, \mathrm{d}x = 0.$$

Hence, (4.18) follows by the arbitrariness of $\eta > 0$. $\square$

The proof of the following proposition is classical, for this reason we only sketch it here, while we refer the reader to the monographs [**7**, **14**] for further details.

Proposition 4.8 (Γ-convergence and measure property of the Γ-limit). *Let F_k be the functionals defined in (3.4). Then there exist a subsequence (k_h) and a functional $F\colon W^{1,\infty}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0,+\infty)$ such that for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$*

$$F(u, A) = F'(u, A) = F''(u, A), \tag{4.20}$$

where F' and F'' are as in (4.4) and (4.5), respectively, with k replaced by k_h.

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Moreover, for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ the set function $F(u,\cdot)$ is the restriction to $\mathcal{A}(Q_0)$ of a Radon measure on Q_0.

Proof. Let (k_h) be the subsequence whose existence is established by proposition 2.4. Thanks to the compactness of Γ-convergence [**14**, theorem 8.5], a standard diagonal argument gives the existence of a further subsequence (not relabelled), such that the corresponding functionals F' and F'' satisfy

$$\begin{aligned}&\sup\{F'(u,B)\colon B \in \mathcal{A}(Q_0), B \subset\subset A\}\\ &\quad = \sup\{F''(u,B)\colon B \in \mathcal{A}(Q_0), B \subset\subset A\} =: F(u,A),\end{aligned}$$

for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and for every $A \in \mathcal{A}(Q_0)$. We note that the set function $F(u,\cdot)$ is inner regular by definition.

Moreover, by virtue of lemma 4.7 we can reason as in [**14**, proposition 18.4] to deduce that $F(u,\cdot)$ is subadditive.

We now prove that (4.20), which will ensure that F is the Γ-limit of F_k on $W^{1,\infty}(Q_0;\mathbb{R}^m)$.

Since by definition of F we have $F \leqslant F' \leqslant F''$, to get (4.20) it suffices to show that

$$F''(u,A) \leqslant F(u,A), \tag{4.21}$$

for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$.

To prove (4.21) we consider the localized functional $H : W^{1,\infty}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0,+\infty)$ defined as

$$H(u,A) := \int_A \lambda_\infty(|\nabla u|^p + 1)\,\mathrm{d}x.$$

Therefore, by lemma 4.3 we immediately obtain that $F''(u, A) \leqslant H(u, A)$, for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$. For every fixed $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ the set function $H(u,\cdot)$ defines a Radon measure on Q_0, hence for every $\eta > 0$ fixed there exists a compact set $K_\eta \subset A$ such that $H(u, A \setminus K_\eta) < \eta$. Let now $A', A'' \in \mathcal{A}(Q_0)$ be such that $K_\eta \subset A' \subset\subset A'' \subset\subset A$ and let $B = A \setminus K_\eta$. By (4.18) we have

$$F''(u,A) \leqslant F''(u,A'') + F''(u,A \setminus K_\eta).$$

Then by definition of F we readily obtain

$$F''(u,A) \leqslant F(u,A) + H(u,A \setminus K_\eta) \leqslant F(u,A) + \eta,$$

thus (4.21) follows by the arbitrariness of $\eta > 0$.

Finally, the inner regularity and subadditivity of $F(u,\cdot)$ together with remark 4.2 allow us to apply the De Giorgi–Letta measure criterion (see, e.g.[**14**, theorem 14.23]) to deduce that $F(u,\cdot)$ is the restriction to $\mathcal{A}(Q_0)$ of a Radon measure on Q_0, and thus to conclude. □

Remark 4.9. We observe that for every $A \in \mathcal{A}(Q_0)$ the functional $F(\cdot, A)$ is invariant under translations in u. Indeed, for given $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$ let $(u_k) \subset W^{1,1}(Q_0;\mathbb{R}^m)$ be such that $u_k \to u$ in $L^1(Q_0;\mathbb{R}^m)$ and $\lim_{k\to\infty} F_k(u_k, A) =$

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$F(u, A)$. Let now $s \in \mathbb{R}^m$, then clearly $(u_k + s)$ converges to $u + s$ in $L^1(Q_0; \mathbb{R}^m)$ and by (4.20)

$$F(u + s, A) \leqslant \liminf_{k\to\infty} F_k(u_k + s, A) = \lim_{k\to\infty} F_k(u_k, A) = F(u, A),$$

since F_k is invariant under translations in u. On the other hand, the argument above also gives

$$F(u, A) = F((u + s) + (-s), A) \leqslant F(u + s, A)$$

and thus the claim.

THEOREM 4.10 (Integral representation). *Let F_k be the functionals defined in (3.4). Then there exist a subsequence (F_{k_h}) and a Borel function $f_\infty : Q_0 \times \mathbb{R}^{m\times n} \to [0, +\infty)$, satisfying (4.2) and (4.3), such that for every $u \in W^{1,\infty}(Q_0; \mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$ there holds*

$$F(u, A) = \Gamma\text{-}\lim_{h\to\infty} F_{k_h}(u, A),$$

where $F : W^{1,\infty}(Q_0; \mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0, +\infty)$ is given by

$$F(u, A) = \int_A f_\infty(x, \nabla u)\, \mathrm{d}x. \tag{4.22}$$

Proof. Proposition 4.8 ensures the existence of a subsequence (F_{k_h}) of (F_k) such that $F_{k_h}(u, A)$ Γ-converges to a functional $F(u, A)$ for every $u \in W^{1,\infty}(Q_0; \mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$. Then, it remains to prove that the functional F admits an integral representation as in (4.22).

We will break up the proof of the integral representation into a number of steps.

Step 1. Definition of f_∞. Let $\xi \in \mathbb{R}^{m\times n}$ be fixed and set $u_\xi(x) := \xi x$. By the measure property of F established in proposition 4.8, the set function $F(u_\xi, \cdot)$ can be extended to a Radon measure on Q_0. Moreover, thanks to lemma 4.3, $F(u_\xi, \cdot)$ is absolutely continuous with respect to the Lebesgue measure. For every $x \in Q_0$ define

$$f_\infty(x, \xi) := \limsup_{\rho\to 0^+} \frac{F(u_\xi, Q_\rho(x))}{|Q_\rho(x)|},$$

where $Q_\rho(x)$ is the cube centred at x, with side length $\rho > 0$, and faces parallel to the coordinate planes. Then, f_∞ is a Borel function and the Lebesgue differentiation theorem guarantees that

$$F(u_\xi, A) = \int_A f_\infty(x, \xi)\, \mathrm{d}x,$$

for every $A \in \mathcal{A}(Q_0)$.

We now show that f_∞ satisfies the growth and coercivity conditions as in (4.2). To this end, we start observing that the growth condition from above readily follows from lemma 4.3. In fact, choosing in (4.6) $u = u_\xi$, $A = Q_\rho(x)$, with x Lebesgue point for λ_∞, the estimate from above in (4.2) follows by dividing both sides of (4.6) by $|Q_\rho(x)|$, and eventually passing to the limit as $\rho \to 0^+$.

To derive the growth condition from below on f_∞ let $u \in W^{1,1}(Q_0;\mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$ be fixed. By the Hölder inequality and by the growth condition from below in (3.2) we get

$$
\begin{aligned}
\left(\int_A |\nabla u| \,\mathrm{d}x\right)^p &\leqslant \left(\int_A \lambda_k |\nabla u|^p \,\mathrm{d}x\right)\left(\int_A \lambda_k^{-1/(p-1)} \,\mathrm{d}x\right)^{p-1}\\
&= \left(\int_A \lambda_k(|\nabla u|^p - 1) \,\mathrm{d}x\right)\left(\int_A \lambda_k^{-1/(p-1)} \,\mathrm{d}x\right)^{p-1}\\
&\quad + \left(\int_A \lambda_k \,\mathrm{d}x\right)\left(\int_A \lambda_k^{-1/(p-1)} \,\mathrm{d}x\right)^{p-1}\\
&\leqslant \frac{1}{\alpha} F_k(u,A)\left(\int_A \lambda_k^{-1/(p-1)} \,\mathrm{d}x\right)^{p-1}\\
&\quad + \left(\int_A \lambda_k \,\mathrm{d}x\right)\left(\int_A \lambda_k^{-1/(p-1)} \,\mathrm{d}x\right)^{p-1},
\end{aligned}
$$

therefore the following lower bound

$$
\alpha\left(\int_A \lambda_k^{-1/(p-1)} \,\mathrm{d}x\right)^{-(p-1)}\left(\int_A |\nabla u| \,\mathrm{d}x\right)^p - \alpha\left(\int_A \lambda_k \,\mathrm{d}x\right) \leqslant F_k(u,A), \tag{4.23}
$$

for every $k \in \mathbb{N}$. Now let $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and let $(u_h) \subset W^{1,1}(Q_0;\mathbb{R}^m)$ be such that

$$
u_h \to u \text{ in } L^1(Q_0;\mathbb{R}^m) \text{ and } \lim_{h\to\infty} F_{k_h}(u_h,A) = F(u,A).
$$

Hence, by the lower semicontinuity of $u \mapsto \int_A |\nabla u| \,\mathrm{d}x$ with respect to the $L^1(Q_0;\mathbb{R}^m)$-topology and by proposition 2.4, evaluating (4.23) in (u_h) and passing to the limit as $h \to \infty$ we find

$$
\alpha\left(\int_A \tilde{\lambda}_\infty^{-1/(p-1)} \,\mathrm{d}x\right)^{-(p-1)}\left(\int_A |\nabla u| \,\mathrm{d}x\right)^p - \alpha\left(\int_A \lambda_\infty \,\mathrm{d}x\right) \leqslant F(u,A), \tag{4.24}
$$

for every $u \in W^{1,\infty}(A;\mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$. Now let $x \in Q_0$ be a Lebesgue point for λ_∞ and $\tilde{\lambda}_\infty$ and choose in (4.24) $u = u_\xi$ and $A = Q_\rho(x)$; then, dividing both sides of (4.24) by $|Q_\rho(x)|$ and passing to the limit as $\rho \to 0^+$ give

$$
\alpha\big(\tilde{\lambda}_\infty(x)|\xi|^p - \lambda_\infty(x)\big) \leqslant f_\infty(x,\xi), \tag{4.25}
$$

for a.e. $x \in Q_0$ and every $\xi \in \mathbb{R}^{m\times n}$. Eventually, (2.6) entails the desired bound from below.

Step 2. Integral representation on piecewise affine functions. Let $A \in \mathcal{A}(Q_0)$ and $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ be piecewise affine on A; i.e. there exists a finite family of

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pairwise disjoint open sets A_j such that $|A \setminus \bigcup_{j=1}^{N} A_j| = 0$ and

$$u(x) = \sum_{j=1}^{N} \chi_{A_j}(x)(u_{\xi^j} + z_j),$$

for every $x \in A$ with $\xi^j \in \mathbb{R}^{m\times n}$, $z_j \in \mathbb{R}^m$ for $j = 1, \ldots, N$. By remark 4.9 and step 1, taking into account the locality of F, we have

$$F(u, A) = \sum_{j=1}^{N} F(u, A_j) = \sum_{j=1}^{N} \int_{A_j} f_\infty(x, \xi^j)\,\mathrm{d}x = \int_A f_\infty(x, \nabla u)\,\mathrm{d}x,$$

that is, the integral representation (4.22) on piecewise affine functions.

Step 3. Convexity properties of f_∞. For every $A \in \mathcal{A}(Q_0)$ the functional $F(\cdot, A)$ is lower semicontinuous on $W^{1,\infty}(Q_0; \mathbb{R}^m)$ with respect to the strong convergence of $L^1(Q_0; \mathbb{R}^m)$, thus, in particular, it is lower semicontinuous with respect to the weak* $W^{1,\infty}(Q_0; \mathbb{R}^m)$-convergence. Therefore, the function $\xi \to f_\infty(x, \xi)$ is $W^{1,\infty}$-quasiconvex (and rank-1-convex) for a.e. $x \in Q_0$ (see, e.g. [**7**, proposition 4.3, corollary 4.12]). Then, it is easy to check that the growth condition (4.2) together with the convexity property of $f_\infty(x, \cdot)$ yield the local Lipschitz continuity in (4.3) (see, e.g. [**7**, remark 4.13]).

Step 4. Integral representation. For $u \in W^{1,p}_{\lambda_\infty}(Q_0; \mathbb{R}^m)$ consider the functional

$$u \mapsto \int_A f_\infty(x, \nabla u)\,\mathrm{d}x. \tag{4.26}$$

We observe that the local Lipschitz condition (4.3) satisfied by f_∞ ensures that, for every $A \in \mathcal{A}(Q_0)$, the functional (4.26) is continuous with respect to the strong $W^{1,p}_{\lambda_\infty}(A; \mathbb{R}^m)$-convergence. Indeed, using Hölder's inequality we easily get

$$\begin{aligned}
&\int_A |f_\infty(x, \nabla u_1) - f_\infty(x, \nabla u_2)|\,\mathrm{d}x \\
&\quad \leqslant 3^{1/(p-1)} L' \left(\int_A \lambda_\infty(|\nabla u_1|^p + |\nabla u_2|^p + 1)\,\mathrm{d}x\right)^{(p-1)/p} \\
&\qquad \times \left(\int_A \lambda_\infty |\nabla u_1 - \nabla u_2|^p\,\mathrm{d}x\right)^{1/p}
\end{aligned}$$

for every $u_1, u_2 \in W^{1,p}_{\lambda_\infty}(Q_0; \mathbb{R}^m)$. Moreover, arguing as in the proof of lemma 4.7 we can deduce that (4.26) is also continuous with respect to the strong convergence of $W^{1,q}(Q_0; \mathbb{R}^m)$, for $q \geqslant p(1+\sigma)/\sigma$.

Let $u \in W^{1,\infty}(Q_0; \mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$ be given; then there exists a sequence $(u_j) \subset W^{1,q}(Q_0; \mathbb{R}^m)$ strongly converging to u in $W^{1,q}(Q_0; \mathbb{R}^m)$ for any $q \in [1, \infty)$ such that its restrictions to A are piecewise affine. Since F is lower semicontinuous with respect to the strong topology of $L^1(Q_0; \mathbb{R}^m)$, appealing to step 2 and to the continuity of (4.26) we then obtain

$$F(u, A) \leqslant \liminf_{j\to\infty} F(u_j, A) = \liminf_{j\to\infty} \int_A f_\infty(x, \nabla u_j)\,\mathrm{d}x = \int_A f_\infty(x, \nabla u)\,\mathrm{d}x.$$

Hence, to represent F in an integral form it only remains to prove the opposite inequality. To this end fix $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and consider the functional $\widetilde{F} : W^{1,\infty}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0, +\infty)$ defined as

$$\widetilde{F}(v, A) := F(u + v, A).$$

We observe that $\widetilde{F}$ satisfies the same properties as F, hence there exists a Carathéodory function $h_\infty : Q_0 \times \mathbb{R}^{m\times n} \to [0, +\infty)$ such that

$$\widetilde{F}(v, A) \leqslant \int_A h_\infty(x, \nabla v)\,\mathrm{d}x,$$

for every $v \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$. Note that the equality holds whenever v is piecewise affine on A.

Let (u_j) be the sequence of piecewise affine functions considered above. Then

$$\begin{aligned}\int_A h_\infty(x, 0)\,\mathrm{d}x = \widetilde{F}(0, A) = F(u, A) \leqslant \int_A f_\infty(x, \nabla u)\,\mathrm{d}x \\ = \lim_{j\to\infty} \int_A f_\infty(x, \nabla u_j)\,\mathrm{d}x = \lim_{j\to\infty} F(u_j, A) = \lim_{j\to\infty} \widetilde{F}(u_j - u, A) \\ \leqslant \lim_{j\to\infty} \int_A h_\infty(x, \nabla(u_j - u))\,\mathrm{d}x = \int_A h_\infty(x, 0)\,\mathrm{d}x,\end{aligned}$$

hence the equality in (4.22) holds for every $u \in W^{1,\infty}(Q_0;\mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$. $\square$

REMARK 4.11. From (4.25) it can be seen that actually f_∞ satisfies the growth conditions

$$\alpha\left(\tilde{\lambda}_\infty(x)|\xi|^p - \lambda_\infty(x)\right) \leqslant f_\infty(x, \xi) \leqslant \beta\, \lambda_\infty(x)\big(|\xi|^p + 1\big),$$

for a.e. $x \in Q_0$ and every $\xi \in \mathbb{R}^{m\times n}$, which then reduce to those established in [**6**, **9**, **30**] when $\lambda_k \equiv 1$.

5. Γ-convergence and integral representation in $W^{1,p}_{\lambda_\infty}$

Consider now the integral functional $F_\infty\colon W^{1,1}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0, +\infty]$ defined as

$$F_\infty(u, A) := \begin{cases} \displaystyle\int_A f_\infty(x, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.1}$$

with f_∞ as in theorem 4.10.

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The purpose of this section is to show that (up to subsequences) there holds

$$F'(u,A) = F''(u,A) = F_\infty(u,A), \tag{5.2}$$

for every $u \in W^{1,1}(Q_0;\mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$, where F' and F'' are as in (4.4) and (4.5), respectively. In other words we will show that, up to subsequences, the functionals F_k defined in (3.4) Γ-converge on the whole space $W^{1,1}(Q_0;\mathbb{R}^m)$ to the functional F_∞, whose domain is the (limit) weighted Sobolev space $W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$.

To do so, we will make use of the following approximation result whose proof follows the line of that of [**1**, theorem II.4] (see also [**17**, theorem 3.1]).

Theorem 5.1. *Let F_k be the functionals defined in (3.4). Let $A \subset\subset Q_0$ be open and with Lipschitz boundary, let $u \in W^{1,1}(Q_0;\mathbb{R}^m)$ and let $(u_k) \subset W^{1,p}_{\lambda_k}(A;\mathbb{R}^m)$ be such that*

$$u_k \to u \text{ in } L^1(A;\mathbb{R}^m) \text{ and } \sup_{k\in\mathbb{N}} \int_A \lambda_k |\nabla u_k|^p \,\mathrm{d}x < +\infty. \tag{5.3}$$

Then, for every $\tau > 0$ there exist: $\beta_\tau > 0$ with $\beta_\tau \to 0$ as $\tau \to 0^+$, $L_\tau > 0$ with $L_\tau \to +\infty$ as $\tau \to 0^+$, a sequence (v_k^τ) and a function v^τ in $W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ with Lipschitz constant $c(n)L_\tau$, for some $c(n) > 0$ depending only on n, such that:

1. *$v_k^\tau \to v^\tau$ in $L^\infty(Q_0;\mathbb{R}^m)$ as $k \to \infty$;*
2. *$|\{x \in A\colon v^\tau(x) \neq u(x)\}| \leqslant (m+1)\tau$;*
3. *the following estimate holds for every $\tau > 0$:*

$$\liminf_{k\to\infty} \int_A f_k(x,\nabla u_k)\,\mathrm{d}x \geqslant \liminf_{k\to\infty} \int_{A_\tau} f_k(x,\nabla v_k^\tau)\,\mathrm{d}x - \beta_\tau, \tag{5.4}$$

for some open set $A_\tau \subset A$ with $|A \setminus A_\tau| < \tau$.

Proof. Without loss of generality, we can assume that liminf in the left-hand side of (5.4) is actually a limit. Moreover, we can also assume that $(u_k) \subset C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$, $\mathrm{supp}(u_k) \subset\subset Q_0$, and

$$\sup_{k\in\mathbb{N}} \int_{\mathbb{R}^n} \lambda_k |\nabla u_k|^p \,\mathrm{d}x < +\infty. \tag{5.5}$$

Indeed, thanks to (2.10) from (5.3) we have

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,p}_{\lambda_k}(A;\mathbb{R}^m)} < +\infty,$$

then, since $A \subset\subset Q_0$, the extension result [**34**, theorem 2.1.13] allows us to replace (u_k) with a sequence of functions in $W^{1,p}_{\lambda_k}(\mathbb{R}^n;\mathbb{R}^m)$, whose support is compactly contained in Q_0, and such that (5.5) holds. Moreover, since for fixed k the space $C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$ is dense in $W^{1,p}_{\lambda_k}(\mathbb{R}^n;\mathbb{R}^m)$ (see, e.g. [**34**, corollary 2.1.6]), a diagonal argument provides us with a sequence $(w_k) \subset C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$, with $\mathrm{supp}(w_k) \subset\subset Q_0$, such that

$$\|u_k - w_k\|_{W^{1,p}_{\lambda_k}(\mathbb{R}^n;\mathbb{R}^m)} < \frac{1}{k}. \tag{5.6}$$

Then, we readily get

$$\sup_{k\in\mathbb{N}} \int_{\mathbb{R}^n} \lambda_k |\nabla w_k|^p \,\mathrm{d}x < +\infty, \tag{5.7}$$

and by the compact embedding of $W^{1,p}_{\lambda_k}(Q_0;\mathbb{R}^m)$ in $L^1(Q_0;\mathbb{R}^m)$, (5.6) also implies that $w_k \to u$ in $L^1(A;\mathbb{R}^m)$.

Furthermore, we observe that u_k and w_k are close in energy so that once we establish the estimate (5.4) along (w_k), the same estimate will hold true along (u_k). In fact, (3.3) gives

$$\begin{aligned} &|F_k(w_k, A) - F_k(u_k, A)| \\ &\quad \leqslant 3^{1/(p-1)} L \left(\int_A \lambda_k (|\nabla w_k|^p + |\nabla u_k|^p + 1) \,\mathrm{d}x \right)^{(p-1)/p} \\ &\times \left(\int_A \lambda_k |\nabla w_k - \nabla u_k|^p \,\mathrm{d}x \right)^{1/p}, \end{aligned} \tag{5.8}$$

hence gathering (5.6)–(5.8) yields

$$|F_k(w_k, A) - F_k(u_k, A)| < \frac{C}{k},$$

for some constant $C > 0$.

Therefore, in all that follows, with a little abuse of notation, (u_k) denotes a sequence in $C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$, with $\mathrm{supp}(u_k) \subset\subset Q_0$, and such that (5.5) holds.

For $k \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$, let $u_k^{(i)}$ denote the i-th component of the vector-valued function u_k. By applying theorem 2.8 to $|\nabla u_k^{(i)}| \in L^1(\mathbb{R}^n)$ we deduce

$$\int_{\mathbb{R}^n} \lambda_k (M|\nabla u_k^{(i)}|)^p \,\mathrm{d}x \leqslant c_4 \int_{\mathbb{R}^n} \lambda_k |\nabla u_k^{(i)}|^p \,\mathrm{d}x, \tag{5.9}$$

for every $k \in \mathbb{N}$, every $i = 1, \ldots, m$, and for some $c_4 > 0$. Hence, by combining (5.5) and (5.9) it follows that the sequence $(\lambda_k (M|\nabla u_h^{(i)}|)^p)$ is bounded in $L^1(\mathbb{R}^n)$, for every $i = 1, \ldots, m$. Let now $\tau > 0$, then lemma 2.11 ensures the existence of a measurable set E_τ, with

$$|E_\tau| < \tau, \tag{5.10}$$

of a constant $\delta_\tau > 0$, and a subsequence (k_j^τ) such that

$$\int_B \lambda_{k_j^\tau} (M|\nabla u_{k_j^\tau}^{(i)}|)^p \,\mathrm{d}x < \tau,$$

for every $j \in \mathbb{N}$, every $i = 1, \ldots, m$, and for every measurable set B with $B \cap E_\tau = \emptyset$ and $|B| < \delta_\tau$.

To simplify the notation we drop the dependence of the sequence on j and τ, thus we write

$$\int_B \lambda_k(M|\nabla u_k^{(i)}|)^p \,\mathrm{d}x < \tau, \tag{5.11}$$

for every $k \in \mathbb{N}$, every $i = 1, \ldots, m$, and every measurable B with $B \cap E_\tau = \emptyset$ and $|B| < \delta_\tau$.

By the Hölder inequality we deduce

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |\nabla u_k^{(i)}| \,\mathrm{d}x\right)^p &= \left(\int_{Q_0} |\nabla u_k^{(i)}| \,\mathrm{d}x\right)^p \\ &\leqslant \left(\int_{Q_0} \lambda_k |\nabla u_k^{(i)}|^p \,\mathrm{d}x\right)\left(\int_{Q_0} \lambda_k^{-1/(p-1)} \,\mathrm{d}x\right)^{p-1},\end{aligned} \tag{5.12}$$

hence by (3.1), (5.5) and (5.12), since λ_k belongs to $A_p(K)$ we get

$$\left(\int_{\mathbb{R}^n} |\nabla u_k^{(i)}| \,\mathrm{d}x\right)^p \leqslant C\frac{K}{c_1}, \tag{5.13}$$

for every $k \in \mathbb{N}$, $i = 1, \ldots, m$, and some $C > 0$. In its turn (5.13) together with (2.11) provide us with a constant $L_\tau \geqslant (\tilde{c}/\tau)(CK/c_1)^{1/p}$ such that for every $k \in \mathbb{N}$, and $i = 1, \ldots, m$

$$|\{x \in \mathbb{R}^n : (M|\nabla u_k^{(i)}|)(x) \geqslant L_\tau\}| \leqslant \min\{\tau, \delta_\tau\}. \tag{5.14}$$

For $k \in \mathbb{N}$, and $i = 1, \ldots, m$ define the sets

$$H_{i,k}^\tau := H_{i,k}^{L_\tau} := \{x \in \mathbb{R}^n : (M|\nabla u_k^{(i)}|)(x) < L_\tau\}, \quad H_k^\tau := \bigcap_{i=1}^m H_{i,k}^\tau.$$

Then lemma 2.9 yields

$$|u_k^{(i)}(x) - u_k^{(i)}(y)| \leqslant c_5(n) L_\tau |x - y|$$

for every $k \in \mathbb{N}$, $i = 1, \ldots, m$, and every x, $y \in H_k^\tau$. Namely, the functions $u_k^{(i)}$ are Lipschitz continuous on H_k^τ with Lipschitz constant $c_5(n)L_\tau$, for every $k \in \mathbb{N}$ and every $i = 1, \ldots, m$.

Appealing to McShane's theorem [**27**] we can extend $u_k^{(i)}$ from $H_k^\tau \cap Q_0$ to $\mathbb{R}^n$ keeping the same Lipschitz constant $c_5(n)L_\tau$. We denote this extension with $v_k^{\tau,(i)}$ and note that we can assume that $v_k^{\tau,(i)}(x) = 0$ if $\mathrm{dist}(x, Q_0) > 1$. We then have

$$v_k^{\tau,(i)} = u_k^{(i)}, \quad \nabla v_k^{\tau,(i)} = \nabla u_k^{(i)} \quad \text{a.e. in } \; H_h^\tau \cap Q_0$$

and

$$\|\nabla v_k^{\tau,(i)}\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \leqslant c_5(n) L_\tau. \tag{5.15}$$

Now, let $x' \in \mathbb{R}^n$ be such that $\mathrm{dist}(x', Q_0) > 1$, then

$$|v_k^{\tau,(i)}(x)| = |v_k^{\tau,(i)}(x) - v_k^{\tau,(i)}(x')| \leqslant c_5(n) L_\tau(\mathrm{diam}(Q_0) + 2), \tag{5.16}$$

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for every $x \in Q_0$. Hence, gathering (5.15) and (5.16) entails

$$\sup_k \|v_k^{\tau,(i)}\|_{W^{1,\infty}(Q_0)} < +\infty,$$

for every $i = 1, \dots, m$, and every $\tau > 0$. Therefore, up to subsequences (not relabelled), for every $\tau > 0$ fixed, we get that in particular

$$v_k^{\tau,(i)} \to v^{\tau,(i)} \text{ in } L^\infty(Q_0)$$

as $k \to \infty$, with

$$\|\nabla v^{\tau,(i)}\|_{L^\infty(Q_0;\mathbb{R}^n)} \leqslant c_5(n)L_\tau,$$

for every $i = 1, \dots, m$. Finally, set

$$v_k^\tau := (v_k^{\tau,(1)}, \dots, v_k^{\tau,(m)}), \quad v^\tau := (v^{\tau,(1)}, \dots, v^{\tau,(m)}).$$

Now, define the set $B_\tau := \{x \in A \colon v^\tau(x) \neq u(x)\}$, then it must hold

$$|B_\tau| \leqslant (m+1)\tau. \tag{5.17}$$

To prove (5.17) we start observing that there exists a subsequence (k_j) such that if

$$E := \left\{x \in A \colon \lim_{j\to\infty} u_{k_j}(x) = u(x)\right\}$$

then $|A \setminus E| = 0$; hence, as a consequence, $|B_\tau \cap E| = |B_\tau|$. Moreover, since $v_k^\tau \to v^\tau$ in $L^\infty(A;\mathbb{R}^m)$ as $k \to \infty$ we have that

$$\lim_{k\to\infty} v_k^\tau(x) = v^\tau(x) \tag{5.18}$$

for every $x \in A$ and hence, in particular, for every $x \in B_\tau$.

Assume by contradiction that $|B_\tau| > (m+1)\tau$, then by (5.14) we obtain

$$|B_\tau \cap E \cap H_{k_j}^\tau| = |B_\tau \cap H_{k_j}^\tau| > \tau, \tag{5.19}$$

for every $j \in \mathbb{N}$. Therefore, by (5.19) and lemma 2.10 there exists $(k_{j_h}) \subset (k_j)$ such that

$$\bigcap_{h\in\mathbb{N}} (B_\tau \cap E \cap H_{k_{j_h}}^\tau) \neq \emptyset.$$

Thus, if x belongs to the set above by (5.18) we get

$$v^\tau(x) = \lim_{h\to\infty} v_{k_{j_h}}^\tau(x) = \lim_{j\to\infty} u_{k_{j_h}}(x) = u(x),$$

which is a contradiction in view of the definition of B_τ. Therefore, (5.17) holds.

To conclude, it only remains to prove the energy estimate (5.4). Let E_τ be as in (5.10); by the nonnegativity of f_k we have

$$\int_A f_k(x, \nabla u_k)\,\mathrm{d}x \geqslant \int_{(A\setminus E_\tau)\cap H_k^\tau} f_k(x, \nabla v_k^\tau)\,\mathrm{d}x$$
$$= \int_{(A\setminus E_\tau)} f_k(x, \nabla v_k^\tau)\,\mathrm{d}x - \int_{(A\setminus E_\tau)\setminus H_k^\tau} f_k(x, \nabla v_k^\tau)\,\mathrm{d}x. \tag{5.20}$$

By (5.14) we get

$$|(A\setminus E_\tau)\setminus H_h^\tau| \leqslant \sum_{i=1}^m |(A\setminus E_\tau)\setminus H_{i,h}^\tau| < m\min\{\tau, \delta_\tau\}, \tag{5.21}$$

hence invoking (3.2), (5.15), (5.14), (5.11), (2.3), proposition 2.5 and (5.21) we obtain

$$\begin{aligned}\int_{(A\setminus E_\tau)\setminus H_k^\tau} f_k(x, \nabla v_k^\tau)\,\mathrm{d}x &\leqslant \beta\int_{(A\setminus E_\tau)\setminus H_k^\tau} \lambda_k(|\nabla v_k^\tau|^p+1)\,\mathrm{d}x\\
&\leqslant \beta m^{p-1}c_5(n)^p L_\tau^p \int_{(A\setminus E_\tau)\setminus H_k^\tau}\lambda_k\,\mathrm{d}x + \beta\int_{A\setminus H_k^\tau}\lambda_k\,\mathrm{d}x\\
&\leqslant \beta m^{p-1}c_5(n)^p L_\tau^p \sum_{i=1}^m \int_{(A\setminus E_\tau)\setminus H_{i,k}^\tau}\lambda_k\,\mathrm{d}x\\
&\quad + \beta c c_2|Q_0|\left(\frac{|A\setminus H_k^\tau|}{|Q_0|}\right)^{\sigma/(1+\sigma)}\\
&\leqslant \beta m^{p-2}c_5(n)^p \sum_{i=1}^m \int_{(A\setminus E_\tau)\setminus H_{i,k}^\tau}\lambda_k(M|\nabla u_k^{(i)}|)^p\,\mathrm{d}x\\
&\quad + \beta c c_2|Q_0|\left(\frac{m\tau}{|Q_0|}\right)^{\sigma/(1+\sigma)}\\
&\leqslant \beta m^{p-1}c_5(n)^p\tau + \alpha_\tau,\end{aligned} \tag{5.22}$$

where $\alpha_\tau := \beta c c_2|Q_0|(m\tau/|Q_0|)^{\sigma/(1+\sigma)}$; thus $\alpha_\tau \to 0$, as $\tau \to 0^+$.

Now let $A_\tau \subset A$ be an open set containing $A\setminus E_\tau$ and such that

$$\left|\int_{A_\tau} f_k(x, \nabla v_k^\tau)\,\mathrm{d}x - \int_{A\setminus E_\tau} f_k(x, \nabla v_k^\tau)\,\mathrm{d}x\right| < \tau. \tag{5.23}$$

We note that this choice is always possible thanks to the growth conditions satisfied by f_k (3.2), to (5.15), and in view of proposition 2.5. Indeed, we have

$$\begin{aligned}\int_{A_\tau\setminus(A\setminus E_\tau)} f_k(x, \nabla v_k^\tau)\,\mathrm{d}x &\leqslant \beta(m^{p-1}c_5(n)^p L_\tau^p+1)\int_{A_\tau\setminus(A\setminus E_\tau)}\lambda_k\,\mathrm{d}x\\
&\leqslant \beta(m^{p-1}c_5(n)^p L_\tau^p+1)c c_2|Q_0|\left(\frac{|A_\tau\setminus(A\setminus E_\tau)|}{|Q_0|}\right)^{\sigma/(1+\sigma)},\end{aligned}$$

moreover, $|A\setminus A_\tau| \leqslant |E_\tau| < \tau$.

Eventually, by combining (5.20), (5.22) and (5.23) we deduce

$$\int_A f_k(x, \nabla u_k)\,\mathrm{d}x \geqslant \int_{A_\tau} f_k(x, \nabla v_k^\tau)\,\mathrm{d}x - \alpha_\tau - \tau(\beta m^{p-1} c_5(n)^p + 1),$$

and hence the claim follows with $\beta_\tau := \alpha_\tau + \tau(\beta m^{p-1} c_5(n)^p + 1)$. □

We are now in a position to show that, up to subsequences, the functionals F_k Γ-converge to F_∞.

THEOREM 5.2. *Let F_k and F_∞ be the functionals defined in (3.4) and (5.1), respectively. Then there exists a subsequence (k_h) such that for every $u \in W^{1,1}(Q_0;\mathbb{R}^m)$ and for every $A \in \mathcal{A}(Q_0)$ with $A \subset\subset Q_0$ there holds*

$$F'(u, A) = F''(u, A) = F_\infty(u, A), \tag{5.24}$$

where F' and F'' are, respectively, as in (4.4) and (4.5) with k replaced by k_h.

Proof. In all that follows (k_h) denotes the subsequence provided by theorem 4.10.

We divide the proof into two main steps.

Step 1: Lower bound. In this step, we prove that

$$F'(u, A) \geqslant F_\infty(u, A), \tag{5.25}$$

for every $u \in W^{1,1}(Q_0;\mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$ with $A \subset\subset Q_0$.

To this end, let $u \in W^{1,1}(Q_0;\mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$, $A \subset\subset Q_0$ be fixed.

Substep 1.1: $u \in W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m)$. By [**14**, proposition 8.1] there exists $(u_h) \subset W^{1,1}(Q_0;\mathbb{R}^m)$ with $u_h \to u$ in $L^1(Q_0;\mathbb{R}^m)$ such that

$$F'(u, A) = \liminf_{h\to\infty} F_{k_h}(u_h, A). \tag{5.26}$$

We observe that lemma 4.3 guarantees that $F'(u, A) < +\infty$; therefore, $(u_h) \subset W^{1,p}_{\lambda_h}(A;\mathbb{R}^m)$ and (up to possibly passing to a subsequence) by (3.2) we get

$$\sup_{h\in\mathbb{N}} \int_A \lambda_{k_h} |\nabla u_h|^p\,\mathrm{d}x < +\infty.$$

Now let $\tau > 0$ be fixed and arbitrary; theorem 5.1 provides us with (β_τ), infinitesimal as $\tau \to 0^+$, $A_\tau \subset A$, with $|A \setminus A_\tau| < \tau$, and (v_h^τ), v^τ in $W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$, such that $v_h^\tau \to v^\tau$ in $L^1(Q_0;\mathbb{R}^m)$, as $h \to \infty$. Moreover, by (5.26) and (5.4) we obtain

$$\begin{aligned} F'(u, A) &= \liminf_{h\to\infty} \int_A f_{k_h}(x, \nabla u_h)\,\mathrm{d}x \\ &\geqslant \liminf_{h\to\infty} \int_{A_\tau} f_{k_h}(x, \nabla v_h^\tau)\,\mathrm{d}x - \beta_\tau \geqslant \int_{A_\tau} f_\infty(x, \nabla v^\tau)\,\mathrm{d}x - \beta_\tau, \end{aligned} \tag{5.27}$$

where the last inequality follows by theorem 4.10, since $v^\tau \in W^{1,\infty}(Q_0;\mathbb{R}^m)$.

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Now let $B_\tau := \{x \in A \colon v^\tau(x) \neq u(x)\}$ be as in the proof of theorem 5.1 and recall that

$$|B_\tau| \leqslant (m+1)\tau. \tag{5.28}$$

By (5.27) and the nonnegativity of f_∞ we have

$$F'(u, A) \geqslant \int_{A_\tau \setminus B_\tau} f_\infty(x, \nabla u)\, \mathrm{d}x - \beta_\tau, \tag{5.29}$$

for every $\tau > 0$. Now, since $|A \setminus A_\tau| < \tau$, using (5.28) we get

$$|A \setminus (A_\tau \setminus B_\tau)| \leqslant (m+2)\tau, \tag{5.30}$$

thus, thanks to (4.2) and (5.30), we can pass to the limit as $\tau \to 0^+$ in (5.29) and obtain

$$F'(u, A) \geqslant \int_A f_\infty(x, \nabla u)\, \mathrm{d}x = F_\infty(u, A), \tag{5.31}$$

hence the lower bound for $u \in W^{1,p}_{\lambda_\infty}(A; \mathbb{R}^m)$.

Substep 1.2: $u \notin W^{1,p}_{\lambda_\infty}(A; \mathbb{R}^m)$. In this case, from (5.1) we have $F_\infty(u,\, A) = +\infty$, hence to conclude we need to show that $F'(u,\, A) = +\infty$. Assume by contradiction that

$$F'(u, A) < +\infty.$$

If this is the case, we may argue exactly as in substep 1.1 and get

$$+\infty > F'(u, A) \geqslant \int_{A_\tau \setminus B_\tau} f_\infty(x, \nabla u)\, \mathrm{d}x - \beta_\tau.$$

By the Fatou lemma, (4.2) and proposition 2.6 this yields $u \in W^{1,p}_{\lambda_\infty}(A; \mathbb{R}^m)$ and hence a contradiction.

Step 2. Upper bound. In this step, we prove that

$$F''(u, A) \leqslant F_\infty(u, A), \tag{5.32}$$

for every $u \in W^{1,1}(Q_0; \mathbb{R}^m)$ and every $A \in \mathcal{A}(Q_0)$ with $A \subset\subset Q_0$.

To this end, let $u \in W^{1,1}(Q_0; \mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$, $A \subset\subset Q_0$ be fixed. We start observing that by the definition of F_∞, if $u \notin W^{1,p}_{\lambda_\infty}(A; \mathbb{R}^m)$ then there is nothing to prove. Therefore, we only consider the case $u \in W^{1,p}_{\lambda_\infty}(A; \mathbb{R}^m)$.

Since A is Lipschitz, by [**34**, theorem 2.1.13] we can find a function $\tilde{u} \in W^{1,p}_{\lambda_\infty}(Q_0; \mathbb{R}^m)$ with $u = \tilde{u}$ a.e. in A. Then, by density (see, e.g. [**34**, corollary 2.1.6]) there exists $(u_j) \subset W^{1,\infty}(Q_0; \mathbb{R}^m)$ such that $u_j \to \tilde{u}$ in $W^{1,p}_{\lambda_\infty}(Q_0; \mathbb{R}^m)$. Then, by the locality of F'' and F_∞, the continuity of F_∞ in $W^{1,p}_{\lambda_\infty}(Q_0; \mathbb{R}^m)$, and the $L^1(Q_0; \mathbb{R}^m)$-lower semicontinuity of F'', invoking theorem 4.10 we deduce

$$F_\infty(u, A) = F_\infty(\tilde{u}, A) = \lim_{j \to \infty} F_\infty(u_j, A) = \liminf_{j \to \infty} F''(u_j, A) \geqslant F''(\tilde{u}, A) = F''(u, A),$$

thus the upper bound.

Eventually (5.24) follows by gathering (5.25) and (5.32). $\square$

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6. Convergence of minimization problems

In this section, we modify the domain of the functionals F_k by prescribing boundary conditions of Dirichlet type. We then study the Γ-convergence of the corresponding functionals and prove a convergence result for the associated minimization problems.

We start by proving a preliminary energy bound.

PROPOSITION 6.1. *Let F_k be the functionals defined in (3.4). Then there exist an exponent $\delta > 0$ and a constant $C > 0$ such that*

$$\left(\int_A |\nabla u|^{1+\delta}\,\mathrm{d}x\right)^{p/(1+\delta)} \leqslant C\big(F_k(u, A) + 1\big), \tag{6.1}$$

for every $A \in \mathcal{A}(Q_0)$, every $u \in W^{1,p}_{\lambda_k}(A;\mathbb{R}^m)$, and every $k \in \mathbb{N}$.

Proof. By theorem 2.2 we can deduce the existence of an exponent $\sigma > 0$ and a constant $c > 0$ such that

$$\left(\fint_Q \lambda_k^{-(1+\sigma)/(p-1)}\,\mathrm{d}x\right)^{1/(1+\sigma)} \leqslant c\left(\fint_Q \lambda_k^{-1/(p-1)}\,\mathrm{d}x\right), \tag{6.2}$$

for every cube Q and for every $k \in \mathbb{N}$.

Now, let $A \in \mathcal{A}(Q_0)$ and $u \in W^{1,p}_{\lambda_k}(A;\mathbb{R}^m)$ be arbitrary, and let $\delta > 0$ to be chosen later. By the Hölder inequality we have

$$\int_A |\nabla u|^{1+\delta}\,\mathrm{d}x \leqslant \left(\int_A \lambda_k |\nabla u|^p\,\mathrm{d}x\right)^{(1+\delta)/p} \left(\int_A \lambda_k^{-(1+\delta)/(p-1-\delta)}\,\mathrm{d}x\right)^{(p-1-\delta)/p}.$$

For $\delta := (p-1)\sigma/(p+\sigma)$ it is immediate to check that

$$\frac{1+\delta}{p-1-\delta} = \frac{1+\sigma}{p-1}; \tag{6.3}$$

hence by (6.2) we readily get

$$\begin{aligned}
&\int_A |\nabla u|^{1+\delta}\,\mathrm{d}x \\
&\quad\leqslant c^{(p-1-\delta)(1+\sigma)/p} |Q_0|^{(p-1-\delta)/p} \left(\int_A \lambda_k |\nabla u|^p\,\mathrm{d}x\right)^{(1+\delta)/p} \\
&\quad\times \left(\fint_{Q_0} \lambda_k^{-1/(p-1)}\,\mathrm{d}x\right)^{(p-1-\delta)(1+\sigma)/p}.
\end{aligned}$$

Moreover, since λ_k belongs to $A_p(K)$, by (3.1) we also deduce that

$$\left(\fint_{Q_0} \lambda_k^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1} \leqslant \frac{K}{c_1}$$

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and therefore

$$\begin{aligned}\int_A |\nabla u|^{1+\delta}\,\mathrm{d}x & \\ &\leqslant c^{(p-1-\delta)(1+\sigma)/p}|Q_0|^{(p-1-\delta)/p}\left(\frac{K}{c_1}\right)^{(p-1-\delta)(1+\sigma)/p(p-1)} \\ &\quad\times\left(\int_A \lambda_k|\nabla u|^p\,\mathrm{d}x\right)^{(1+\delta)/p}.\end{aligned}$$

Eventually, gathering (3.1), (3.2) and (6.3) gives

$$\begin{aligned}&\left(\int_A |\nabla u|^{1+\delta}\,\mathrm{d}x\right)^{p/(1+\delta)} \\ &\quad\leqslant c^{p-1}|Q_0|^{(p-1)/(1+\sigma)}\left(\frac{K}{c_1}\right)\int_A \lambda_k(x)(|\nabla u|^p-1)\,\mathrm{d}x \\ &\qquad + c^{p-1}|Q_0|^{(p-1)/(1+\sigma)}\left(\frac{K}{c_1}\right)\int_A \lambda_k(x)\,\mathrm{d}x \\ &\quad\leqslant c^{p-1}|Q_0|^{(p-1)/(1+\sigma)}\left(\frac{K}{c_1\alpha}\right)F_k(u,A)+c^{p-1}|Q_0|^{(p-1)/(1+\sigma)}\left(\frac{K}{c_1}\right)c_2|Q_0|,\end{aligned}$$

for every $k\in\mathbb{N}$. Hence, (6.1) immediately follows by choosing $C:=\max\{C_1,\,C_2\}$ with

$$C_1:=c^{p-1}|Q_0|^{(p-1)/(1+\sigma)}\left(\frac{K}{c_1\alpha}\right)\ \text{and}\ C_2:=\alpha c_2|Q_0|C_1.$$

□

Let F_k be functionals defined in (3.4). We consider $F_k^{\psi}: W^{1,1}(Q_0;\mathbb{R}^m)\times\mathcal{A}(Q_0)\longrightarrow[0,+\infty]$ given by

$$F_k^{\psi}(u,A):=\begin{cases}F_k(u,A) & \text{if } u\in W^{1,p}_{0,\lambda_k}(A;\mathbb{R}^m)+\psi,\\ +\infty & \text{otherwise,}\end{cases}\tag{6.4}$$

with $\psi\in W^{1,\infty}(Q_0;\mathbb{R}^m)$.

We are now in a position to prove a Γ-convergence result for the functionals F_k^{ψ}.

Theorem 6.2 (Γ-convergence with boundary data). *Let F_k^{ψ} be the functionals defined in (6.4). Then there exists a subsequence (k_h) such that for every $A\in\mathcal{A}(Q_0)$, $A\subset\subset Q_0$*

$$F_\infty^{\psi}(\cdot,A)=\Gamma\text{-}\lim_{h\to\infty}F_{k_h}^{\psi}(\cdot,A),$$

where $F_\infty^{\psi}: W^{1,1}(Q_0;\mathbb{R}^m)\times\mathcal{A}(Q_0)\longrightarrow[0,+\infty]$ is given by

$$F_\infty^{\psi}(u,A):=\begin{cases}F_\infty(u,A) & \text{if } u\in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m)+\psi,\\ +\infty & \text{otherwise,}\end{cases}\tag{6.5}$$

with F_∞ as in (5.1).

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Proof. Let $u \in W^{1,1}(Q_0;\mathbb{R}^m)$ and $A \in \mathcal{A}(Q_0)$, with $A \subset\subset Q_0$ be fixed and let (k_h) be the subsequence whose existence is guaranteed by theorem 5.2.

We divide the proof into two main steps.

Step 1: Lower bound. Let $(u_h) \subset W^{1,1}(Q_0;\mathbb{R}^m)$ be such that $u_h \to u$ in $L^1(Q_0;\mathbb{R}^m)$. In this step, we want to show that

$$\liminf_{h\to\infty} F^{\psi}_{k_h}(u_h, A) \geqslant F^{\psi}_{\infty}(u, A). \tag{6.6}$$

We note that we can always assume that

$$\liminf_{h\to\infty} F^{\psi}_{k_h}(u_h, A) < +\infty, \tag{6.7}$$

otherwise there is nothing to prove. Moreover, without loss of generality, we may also assume that the liminf in (6.7) is actually a limit. Then, by the definition of $F^{\psi}_{k_h}$ we have that $(u_h) \subset W^{1,p}_{0,\lambda_{k_h}}(A;\mathbb{R}^m) + \psi$; while by theorem 5.2 we get that $u \in W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m)$ and

$$F_\infty(u, A) \leqslant \liminf_{h\to\infty} F_{k_h}(u_h, A) = \liminf_{h\to\infty} F^{\psi}_{k_h}(u_h, A).$$

Since $W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) = W^{1,1}_0(A;\mathbb{R}^m) \cap W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m)$, to conclude it is enough to show that u belongs to $W^{1,1}_0(A;\mathbb{R}^m) + \psi$.

To this end, we start observing that thanks to (6.7), proposition 6.1 yields the existence of an exponent $\delta > 0$ and of a constant $C > 0$ such that

$$\int_A |\nabla u_h|^{1+\delta}\,\mathrm{d}x \leqslant C,$$

for every $h \in \mathbb{N}$. Then, by Poincaré's inequality the sequence (u_h) is bounded in $W^{1,1+\delta}(A;\mathbb{R}^m)$. This readily implies that, up to subsequences, $u_h \rightharpoonup u$ in $W^{1,1+\delta}(A;\mathbb{R}^m)$. Since $(u_h) \subset W^{1,1+\delta}_0(A;\mathbb{R}^m) + \psi$ and this space is weakly closed, we immediately get $u \in W^{1,1+\delta}_0(A;\mathbb{R}^m) + \psi$, and therefore the claim.

Step 2: Upper bound. We start by considering the case $u \in C^\infty_0(A;\mathbb{R}^m) + \psi$.

By proposition 4.5 and theorem 5.2 there exists a sequence $(u_h) \subset W^{1,p}_{\lambda_{k_h}}(A;\mathbb{R}^m)$ such that $u_h \to u$ in $L^q(Q_0;\mathbb{R}^m)$ for every $1 \leqslant q < +\infty$ and

$$\limsup_{h\to\infty} F_{k_h}(u_h, A) \leqslant F_\infty(u, A) = F^{\psi}_{\infty}(u, A). \tag{6.8}$$

Starting from u_h we now want to construct a recovery sequence which also satisfies the boundary condition. To this purpose, let $\eta > 0$ be fixed. By the equi-integrability of the sequence (λ_{k_h}) (cf. proposition 2.5) there exists a compact set $K_\eta \subset A$ such that

$$\int_{A\setminus K_\eta} \lambda_{k_h} |\nabla u|^p\,\mathrm{d}x \leqslant \|\nabla u\|^p_{L^\infty(A;\mathbb{R}^{m\times n})} \int_{A\setminus K_\eta} \lambda_{k_h}\,\mathrm{d}x < \|\nabla u\|^p_{L^\infty(A;\mathbb{R}^{m\times n})}\eta, \tag{6.9}$$

for every $h \in \mathbb{N}$.

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Choose A', $A'' \in \mathcal{A}(Q_0)$ such that $K_\eta \subset A' \subset\subset A'' \subset\subset A$. Then, proposition 4.6 ensures the existence of a positive constant M_η and a sequence (φ_h) of cut-off functions between A' and A'' such that

$$\begin{aligned} & F_{k_h}(\varphi_h u_h + (1-\varphi_h)u, A) \\ & \quad \leqslant (1+\eta)\left(F_{k_h}(u_h, A'') + F_{k_h}(u, A \setminus K_\eta)\right) + M_\eta \int_A \lambda_{k_h} |u_h - u|^p \,\mathrm{d}x + \eta. \end{aligned} \tag{6.10}$$

Set $w_h := \varphi_h u_h + (1-\varphi_h)u$; then by definition $(w_h) \subset W^{1,p}_{0,\lambda_{k_h}}(A;\mathbb{R}^m) + \psi$ and $w_h \to u$ in $L^q(Q_0;\mathbb{R}^m)$ for every $1 \leqslant q < +\infty$. Moreover, by (6.10) and (3.2) we get

$$\begin{aligned} F_{k_h}(w_h, A) & \leqslant (1+\eta) F_{k_h}(u_h, A) \\ & \quad + (1+\eta)\beta \int_{A\setminus K_\eta} \lambda_{k_h} (|\nabla u|^p + 1)\,\mathrm{d}x + M_\eta \int_A \lambda_{k_h} |u_h - u|^p \,\mathrm{d}x + \eta. \end{aligned} \tag{6.11}$$

Hence, by (6.8), (6.9) and (6.11) we have

$$\begin{aligned} \Gamma\text{-}\limsup_{h\to\infty} F^\psi_{k_h}(u, A) & \leqslant \limsup_{h\to\infty} F^\psi_{k_h}(w_h, A) \\ & \leqslant (1+\eta) \limsup_{h\to\infty} F_{k_h}(u_h, A) \\ & \quad + (1+\eta)\beta(\|\nabla u\|^p_{L^\infty(A;\mathbb{R}^{m\times n})} + 1)\eta + \eta \\ & \leqslant (1+\eta) F^\psi_\infty(u, A) + (1+\eta)\beta(\|\nabla u\|^p_{L^\infty(A;\mathbb{R}^{m\times n})} + 1)\eta + \eta. \end{aligned}$$

Therefore, by the arbitrariness of $\eta > 0$ we conclude that

$$\Gamma\text{-}\limsup_{h\to\infty} F^\psi_{k_h}(u, A) \leqslant F^\psi_\infty(u, A), \tag{6.12}$$

for every $u \in C^\infty_0(A;\mathbb{R}^m) + \psi$.

Now, let $u \in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) + \psi$. We extend u to ψ outside A; we clearly have that the extended function (still denoted by u) belongs to $W^{1,p}_{0,\lambda_\infty}(Q_0;\mathbb{R}^m) + \psi$. Now, let $(u_j) \subset C^\infty_0(Q_0;\mathbb{R}^m)$ be such that $u_j \to u$ in $W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$, hence, in particular, $u_j \to u$ strongly in $L^1(Q_0;\mathbb{R}^m)$. By the $W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$-continuity of F^ψ_∞, by (6.12), and by the lower semicontinuity of the Γ-limsup with respect to the strong topology of $L^1(Q_0;\mathbb{R}^m)$ we get

$$F^\psi_\infty(u, A) = \lim_{j\to\infty} F^\psi_\infty(u_j, A) \geqslant \lim_{j\to\infty} \Gamma\text{-}\limsup_{h\to\infty} F^\psi_{k_h}(u_j, A) \geqslant \Gamma\text{-}\limsup_{h\to\infty} F^\psi_{k_h}(u, A),$$

for every $u \in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) + \psi$, and therefore the upper bound. □

The following result shows that the functionals F^ψ_k are equi-coercive with respect to the strong $L^1(Q_0;\mathbb{R}^m)$-topology.

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PROPOSITION 6.3 (Equi-coerciveness). *Let F_k^{ψ} be functionals defined in (6.4), let $A \in \mathcal{A}(Q_0)$, $A \subset\subset Q_0$, and let $(u_k) \subset W^{1,1}(A;\mathbb{R}^m)$ be such that*

$$\sup_{k\in\mathbb{N}} F_k^{\psi}(u_k, A) < +\infty. \tag{6.13}$$

Then there exist a subsequence $(u_{k_h}) \subset (u_k)$ and an exponent $\delta > 0$ such that

$$u_{k_h} \rightharpoonup u \quad \text{weakly in } W^{1,1+\delta}(A;\mathbb{R}^m),$$

with $u \in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) + \psi$. Moreover, if we extend u_{k_h} and u to Q_0 by setting $u_{k_h} := \psi$ and $u := \psi$ in $Q_0 \setminus A$, respectively, then $u_{k_h} \to u$ in $L^1(Q_0;\mathbb{R}^m)$.

Proof. By (6.13) and by (6.4) we have $u_k \in W^{1,p}_{0,\lambda_k}(A;\mathbb{R}^m) + \psi$, for every $k \in \mathbb{N}$. Then, arguing exactly as in the proof of theorem 6.2 we may deduce the existence of a subsequence $(u_{k_h}) \subset (u_k)$ which weakly converges in $W^{1,1+\delta}(A;\mathbb{R}^m)$ to a function $u \in W_0^{1,1}(A;\mathbb{R}^m) + \psi$. Furthermore, by the compact embedding of $W^{1,1+\delta}(A;\mathbb{R}^m)$ in $L^{1,1+\delta}(A;\mathbb{R}^m)$ we have that, in particular, $u_{k_h} \to u$ in $L^1(A;\mathbb{R}^m)$. Now, extend u_{k_h} and u by setting $u_{k_h} := \psi$, $u := \psi$ in $Q_0 \setminus A$. Then, clearly $u_{k_h} \to u$ in $L^1(Q_0;\mathbb{R}^m)$. Hence, by theorem 6.2 and by (6.13) there holds

$$F_\infty^{\psi}(u, A) \leqslant \liminf_{h\to\infty} F_{k_h}^{\psi}(u_{k_h}, A) < +\infty,$$

thus by (6.5) we get $u \in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) + \psi$. □

Thanks to the fundamental property of Γ-convergence, by combining theorem 6.2 and proposition 6.3 we obtain the following convergence result for the associated minimization problems.

THEOREM 6.4. *Let $A \subset \mathcal{A}(Q_0)$ with $A \subset\subset Q_0$. Let f_k be functions satisfying (3.2) and (3.3) and set*

$$M_k := \inf\left\{\int_A f_k(x, \nabla u)\,\mathrm{d}x \colon u \in W^{1,p}_{0,\lambda_k}(A;\mathbb{R}^m) + \psi\right\}.$$

Let $(u_k) \subset W^{1,p}_{0,\lambda_k}(A;\mathbb{R}^m) + \psi$ be such that

$$\lim_{k\to\infty}(F_k^{\psi}(u_k, A) - M_k) = 0.$$

Then, up to subsequences (not relabelled), $u_k \to u_\infty$ in $L^1(A;\mathbb{R}^m)$ with u_∞ solution to

$$M_\infty := \min\left\{\int_A f_\infty(x, \nabla u)\,\mathrm{d}x \colon u \in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) + \psi\right\},$$

Moreover, we have $M_k \to M_\infty$, as $k \to +\infty$.

7. Asymptotic formula for f_∞

In this section, we derive an asymptotic formula for the integrand of the Γ-limit, f_∞. This formula will be particularly useful when proving the homogenization result in § 8.

In all that follows $F_\infty \colon W^{1,1}(Q_0;\mathbb{R}^m) \times \mathcal{A}(Q_0) \longrightarrow [0, +\infty]$ denotes the Γ-limit of (F_{k_h}) where (k_h) as in theorem 5.2. That is, F_∞ coincides with the integral functional

$$F_\infty(u, A) = \int_A f_\infty(x, \nabla u)\,\mathrm{d}x,$$

for every $u \in W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$, where for a.e. $x \in Q_0$ and for every $\xi \in \mathbb{R}^{m\times n}$

$$f_\infty(x, \xi) = \limsup_{\rho\to 0^+} \frac{F_\infty(u_\xi, Q_\rho(x))}{|Q_\rho(x)|}; \tag{7.1}$$

moreover, f_∞ satisfies (4.2) and (4.3) (cf. theorem 4.1). We also recall that, being F_∞ a Γ-limit, it is lower semicontinuous with respect to the strong $L^1(Q_0;\mathbb{R}^m)$-convergence.

The following theorem is the main result of this section.

Theorem 7.1. *For almost every $x \in Q_0$ and every $\xi \in \mathbb{R}^{m\times n}$ there holds*

$$f_\infty(x, \xi) := \limsup_{\rho\to 0^+} \frac{m_{F_\infty}(u_\xi, Q_\rho(x))}{\rho^n}, \tag{7.2}$$

where, for every $A \in \mathcal{A}(Q_0)$,

$$m_{F_\infty}(u_\xi, A) := \min\left\{F_\infty(v, A) \colon v \in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) + u_\xi\right\}.$$

The proof of theorem 7.1 will be achieved by combining lemmas 7.3–7.5 below, by following the same strategy as in [**4**, § 3] (see also [**5**, § 2.2]).

As an immediate corollary of theorems 6.4 and 7.1 we also obtain the following asymptotic formula for f_∞.

Corollary 7.2 (Asymptotic formula for f_∞). *For almost every $x \in Q_0$ and every $\xi \in \mathbb{R}^{m\times n}$ there holds*

$$f_\infty(x, \xi) := \limsup_{\rho\to 0^+} \lim_{h\to\infty} \frac{m_{F_{k_h}}(u_\xi, Q_\rho(x))}{\rho^n}, \tag{7.3}$$

where, for every $A \in \mathcal{A}(Q_0)$,

$$m_{F_{k_h}}(u_\xi, A) := \inf\left\{F_{k_h}(v, A) \colon v \in W^{1,p}_{0,\lambda_{k_h}}(A;\mathbb{R}^m) + u_\xi\right\}.$$

We now turn to the proof of theorem 7.1; to this end, we need to introduce the following notation. Set $\mathcal{A}^* := \{Q_\rho(x) \colon x \in Q_0,\ \rho > 0\}$ and let $\delta > 0$. For $A \in \mathcal{A}(Q_0)$

define

$$m^{\delta}_{F_\infty}(u_\xi, A) := \inf\left\{\sum_{i=1}^{\infty} m_{F_\infty}(u_\xi, Q_i) \colon Q_i \in \mathcal{A}^*,\ Q_i \cap Q_j = \emptyset,\right.$$
$$\left.\operatorname{diam}(Q_i) < \delta, |A \setminus \cup_{i=1}^{\infty} Q_i| = 0\right\}.$$

We note that $m^{\delta}_{F_\infty}$ is decreasing in δ; hence for every $A \in \mathcal{A}(Q_0)$ we can consider

$$m^{*}_{F_\infty}(u_\xi, A) := \lim_{\delta \to 0^+} m^{\delta}_{F_\infty}(u_\xi, A). \tag{7.4}$$

We start by proving the following technical lemma which is an adaptation from [**4**, lemma 3.3] to the setting of weighted Sobolev spaces.

LEMMA 7.3. *Let $A \in \mathcal{A}(Q_0)$, $A \subset\subset Q_0$; there holds*

$$F_\infty(u_\xi, A) = m^{*}_{F_\infty}(u_\xi, A).$$

Proof. We observe that the inequality

$$F_\infty(u_\xi, A) \geqslant m^{*}_{F_\infty}(u_\xi, A) \tag{7.5}$$

is an immediate consequence of the definition of $m^{*}_{F_\infty}$. Indeed, let $\delta > 0$ be fixed and let (Q_i) be an admissible sequence in the sense of the definition of $m^{\delta}_{F_\infty}(u_\xi,\ A)$, then

$$m^{\delta}_{F_\infty}(u_\xi, A) \leqslant \sum_{i=1}^{\infty} m_{F_\infty}(u_\xi, Q_i) \leqslant \sum_{i=1}^{\infty} F_\infty(u_\xi, Q_i) = F_\infty(u_\xi, A),$$

thus (7.5) follows by taking the limit as $\delta \to 0^+$.

We now prove the converse inequality; i.e.

$$F_\infty(u_\xi, A) \leqslant m^{*}_{F_\infty}(u_\xi, A). \tag{7.6}$$

To this end, let $\delta > 0$ be fixed and let (Q^{δ}_i) be an admissible sequence in the definition of $m^{\delta}_{F_\infty}(u_\xi,\ A)$ such that

$$\sum_{i=1}^{\infty} m_{F_\infty}(u_\xi, Q^{\delta}_i) \leqslant m^{\delta}_{F_\infty}(u_\xi, A) + \delta. \tag{7.7}$$

By definition of m_{F_∞}, for every $i \in \mathbb{N}$ we can choose $v^{\delta}_i \in W^{1,p}_{0,\lambda_\infty}(Q^{\delta}_i; \mathbb{R}^m) + u_\xi$ such that

$$F_\infty(v^{\delta}_i, Q^{\delta}_i) \leqslant m_{F_\infty}(u_\xi, Q^{\delta}_i) + \delta |Q^{\delta}_i|. \tag{7.8}$$

Set

$$v^{\delta} := \sum_{i=1}^{\infty} v^{\delta}_i \chi_{Q^{\delta}_i} + u_\xi \chi_{Q_0 \setminus \cup_{i=1}^{\infty} Q^{\delta}_i};$$

we claim that $v^\delta \in W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$. To prove the claim define

$$v^{\delta,N} := \sum_{i=1}^{N} v_i^\delta \chi_{Q_i^\delta} + u_\xi \chi_{Q_0 \setminus \cup_{i=1}^N Q_i^\delta},$$

clearly $v^{\delta,N} \in W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$ and $v^{\delta,N} \to v^\delta$ a.e. in Q_0, as $N \to \infty$. Since $v_i^\delta \in W^{1,p}_{0,\lambda_\infty}(Q_i^\delta;\mathbb{R}^m) + u_\xi$ for every $i = 1, \ldots, N$, by the Poincaré inequality in weighted Sobolev spaces (see, e.g. [**23**, corollary 1]) we have

$$\begin{aligned}\|v^{\delta,N} - u_\xi\|^p_{W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)} &= \sum_{i=1}^{N} \|v_i^\delta - u_\xi\|^p_{W^{1,p}_{\lambda_\infty}(Q_i^\delta;\mathbb{R}^m)} \\ &\leqslant C(\delta,p) \sum_{i=1}^{N} \|\nabla v_i^\delta - \xi\|^p_{L^p_{\lambda_\infty}(Q_i^\delta;\mathbb{R}^{m\times n})},\end{aligned}$$

for some $C(\delta, p) > 0$. By (4.2), (7.7) and (7.8) we get

$$\begin{aligned}&\|v^{\delta,N} - u_\xi\|^p_{W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)} \\ &\quad\leqslant C(\delta,p)\left(K\sum_{i=1}^{N}\int_{Q_i^\delta} \lambda_\infty \left(\frac{1}{K}|\nabla v_i^\delta|^p - 1\right) \mathrm{d}x + (K + |\xi|^p)\sum_{i=1}^{N}\int_{Q_i^\delta} \lambda_\infty \,\mathrm{d}x\right) \\ &\quad\leqslant C(\delta,p)\left(\frac{K}{\alpha}\sum_{i=1}^{\infty} F_\infty(v_i^\delta, Q_i^\delta) + (K + |\xi|^p)\|\lambda_\infty\|_{L^1(A)}\right) \\ &\quad\leqslant C(\delta,p,K,\alpha)\left(m^\delta_{F_\infty}(u_\xi, A) + \delta + \delta|A| + (1 + |\xi|^p)\|\lambda_\infty\|_{L^1(A)}\right).\end{aligned}$$

Hence, for $\delta > 0$ fixed, the sequence $(v^{\delta,N})$ is bounded in $W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$, uniformly in N. Then, by [**25**, theorem 1.32] v^δ belongs to $W^{1,p}_{\lambda_\infty}(Q_0;\mathbb{R}^m)$ and the claim is proven. Moreover, we have

$$F_\infty(v^\delta, A \setminus \cup_{i=1}^{\infty} Q_i^\delta) = 0; \tag{7.9}$$

indeed, by (4.2)

$$F_\infty(v^\delta, A \setminus \cup_{i=1}^{\infty} Q_i^\delta) \leqslant \beta(|\xi|^p + 1)\int_{A \setminus \cup_{i=1}^{\infty} Q_i^\delta} \lambda_\infty \,\mathrm{d}x = 0$$

since $\lambda_\infty \in L^1(Q_0)$ and $|A \setminus \cup_{i=1}^{\infty} Q_i^\delta| = 0$. By combining (7.7)–(7.9) we deduce that

$$\begin{aligned}F_\infty(v^\delta, A) &= \sum_{i=1}^{\infty} F_\infty(v_i^\delta, Q_i^\delta) + F_\infty(v^\delta, A \setminus \cup_{i=1}^{\infty} Q_i^\delta) \leqslant \sum_{i=1}^{\infty} m_{F_\infty}(u_\xi, Q_i^\delta) + \delta\sum_{i=1}^{\infty}|Q_i^\delta| \\ &\leqslant m^\delta_{F_\infty}(u_\xi, A) + \delta + \delta|A|.\end{aligned} \tag{7.10}$$

We now claim that $v^\delta \to u_\xi$ in $L^1(Q_0;\mathbb{R}^m)$. If so, by virtue of the lower semicontinuity of F_∞ with respect to the strong $L^1(Q_0;\mathbb{R}^m)$-convergence, passing to the

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limit as $\delta \to 0^+$ in (7.10) would give

$$F_\infty(u_\xi, A) \leqslant \liminf_{\delta\to 0^+} F_\infty(v^\delta, A) \leqslant \lim_{\delta\to 0^+} m^\delta_{F_\infty}(u_\xi, A) = m^*_{F_\infty}(u_\xi, A)$$

and therefore (7.6). Hence, to conclude the proof it only remains to show that $v^\delta \to u_\xi$ in $L^1(Q_0;\mathbb{R}^m)$. Since, in particular, $v_i^\delta \in W_0^{1,1}(Q_i^\delta;\mathbb{R}^m) + u_\xi$, by the Poincaré inequality in $W^{1,1}(Q_i;\mathbb{R}^m)$ there exists a constant $C > 0$ such that

$$\begin{aligned}\|v^\delta - u_\xi\|_{L^1(Q_0;\mathbb{R}^m)} = \|v^\delta - u_\xi\|_{L^1(A;\mathbb{R}^m)} &= \sum_{i=1}^\infty \|v_i^\delta - u_\xi\|_{L^1(Q_i^\delta;\mathbb{R}^m)} \\ &\leqslant C\delta \sum_{i=1}^\infty \|\nabla v_i^\delta - \xi\|_{L^1(Q_i^\delta;\mathbb{R}^{m\times n})}. \end{aligned} \tag{7.11}$$

Moreover, arguing similarly as above, by (4.2), (7.7) and (7.8) we deduce

$$\begin{aligned}\sum_{i=1}^\infty \|\nabla v_i^\delta - \xi\|^p_{L^p_{\lambda_\infty}(Q_i^\delta;\mathbb{R}^{m\times n})} &\leqslant \frac{K}{\alpha}\sum_{i=1}^\infty F_\infty(v_i^\delta, Q_i^\delta) + (K + |\xi|^p)\|\lambda_\infty\|_{L^1(A)} \\ &\leqslant m^\delta_{F_\infty}(u_\xi, A) + \delta + \delta|A| + (1+|\xi|^p)\|\lambda_\infty\|_{L^1(A)}. \end{aligned} \tag{7.12}$$

Therefore, gathering (7.4), (7.11) and (7.12) gives the desired convergence and completes the proof. □

We also need the following lemma.

LEMMA 7.4. *Let $A \in \mathcal{A}(Q_0)$, $\delta > 0$ and define $A_\delta := \{x \in A\colon \operatorname{dist}(x, \partial A) > \delta\}$. Then*

$$\lim_{\delta\to 0^+} m_{F_\infty}(u_\xi, A_\delta) = m_{F_\infty}(u_\xi, A).$$

Proof. Let $\delta, \eta > 0$. By the definition of m_{F_∞}, we can choose $v \in W^{1,p}_{0,\lambda_\infty}(A_\delta;\mathbb{R}^m) + u_\xi$ such that

$$F_\infty(v, A_\delta) \leqslant m_{F_\infty}(u_\xi, A_\delta) + \eta. \tag{7.13}$$

Set

$$w = \begin{cases} v & \text{in } A_\delta, \\ u_\xi & \text{in } Q_0 \setminus \overline{A_\delta}, \end{cases}$$

clearly $w \in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) + u_\xi$. Using (4.2) and (7.13) we have

$$\begin{aligned} m_{F_\infty}(u_\xi, A) \leqslant F_\infty(w, A) &= F_\infty(v, A_\delta) + F_\infty(u_\xi, A\setminus A_\delta) \\ &\leqslant F_\infty(v, A_\delta) + \beta(|\xi|^p + 1)\int_{A\setminus A_\delta} \lambda_\infty \, \mathrm{d}x \\ &\leqslant m_{F_\infty}(u_\xi, A_\delta) + \eta + \beta(|\xi|^p + 1)\int_{A\setminus A_\delta} \lambda_\infty \, \mathrm{d}x. \end{aligned}$$

Letting $\eta \to 0^+$ we conclude that

$$m_{F_\infty}(u_\xi, A) \leqslant m_{F_\infty}(u_\xi, A_\delta) + \beta(|\xi|^p + 1)\int_{A\setminus A_\delta} \lambda_\infty \,\mathrm{d}x,$$

and therefore

$$m_{F_\infty}(u_\xi, A) \leqslant \liminf_{\delta\to 0^+} m_{F_\infty}(u_\xi, A_\delta),$$

since $\lambda_\infty \in L^1(A)$.

Conversely, let $\eta > 0$ and choose $v \in W^{1,p}_{0,\lambda_\infty}(A;\mathbb{R}^m) + u_\xi$ such that

$$F_\infty(v, A) \leqslant m_{F_\infty}(u_\xi, A) + \eta. \tag{7.14}$$

Let $(v_j) \subset C_0^\infty(A;\mathbb{R}^m) + u_\xi$ be such that $v_j \to v$ in $W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m)$. We can find $\delta_0 > 0$ small enough so that $(v_j) \subset C_0^\infty(A_\delta;\mathbb{R}^m) + u_\xi$ for every $0 < \delta < \delta_0$, hence

$$m_{F_\infty}(u_\xi, A_\delta) \leqslant F_\infty(v_j, A_\delta) \tag{7.15}$$

for every $j \in \mathbb{N}$. By the continuity of $F_\infty(\cdot, A)$ with respect to the strong convergence of $W^{1,p}_{\lambda_\infty}(A;\mathbb{R}^m)$, (7.14) and (7.15) we deduce that

$$m_{F_\infty}(u_\xi, A_\delta) \leqslant \lim_{j\to\infty} F_\infty(v_j, A_\delta) = F_\infty(v, A_\delta) \leqslant F_\infty(v, A) \leqslant m_{F_\infty}(u_\xi, A) + \eta. \tag{7.16}$$

Letting first $\delta \to 0^+$ and then $\eta \to 0^+$ we eventually get

$$\limsup_{\delta\to 0^+} m_{F_\infty}(u_\xi, A_\delta) \leqslant m_{F_\infty}(u_\xi, A).$$

□

Eventually, we are in a position to prove the following lemma which, in its turn, yields the desired derivation formula (7.2) (cf. (7.1)).

Lemma 7.5. *For a.e. $x \in Q_0$ there holds*

$$\limsup_{\rho\to 0^+} \frac{F_\infty(u_\xi, Q_\rho(x))}{|Q_\rho(x_0)|} = \limsup_{\rho\to 0^+} \frac{m_{F_\infty}(u_\xi, Q_\rho(x))}{|Q_\rho(x)|}.$$

Proof. The proof follows arguing exactly as in [**4**, lemma 3.5], now using lemmas 7.3 and 7.4. □

8. Stochastic homogenization

In this last section, we illustrate an application of the Γ-convergence result theorem 3.2 to the case of stochastic homogenization.

We start by recalling some basic notions and results from ergodic theory.

8.1. Ergodic theory

Let $d \geqslant 1$ be an integer; in all that follows $\mathcal{B}^d$ denotes the Borel σ-algebra of $\mathbb{R}^d$; if $d = 1$ we set $\mathcal{B} := \mathcal{B}^1$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\tau = (\tau_y)_{y\in\mathbb{R}^n}$ denote a group of P-preserving transformations on $(\Omega, \mathcal{F}, P)$; i.e. τ is a family of measurable mappings $\tau_y : \Omega \to \Omega$ satisfying the following properties:

- $\tau_y\tau_{y'} = \tau_{y+y'}$, $\tau_y^{-1} = \tau_{-y}$, for every y, $y' \in \mathbb{R}^n$;
- the map τ_y preserves the probability measure P; i.e. $P(\tau_y E) = P(E)$, for every $y \in \mathbb{R}^n$ and every $E \in \mathcal{F}$;
- for any measurable function φ on Ω, the function $\phi(\omega, x) := \varphi(\tau_y\omega)$ is $\mathcal{F} \otimes \mathcal{B}^n$-measurable on $\Omega \times \mathbb{R}^n$.

If in addition every τ-invariant set $E \in \mathcal{F}$ has either probabilty 0 or 1, then τ is called *ergodic*.

We also need to recall the notion of subadditive process. In what follows $\mathcal{A}_0$ denotes the family of all open, bounded subsets of $\mathbb{R}^n$ with Lipschitz boundary.

Definition 8.1 (Subadditive process). *Let $\tau = (\tau_y)_{y\in\mathbb{R}^n}$ be a group of P-preserving transformations on $(\Omega, \mathcal{F}, P)$. A subadditive process is a function $\mu : \Omega \times \mathcal{A}_0 \to [0, +\infty)$ satisfying the following properties:*

(1) *for every $A \in \mathcal{A}_0$, $\mu(\cdot, A)$ is $\mathcal{F}$-measurable;*

(2) *for every $\omega \in \Omega$, $A \in \mathcal{A}_0$, and $y \in \mathbb{R}^n$*

$$\mu(\omega, A + y) = \mu(\tau_y\omega, A);$$

(3) *for every $\omega \in \Omega$, for every $A \in \mathcal{A}_0$, and for every finite family $(A_i)_{i\in I} \subset \mathcal{A}_0$ of pairwise disjoint sets such that $A_i \subset A$ for every $i \in I$ and $|A \setminus \cup_{i\in I} A_i| = 0$, there holds*

$$\mu(\omega, A) \leqslant \sum_{i\in I} \mu(\omega, A_i);$$

(4) *there exists a constant $c > 0$ such that for every $A \in \mathcal{A}_0$*

$$0 \leqslant \int_\Omega \mu(\omega, A)\,\mathrm{d}P \leqslant c|A|.$$

Moreover, if $\tau := (\tau_y)_{y\in\mathbb{R}^n}$ is ergodic then μ is called a subadditive ergodic process.

We now state a version of the subadditive ergodic theorem, originally proven by Akcoglu and Krengel [2], which is suitable for our purposes (see [26, theorem 4.3]).

Theorem 8.2. *Let $\mu : \Omega \times \mathcal{A}_0 \to [0, +\infty)$ be a subadditive process. Then there exist a $\mathcal{F}$-measurable function $\phi : \Omega \to [0, +\infty)$ and a set $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such*

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that

$$\lim_{t\to\infty}\frac{\mu(\omega,tQ)}{|tQ|}=\phi(\omega),$$

for every $\omega\in\Omega'$ and for every cube Q in $\mathbb{R}^n$ with faces parallel to the coordinate planes.

If in addition μ is ergodic, then ϕ is constant.

For later use we also recall the Birkhoff ergodic theorem. To this end, we preliminarily need to fix some notation. Let φ be a measurable function on $(\Omega,\,\mathcal{F},\,P)$; we denote with $\mathbb{E}[\varphi]$ the expected value of φ; i.e.

$$\mathbb{E}[\varphi]:=\int_\Omega\varphi(\omega)\mathrm{d}P.$$

For every $\varphi\in L^1(\Omega)$ and for every σ-algebra $\mathcal{F}'\subset\mathcal{F}$, we denote with $\mathbb{E}[\varphi|\mathcal{F}']$ the conditional expectation of φ with respect to $\mathcal{F}'$. We recall that $\mathbb{E}[\varphi|\mathcal{F}']$ is the unique $L^1(\Omega)$-function satisfying

$$\int_E\mathbb{E}[\varphi|\mathcal{F}'](\omega)\,\mathrm{d}P=\int_E\varphi(\omega)\,\mathrm{d}P,$$

for every $E\in\mathcal{F}'$.

We now state the following version of the Birkhoff ergodic theorem which is convenient for our purposes.

THEOREM 8.3 (Birkhoff's ergodic theorem). *Let $\varphi\in L^1(\Omega)$, let $\tau=(\tau_y)_{y\in\mathbb{R}^n}$ be a group of P-preserving transformations on $(\Omega,\,\mathcal{F},\,P)$, and let $\mathcal{F}_\tau$ denote the σ-algebra of τ-invariants sets. Then there exists a set $\widetilde{\Omega}\in\mathcal{F}$ with $P(\widetilde{\Omega})=1$ such that*

$$\lim_{t\to\infty}\fint_B\varphi(\tau_{ty}\omega)\,\mathrm{d}y=\mathbb{E}[\varphi|\mathcal{F}_\tau](\omega),\tag{8.1}$$

for every $\omega\in\widetilde{\Omega}$ and for every measurable bounded set $B\subset\mathbb{R}^n$ with $|B|>0$.

REMARK 8.4. We note that if τ is ergodic, then $\mathcal{F}_\tau$ reduces to the trivial σ-algebra, therefore (8.1) becomes

$$\lim_{t\to\infty}\fint_B\varphi(\tau_{ty}\omega)\,\mathrm{d}y=\mathbb{E}[\varphi].\tag{8.2}$$

8.2. Setting of the problem and main results

In this section, we introduce the random integral functionals we are going to analyse. To this end, we preliminarily need to define the class of admissible random weights.

ASSUMPTION 8.5 (Admissible random weights). Let $\tau=(\tau_y)_{y\in\mathbb{R}^n}$ be a group of P-preserving transformations on $(\Omega,\,\mathcal{F},\,P)$. A function $\lambda:\Omega\times\mathbb{R}^n\to[0,\,+\infty)$ is an admissible weight if:

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- λ is $\mathcal{F}\otimes\mathcal{B}^n$-measurable;
- λ is stationary; i.e. $\lambda(\omega,\, x+y)=\lambda(\tau_y\omega,\, x)$, for every $\omega\in\Omega$, $x,\, y\in\mathbb{R}^n$;
- $\lambda(\omega,\,\cdot)\in A_p(K)$, for every $\omega\in\Omega$;
- $\lambda(\cdot,\, 0)>0$ in Ω;
- $\lambda(\cdot,\, 0),\, \lambda(\cdot,\, 0)^{-1/(p-1)}\in L^1(\Omega)$.

REMARK 8.6. We note that $\lambda=\lambda(\omega,\, x)$ is τ-stationary if and only if for every $\omega\in\Omega$ and every $x\in\mathbb{R}^n$ there holds

$$\lambda(\omega,x)=\hat{\lambda}(\tau_x\omega), \tag{8.3}$$

with $\hat{\lambda}(\omega):=\lambda(\omega,\, 0)$.

Since by assumption $\lambda(\omega,\, 0)>0$ for every $\omega\in\Omega$, we then have $\mathbb{E}[\hat{\lambda}|\mathcal{F}'](\omega)>0$, for every $\mathcal{F}'\subset\mathcal{F}$. Moreover, we also observe that if we assume $\hat{\lambda},\, \hat{\lambda}^{-1/(p-1)}\in L^1(\Omega)$, then the Fubini theorem yields $\lambda(\omega,\,\cdot),\, \lambda^{-1/(p-1)}(\omega,\,\cdot)\in L^1_{\rm loc}(\mathbb{R}^n)$, for every $\omega\in\Omega$. However, in order to apply theorem 3.2 we need the stronger condition $\lambda(\omega,\,\cdot)\in A_p(K)$, for every $\omega\in\Omega$.

Below we introduce the notion of stationary random integrand.

DEFINITION 8.7 (Stationary random integrand). *Let $\tau=(\tau_y)_{y\in\mathbb{R}^n}$ be a group of P-preserving transformations on $(\Omega,\, \mathcal{F},\, P)$ and let $\lambda\colon\Omega\times\mathbb{R}^n\to[0,\,+\infty)$ satisfy assumption 8.5.*

(i) *We say that $f:\Omega\times\mathbb{R}^n\times\mathbb{R}^{m\times n}\to[0,\,+\infty)$ is a random integrand if:*

- *f is $(\mathcal{F}\otimes\mathcal{B}^n\otimes\mathcal{B}^{m\times n},\, \mathcal{B})$-measurable;*
- *for every $\omega\in\Omega$ and for every $x\in\mathbb{R}^n$, the two following conditions hold:*

$$\alpha\lambda(\omega,x)(|\xi|^p-1)\leqslant f(\omega,x,\xi)\leqslant\beta\lambda(\omega,x)(|\xi|^p+1), \tag{8.4}$$

for every $\xi\in\mathbb{R}^{m\times n}$ and for some $0<\alpha\leqslant\beta<+\infty$, and

$$|f(\omega,x,\xi_1)-f(\omega,x,\xi_2)|\leqslant L\lambda(\omega,x)(|\xi_1|^{p-1}+|\xi_2|^{p-1}+1)|\xi_1-\xi_2|, \tag{8.5}$$

for every $\xi_1,\, \xi_2\in\mathbb{R}^{m\times n}$ and for some $L>0$.

(ii) *We say that a random integrand f is stationary if for every $\omega\in\Omega$, for every $x,\, y\in\mathbb{R}^n$ and every $\xi\in\mathbb{R}^{m\times n}$ it holds:*

- *$f(\omega,\, x+y,\, \xi)=f(\tau_y\omega,\, x,\, \xi)$.*

(iii) *We say that a stationary random integrand f is ergodic if $\tau=(\tau_y)_{y\in\mathbb{R}^n}$ is ergodic.*

Let f be a stationary random integrand in the sense of definition 8.7. Let $\omega\in\Omega$ be fixed and consider the integral functional $F(\omega):W^{1,1}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)\times\mathcal{A}_0\longrightarrow[0,\,+\infty]$

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defined as

$$F(\omega)(u, A) := \begin{cases} \int_A f(\omega, x, \nabla u)\, \mathrm{d}x & \text{if } u \in W^{1,p}_\lambda(A; \mathbb{R}^m), \\ +\infty & \textit{otherwise}. \end{cases} \tag{8.6}$$

Moreover, for every $\omega \in \Omega$, $A \in \mathcal{A}_0$, and $\xi \in \mathbb{R}^{m\times n}$ set

$$\begin{aligned} m_{F(\omega)}(u_\xi, A) &:= \inf\left\{ \int_A f(\omega, x, \nabla u)\, \mathrm{d}x \colon u \in W^{1,p}_{0,\lambda}(A; \mathbb{R}^m) + u_\xi \right\} \\ &= \inf\left\{ \int_A f(\omega, x, \nabla u + \xi)\, \mathrm{d}x \colon u \in W^{1,p}_{0,\lambda}(A; \mathbb{R}^m) \right\}. \end{aligned} \tag{8.7}$$

The following proposition shows that for every fixed $\xi \in \mathbb{R}^{m\times n}$, the minimization problem in (8.7) defines a subadditive process.

PROPOSITION 8.8. *Let f be a stationary random integrand; let $F(\omega)$ and $m_{F(\omega)}$ be as in* (8.6) *and* (8.7), *respectively. Then for every $\xi \in \mathbb{R}^{m\times n}$ the function*

$$(\omega, A) \mapsto m_{F(\omega)}(u_\xi, A)$$

defines a subadditive process on $(\Omega, \mathcal{F}, P)$.

Moreover, for every $\xi \in \mathbb{R}^{m\times n}$ and $A \in \mathcal{A}_0$

$$0 \leqslant \int_\Omega m_{F(\omega)}(u_\xi, A)\, \mathrm{d}P \leqslant \beta(|\xi|^p + 1)\mathbb{E}[\hat{\lambda}]|A|, \tag{8.8}$$

where $\hat{\lambda}$ is as in (8.3).

Proof. Let $\xi \in \mathbb{R}^{m\times n}$ and $A \in \mathcal{A}_0$ be fixed. We first show that $\omega \mapsto m_{F(\omega)}(u_\xi, A)$ is $\mathcal{F}$-measurable. To this end fix $u \in W^{1,p}_\lambda(A; \mathbb{R}^m)$, then the function $(\omega, x) \mapsto f(\omega, x, \nabla u + \xi)$ is $\mathcal{F} \otimes \mathcal{L}^n$-measurable, hence by Fubini's theorem

$$\omega \mapsto F(\omega)(u + u_\xi, A) = \int_A f(\omega, x, \nabla u + \xi)\, \mathrm{d}x$$

is $\mathcal{F}$-measurable. Observe now that $W^{1,p}_{0,\lambda}(A; \mathbb{R}^m)$ endowed with the norm $\|\nabla \cdot \|_{L^p_\lambda(A;\mathbb{R}^{m\times n})}$ is a separable Banach space and that, by virtue of (8.5), the map $u \mapsto F(\omega)(u + u_\xi, A)$ is continuous with respect to the same norm. Then there exists a countable dense set $D \subset W^{1,p}_{0,\lambda}(A; \mathbb{R}^m)$ such that

$$m_{F(\omega)}(u_\xi, A) = \inf_{u\in D} F(\omega)(u + u_\xi, A),$$

hence the map $\omega \mapsto m_{F(\omega)}(u_\xi, A)$ is $\mathcal{F}$-measurable.

We now show that for every fixed $\xi \in \mathbb{R}^{m\times n}$ there holds

$$m_{F(\omega)}(u_\xi, A+y) = m_{F(\tau_y\omega)}(u_\xi, A),$$

for every $\omega \in \Omega$, $A \in \mathcal{A}_0$ and $y \in \mathbb{R}^n$. Indeed, a change of variables and the stationarity of f yield

$$\begin{aligned} m_{F(\omega)}(u_\xi, A+y) &= \inf\left\{\int_A f(\omega, x+y, \nabla u + \xi)\,\mathrm{d}x \colon u \in W^{1,p}_{0,\lambda(\omega,x+y)}(A;\mathbb{R}^m)\right\} \\ &= \inf\left\{\int_A f(\tau_y\omega, x, \nabla u + \xi)\,\mathrm{d}x \colon u \in W^{1,p}_{0,\lambda(\tau_y\omega,x)}(A;\mathbb{R}^m)\right\} \\ &= m_{F(\tau_y\omega)}(u_\xi, A). \end{aligned}$$

Let $\xi \in \mathbb{R}^{m\times n}$ and $\omega \in \Omega$ be fixed. We now prove that the function $A \mapsto m_{F(\omega)}(u_\xi,\, A)$ is subadditive in the sense of definition 8.1. To this end, let $A \in \mathcal{A}_0$ and let $(A_i)_{i\in I}$ be a finite family of pairwise disjoint sets in $\mathcal{A}_0$ such that $A_i \subset A$, for every $i \in I$, and $|A \setminus \cup_{i\in I} A_i| = 0$. Let $\eta > 0$ and choose $u_i \in W^{1,p}_{0,\lambda}(A_i,\, \mathbb{R}^m)$ such that $F(\omega)(u_i + u_\xi,\, A_i) \leqslant m_{F(\omega)}(u_\xi,\, A_i) + \eta$. Define $u \in W^{1,p}_{0,\lambda}(A;\mathbb{R}^m)$ by setting $u := \sum_{i\in I} u_i\chi_{A_i}$. Then by the locality of $F(\omega)$ we have

$$m_{F(\omega)}(u_\xi, A) \leqslant F(\omega)(u + u_\xi, A) = \sum_{i\in I} F(\omega)(u_i + u_\xi, A_i) \leqslant \sum_{i\in I} m_{F(\omega)}(u_\xi, A_i) + \eta,$$

which proves the subadditivity thanks to the arbitrariness of $\eta > 0$.

Finally, by definition of $m_F(u_\xi,\, A)$, choosing $u = 0$, by (8.4) we have

$$0 \leqslant m_{F(\omega)}(u_\xi, A) \leqslant \beta(|\xi|^p + 1)\int_A \lambda(\omega, x)\,\mathrm{d}x, \tag{8.9}$$

for every $\xi \in \mathbb{R}^{m\times n}$, every $\omega \in \Omega$, and every $A \in \mathcal{A}_0$. Therefore, integrating on Ω both sides of (8.9) and using the stationarity of λ we get

$$0 \leqslant \int_\Omega m_{F(\omega)}(u_\xi, A)\,\mathrm{d}P \leqslant \beta(|\xi|^p + 1)\int_\Omega\int_A \hat{\lambda}(\tau_x\omega)\,\mathrm{d}x\,\mathrm{d}P = \beta(|\xi|^p + 1)\mathbb{E}[\hat{\lambda}]|A|,$$

where to establish the last equality we have used the Tonelli theorem together with a change of variables in ω. Eventually, we deduce both (8.8) and that $(\omega,\, A) \mapsto m_{F(\omega)}(u_\xi,\, A)$ is a subadditive process. □

By combining proposition 8.8 together with the subadditive ergodic theorem 8.2 we are now able to establish the existence of the homogenization formula which will eventually define the integrand of the Γ-limit (cf. theorem 8.12 below).

Proposition 8.9. *Let f be a stationary random integrand. Then there exist a set $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ and a $\mathcal{F}\otimes\mathcal{B}^{m\times n}$-measurable function $f_{\mathrm{hom}} : \Omega \times \mathbb{R}^{m\times n} \to$*

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$[0, +\infty)$ such that for every $\omega \in \Omega'$, $\xi \in \mathbb{R}^{m\times n}$, and every cube Q in $\mathbb{R}^n$ there holds

$$f_{\text{hom}}(\omega,\xi) := \lim_{t\to\infty} \frac{m_{F(\omega)}(u_\xi, tQ)}{t^n|Q|}.$$

Moreover, for every $\omega \in \Omega'$, f_{hom} satisfies the following conditions:

$$\alpha\, \mathbb{E}[\hat{\lambda}|\mathcal{F}_\tau](\omega)\left(\frac{1}{K}|\xi|^p - 1\right) \leqslant f_{\text{hom}}(\omega,\xi) \leqslant \beta\, \mathbb{E}[\hat{\lambda}|\mathcal{F}_\tau](\omega)(|\xi|^p + 1), \tag{8.10}$$

$$|f_{\text{hom}}(\omega,\xi_1) - f_{\text{hom}}(\omega,\xi_2)| \leqslant L'\, \mathbb{E}[\hat{\lambda}|\mathcal{F}_\tau](\omega)(|\xi_1|^{p-1} + |\xi_2|^{p-1} + 1)|\xi_1 - \xi_2|, \tag{8.11}$$

for every ξ, ξ_1, $\xi_2 \in \mathbb{R}^{m\times n}$ and for some $L' > 0$, where $\hat{\lambda}$ is as in (8.3).

If in addition f is ergodic, then f_{hom} does not depend on ω and

$$f_{\text{hom}}(\xi) = \lim_{t\to\infty} \frac{1}{t^n} \int_\Omega m_{F(\omega)}(u_\xi, Q_t(0))\, \mathrm{d}P, \tag{8.12}$$

for every $\xi \in \mathbb{R}^{m\times n}$. Moreover, in this case (8.10) and (8.11) become, respectively,

$$\alpha\, \mathbb{E}[\hat{\lambda}]\left(\frac{1}{K}|\xi|^p - 1\right) \leqslant f_{\text{hom}}(\xi) \leqslant \beta\, \mathbb{E}[\hat{\lambda}](|\xi|^p + 1) \tag{8.13}$$

and

$$|f_{\text{hom}}(\xi_1) - f_{\text{hom}}(\xi_2)| \leqslant L'\, \mathbb{E}[\hat{\lambda}](|\xi_1|^{p-1} + |\xi_2|^{p-1} + 1)|\xi_1 - \xi_2|, \tag{8.14}$$

for every ξ, ξ_1, $\xi_2 \in \mathbb{R}^{m\times n}$.

Proof. Let $\xi \in \mathbb{R}^{m\times n}$ be fixed; theorem 8.2 and proposition 8.8 ensure the existence of a set $\Omega^\xi \in \mathcal{F}$ with $P(\Omega^\xi) = 1$ and of a $\mathcal{F}$-measurable function $\phi^\xi : \Omega \longrightarrow [0, +\infty)$ such that

$$\phi^\xi(\omega) = \lim_{t\to\infty} \frac{m_{F(\omega)}(u_\xi, tQ)}{|tQ|}, \tag{8.15}$$

for every $\omega \in \Omega^\xi$ and for every cube Q in $\mathbb{R}^n$.

Now, let $t > 0$ and denote with $Q_t = Q_t(0)$ the cube centred at the origin and with side length t. Let $f_{\text{hom}} : \Omega \times \mathbb{R}^{m\times n} \to [0, +\infty)$ be the function defined as

$$f_{\text{hom}}(\omega,\xi) := \limsup_{t\to\infty} \frac{m_{F(\omega)}(u_\xi, Q_t)}{|Q_t|}.$$

Let $\omega \in \Omega$ and $A \in \mathcal{A}_0$ be fixed; we start by showing that the function

$$\xi \mapsto \frac{m_{F(\omega)}(u_\xi, A)}{|A|}$$

is locally Lipschitz continuous. To this end, let ξ_1, $\xi_2 \in \mathbb{R}^{m\times n}$, let $\eta > 0$ be arbitrary, and let $u \in W^{1,p}_{0,\lambda}(A;\mathbb{R}^m)$ be such that

$$F(\omega)(u + u_{\xi_2}, A) \leqslant m_{F(\omega)}(u_{\xi_2}, A) + \eta.$$

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Then, appealing to (8.5) and to the Hölder inequality we deduce

$$
\begin{aligned}
&\frac{m_{F(\omega)}(u_{\xi_1},A)}{|A|}-\frac{m_{F(\omega)}(u_{\xi_2},A)}{|A|}\\
&\quad\leqslant \frac{1}{|A|}\big(F(\omega)(u+u_{\xi_1},A)-F(\omega)(u+u_{\xi_2},A)+\eta\big)\\
&\quad\leqslant \frac{1}{|A|}\int_A |f(\omega,x,\nabla u+\xi_1)-f(\omega,x,\nabla u+\xi_2)|\,\mathrm{d}x+\frac{\eta}{|A|}\\
&\quad\leqslant \frac{L}{|A|}\int_A \lambda(\omega,x)|\xi_1-\xi_2|(|\nabla u+\xi_1|^{p-1}+|\nabla u+\xi_2|^{p-1}+1)\,\mathrm{d}x+\frac{\eta}{|A|}\\
&\quad\leqslant \frac{L}{|A|}C(p)|\xi_1-\xi_2|\left(\int_A\lambda(\omega,x)\,\mathrm{d}x\right)^{1/p}\\
&\qquad\times\left(\int_A\lambda(\omega,x)(|\xi_1|^p+|\xi_2|^p+|\nabla u+\xi_2|^p+1)\,\mathrm{d}x\right)^{\frac{p-1}{p}}+\frac{\eta}{|A|},
\end{aligned}
\tag{8.16}
$$

where $C(p)>0$ depends only on p. By using (8.4) (see also (8.9)) we get

$$
\begin{aligned}
\alpha\int_A\lambda(\omega,x)(|\nabla u+\xi_2|^p-1)\,\mathrm{d}x &\leqslant F(\omega)(u+u_{\xi_2},A)\leqslant m_{F(\omega)}(u_{\xi_2},A)+\eta\\
&\leqslant \beta(|\xi_2|^p+1)\int_A\lambda(\omega,x)\,\mathrm{d}x+\eta.
\end{aligned}
\tag{8.17}
$$

Therefore, plugging (8.17) into (8.16) gives

$$
\begin{aligned}
&\frac{m_{F(\omega)}(u_{\xi_1},A)}{|A|}-\frac{m_{F(\omega)}(u_{\xi_2},A)}{|A|}\\
&\quad\leqslant \frac{C\,L}{|A|}|\xi_1-\xi_2|\left(\int_A\lambda(\omega,x)\,\mathrm{d}x\right)^{1/p}\\
&\qquad\times\left((|\xi_1|^{p-1}+|\xi_2|^{p-1}+1)\left(\int_A\lambda(\omega,x)\,\mathrm{d}x\right)^{(p-1)/p}+\eta^{(p-1)/p}\right)+\frac{\eta}{|A|},
\end{aligned}
$$

where $C>0$ depends on p, α, β. Hence, by the arbitrariness of $\eta>0$ we get

$$
\frac{m_{F(\omega)}(u_{\xi_1},A)}{|A|}-\frac{m_{F(\omega)}(u_{\xi_2},A)}{|A|}\leqslant L'\fint_A\lambda(\omega,x)\,\mathrm{d}x\,\big(|\xi_1|^{p-1}+|\xi_2|^{p-1}+1\big)|\xi_1-\xi_2|.
\tag{8.18}
$$

Thus, the claim simply follows by interchanging the role of ξ_1 and ξ_2.

Now, choose $A=tQ$ in (8.18) with Q cube of $\mathbb{R}^n$ and $t>0$. By the stationarity of λ and a change of variables we obtain

$$
\begin{aligned}
&\frac{m_{F(\omega)}(u_{\xi_1},tQ)}{|tQ|}-\frac{m_{F(\omega)}(u_{\xi_2},tQ)}{|tQ|}\\
&\quad\leqslant L'\fint_Q\hat{\lambda}(\tau_{tx}\omega)\,\mathrm{d}x\,\big(|\xi_1|^{p-1}+|\xi_2|^{p-1}+1\big)|\xi_1-\xi_2|
\end{aligned}
\tag{8.19}
$$

and, as above, the other inequality follows by exchanging the role of ξ_1 and ξ_2. Therefore, taking the lim sup as $t \to \infty$ and invoking theorem 8.3, we deduce the existence of a set $\widetilde{\Omega} \in \mathcal{F}$ with $P(\widetilde{\Omega}) = 1$ such that

$$\limsup_{t\to\infty} \left| \frac{m_{F(\omega)}(u_{\xi_1}, tQ)}{|tQ|} - \frac{m_{F(\omega)}(u_{\xi_2}, tQ)}{|tQ|} \right| \leqslant L'\, \mathbb{E}[\hat{\lambda}|\mathcal{F}_\tau](\omega) \left(|\xi_1|^{p-1} + |\xi_2|^{p-1} + 1\right)|\xi_1 - \xi_2|, \tag{8.20}$$

for every $\omega \in \widetilde{\Omega}$. We also observe that choosing in (8.19) $Q = Q_1(0)$, it is immediate to check that $f_{\text{hom}}(\omega, \cdot)$ satisfies the local Lipschitz condition (8.20), for every $\omega \in \widetilde{\Omega}$.

Set $\Omega' := (\cap_{\xi \in \mathbb{Q}^{m\times n}} \Omega^\xi) \cap \widetilde{\Omega}$, clearly $P(\Omega') = 1$ and (8.15) holds true for every fixed $\xi \in \mathbb{Q}^{m\times n}$ and every $\omega \in \Omega'$. Let now $\xi \in \mathbb{R}^{m\times n}$ be fixed and let $(\xi_j) \subset \mathbb{Q}^{m\times n}$ be such that $\xi_j \to \xi$, as $j \to \infty$. For $\omega \in \Omega'$ we have

$$\left| f_{\text{hom}}(\omega, \xi) - \frac{m_{F(\omega)}(u_\xi, tQ)}{|tQ|} \right| \leqslant |f_{\text{hom}}(\omega, \xi) - f_{\text{hom}}(\omega, \xi_j)| + \left| f_{\text{hom}}(\omega, \xi_j) - \frac{m_{F(\omega)}(u_{\xi_j}, tQ)}{|tQ|} \right| + \left| \frac{m_{F(\omega)}(u_{\xi_j}, tQ)}{|tQ|} - \frac{m_{F(\omega)}(u_\xi, tQ)}{|tQ|} \right|.$$

Then, view of (8.11), (8.15), and (8.20) we get that for every $j \in \mathbb{N}$ there holds

$$\limsup_{t\to\infty} \left| f_{\text{hom}}(\omega, \xi) - \frac{m_{F(\omega)}(u_\xi, tQ)}{|tQ|} \right| \leqslant 2L'\, \mathbb{E}[\hat{\lambda}|\mathcal{F}_\tau](\omega) \left(|\xi|^{p-1} + |\xi_j|^{p-1} + 1\right)|\xi - \xi_j|.$$

Thus, by letting $j \to \infty$ we obtain

$$f_{\text{hom}}(\omega, \xi) = \lim_{t\to\infty} \frac{m_{F(\omega)}(u_\xi, tQ)}{t^n|Q|},$$

for every $\omega \in \Omega'$ and every $\xi \in \mathbb{R}^{m\times n}$, as desired.

Then, it only remains to show that $f_{\text{hom}}(\omega, \cdot)$ satisfies the growth condition (8.10) for every $\omega \in \Omega'$. The growth condition from above readily follows from

$$\frac{m_{F(\omega)}(u_\xi, tQ)}{t^n|Q|} \leqslant \beta(|\xi|^p + 1) ⨍_{tQ} \lambda(\omega, x)\, \mathrm{d}x = \beta(|\xi|^p + 1) ⨍_{Q} \hat{\lambda}(\tau_{tx}\omega)\, \mathrm{d}x \tag{8.21}$$

passing to the limit as $t \to \infty$, and using theorem 8.3.

We now establish the growth condition from below. To this end let $u \in W^{1,p}_{0,\lambda}(tQ; \mathbb{R}^m)$ be arbitrary; then Hölder's inequality and (8.4) give

$$\begin{aligned} \alpha\, |\xi|^p |tQ|^p &= \alpha \left(\int_{tQ} |\nabla u + \xi|\, \mathrm{d}x \right)^p \\ &\leqslant F(\omega)(u + u_\xi, tQ) \left(\int_{tQ} \lambda(\omega, x)^{-1/(p-1)}\, \mathrm{d}x \right)^{p-1} \\ &\quad + \alpha \left(\int_{tQ} \lambda(\omega, x)\, \mathrm{d}x \right) \left(\int_{tQ} \lambda(\omega, x)^{-1/(p-1)}\, \mathrm{d}x \right)^{p-1}. \end{aligned}$$

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Dividing both sides by $|tQ|^p$ and taking the infimum over $W^{1,p}_{0,\lambda}(tQ;\mathbb{R}^m)$ we get

$$\alpha|\xi|^p \left(\fint_{tQ} \lambda(\omega,x)^{-1/(p-1)}\,\mathrm{d}x\right)^{-(p-1)} - \alpha\left(\fint_{tQ}\lambda(\omega,x)\,\mathrm{d}x\right) \leqslant \frac{m_{F(\omega)}(u_\xi,tQ)}{|tQ|},$$

then, recalling that $\lambda(\omega,\cdot)\in A_p(K)$ we find

$$\alpha\left(\fint_{tQ}\lambda(\omega,x)\,\mathrm{d}x\right)\left(\frac{1}{K}|\xi|^p-1\right) \leqslant \frac{m_{F(\omega)}(u_\xi,tQ)}{|tQ|}.$$

Therefore, passing to the limit as $t\to\infty$ and using again the Birkhoff ergodic theorem we finally obtain the growth conditions from below in (8.10).

We note that the $\mathcal{F}\otimes\mathcal{B}^{m\times n}$-measurability of f_{hom} follows from the $\mathcal{F}$-measurability of $\omega\mapsto f_{\mathrm{hom}}(\omega,\xi)$ and the continuity of $\xi\mapsto f_{\mathrm{hom}}(\omega,\xi)$.

If f is ergodic, then theorem 8.2 ensures that f_{hom} does not depend on ω. Moreover, by (8.21) and the Birkhoff ergodic theorem we can invoke a generalized version of the dominated convergence theorem to deduce (8.12). Eventually, (8.13) and (8.14) follow, respectively, by integrating (8.10) and (8.11) on Ω and using the definition of conditional expectation. □

Remark 8.10. From the proof of proposition 8.9 it can be actually seen that in the ergodic case f_{hom} satisfies the standard growth conditions

$$\alpha\,\mathbb{E}[\hat{\lambda}^{-1/(p-1)}]^{1-p}(|\xi|^p-1)\leqslant f_{\mathrm{hom}}(\xi)\leqslant\beta\,\mathbb{E}[\hat{\lambda}](|\xi|^p+1),$$

for every $\xi\in\mathbb{R}^{m\times n}$ (and similarly in the general stationary case), which then reduce to those established in [**15**, **16**, **28**] when $\lambda\equiv 1$.

Now, let $(\varepsilon_k)\searrow 0$ be a vanishing sequence of strictly positive real numbers and let f be a stationary random integrand. For $\omega\in\Omega$ let $F_k(\omega): W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)\times\mathcal{A}_0\longrightarrow [0,+\infty]$ be the functionals defined as

$$F_k(\omega)(u,A) := \begin{cases}\displaystyle\int_A f\left(\omega,\frac{x}{\varepsilon_k},\nabla u\right)\,\mathrm{d}x & \text{if } u\in W^{1,p}_{\lambda_k}(A;\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases} \tag{8.22}$$

where for every $\omega\in\Omega$ and $x\in\mathbb{R}^n$ we set

$$\lambda_k(\omega,x) := \lambda\left(\omega,\frac{x}{\varepsilon_k}\right), \tag{8.23}$$

with λ satisfying assumption 8.5.

Remark 8.11. If λ_k is as in (8.23) then by assumption 8.5 and the Birkhoff ergodic theorem there exists $\Omega'\in\mathcal{F}$ with $P(\Omega')=1$ such that (3.8) holds true for every $\omega\in\Omega'$.

The following homogenization theorem is the main result of this section.

THEOREM 8.12 (Stochastic homogenization). *Let f be a stationary random integrand and let $F_k(\omega)$ be as in (8.22). Then there exists $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such that for every $\omega \in \Omega'$ and every $A \in \mathcal{A}_0$*

$$\Gamma(L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m))\text{-}\lim_{k\to\infty} F_k(\omega)(u, A) = F_{\mathrm{hom}}(\omega)(u, A),$$

where $F_{\mathrm{hom}}(\omega) : W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m) \times \mathcal{A}_0 \longrightarrow [0, +\infty]$ is the random functional defined as

$$F_{\mathrm{hom}}(\omega)(u, A) := \begin{cases} \displaystyle\int_A f_{\mathrm{hom}}(\omega, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(A;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \tag{8.24}$$

with f_{hom} as in proposition 8.9.

If in addition f is ergodic, then F_{hom} is deterministic with integrand given by

$$f_{\mathrm{hom}}(\xi) = \lim_{t\to\infty} \frac{1}{t^n} \int_\Omega \inf\left\{ \int_{Q_t(0)} f(\omega, x, \nabla u + \xi)\,\mathrm{d}x \colon u \in W^{1,p}_{0,\lambda}(Q_t(0);\mathbb{R}^m) \right\} \mathrm{d}P,$$

for every $\xi \in \mathbb{R}^{m\times n}$.

Proof. Let $\Omega' \in \mathcal{F}$ be the measurable set whose existence is ensured by proposition 8.9. In all that follows we fix ω in Ω'.

In view of remark 8.11, theorem 3.2 provides us with a subsequence (k_h) such that for every $A \in \mathcal{A}_0$ the functionals $F_{k_h}(\omega)(\cdot, A)$ Γ-converge to the integral functional $F_\infty(\omega)(\cdot, A)$ with respect to the strong $L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$-convergence, where $F_\infty(\omega) : W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m) \times \mathcal{A}_0 \longrightarrow [0, +\infty]$ is given by

$$F_\infty(\omega)(u, A) := \begin{cases} \displaystyle\int_A f_\infty(\omega, x, \nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(A;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

we note that f_∞ is nondegenerate since, by assumption, $\mathbb{E}[\hat{\lambda}|\mathcal{F}_\tau](\omega) > 0$. Moreover, again invoking theorem 3.2, we have

$$f_\infty(\omega, x, \xi) = \limsup_{\rho\to 0^+} \frac{1}{\rho^n} \lim_{h\to\infty} m_{F_{k_h}(\omega)}(u_\xi, Q_\rho(x)), \tag{8.25}$$

for a.e. $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^{m\times n}$. Hence, from (8.25) by a change of variables we immediately get

$$\begin{aligned} f_\infty(\omega, x, \xi) &= \limsup_{\rho\to 0^+} \lim_{h\to\infty} \frac{\varepsilon_{k_h}^n}{\rho^n} m_{F(\omega)}\left(u_\xi, Q_{\rho/\varepsilon_{k_h}}\left(\frac{x}{\varepsilon_{k_h}}\right)\right) \\ &= \lim_{t\to\infty} \frac{1}{t^n} m_{F(\omega)}(u_\xi, Q_t(0)) = f_{\mathrm{hom}}(\omega, \xi), \end{aligned} \tag{8.26}$$

where (8.26) follows by proposition 8.9 by setting $t := \rho/\varepsilon_{k_h} \to \infty$, as $h \to \infty$.

As a consequence, we deduce that f_∞ is independent of the subsequence (k_h) and hence the Urysohn property of Γ-convergence (see [**14**, proposition 8.3]) allows us to conclude that the whole sequence $(F_k(\omega))$ Γ-converges to $F_{\text{hom}}(\omega)$, for every $\omega \in \Omega'$.

Eventually, in the ergodic case the claim readily follows from the corresponding statement in proposition 8.9. □

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