1 Introduction to general relativity 1: Kinematics and Einstein equations

In this chapter, I will give a lightning review of the basics of general relativity, from how it is built, to its kinematics, and finally to its dynamics, given by the Einstein equation.

1.1 Intrinsically curved spacetime and the geometry of general relativity

I will start with the need for and meaning of intrinsically curved spacetime, which will lead us to the geometry of general relativity.

But since general relativity is a generalization of special relativity, I will review its basic ideas in order to be able to generalize it.

1.1.1 Special relativity

Special relativity was developed as a result of the experimental observation that the speed of light in a vacuum is equal to a constant in all inertial reference frames, where the constant can be put to 1, so that $c = 1$. This then becomes a postulate of special relativity.

As a result, we find that the line element, or the infinitesimal distance between two points, must be taken in *spacetime*, not just in space, in order to be invariant under transformations of coordinates between any inertial reference frames. This invariant distance is then

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{1.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$ is the Minkowski metric. This now takes the role of the invariant length $d\vec{x}^2$ in Newtonian physics, which is invariant under rotations of space at a given time.

The symmetry group that leaves ds^2 invariant is $SO(1, 3)$, or in a general dimension $SO(1, d-1)$, called the Lorentz group. It is a generalization of the group $SO(d-1)$ of spatial rotations that leaves $d\vec{x}^2$ invariant. The corresponding Lorentz transformations are linear transformations of the coordinates that generalize rotations, $x'^i = \Lambda^i{}_j x^j$, where $\Lambda \in SO(d-1)$, which leaves invariant $d\vec{x}^2$. Now, instead, we have

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu; \quad \Lambda^\mu{}_\nu \in SO(1, 3), \tag{1.2}$$

which leaves invariant ds^2.

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In conclusion, special relativity is defined by the following statement: Physics is Lorentz invariant or covariant (under $SO(1, d-1)$ transformations). It replaces the statement of Newtonian or Galilean physics that physics is invariant under the Galilean group, of spatial rotations, with no action on time.

1.1.2 General relativity

Now to define general relativity, we need to consider the most general line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \tag{1.3}$$

where $g_{\mu\nu}(x)$ is a symmetric matrix of functions called "the metric." By extension, sometimes one calls the corresponding ds^2 the metric. Moreover, consider here that x^μ make up an arbitrary parametrization of spacetime, that is, are arbitrary coordinates on a manifold.

Example 1 $\boldsymbol{S^2}$ **in angular coordinates.** To understand the notation, consider the usual case of a two-dimensional sphere, described in terms of angles. Then the line element is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2, \tag{1.4}$$

so $x^\mu = (\theta, \phi)$. Then it follows that $g_{\theta\theta} = 1, g_{\phi\phi} = \sin\theta$, and $g_{\theta\phi} = 0$.

Example 2 $\boldsymbol{S^2}$ **as an embedding in three-dimensional Euclidean space.** We can describe the sphere also by embedding it in three Euclidean dimensions, meaning as we usually understand it, as an object in three-dimensional space, with the metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{1.5}$$

defined by the constraint

$$x_1^2 + x_2^2 + x_3^2 = R^2. \tag{1.6}$$

Differentiating the constraint, we obtain

$$\begin{aligned} &2(x_1dx_1 + x_2dx_2 + x_3dx_3) = 0 \\ &\Rightarrow dx_3 = -\frac{x_1dx_1 + x_2dx_2}{x_3} = -\frac{x_1}{\sqrt{R^2 - x_1^2 - x_2^2}}dx_1 - \frac{x_2}{\sqrt{R^2 - x_1^2 - x_2^2}}dx_2, \end{aligned} \tag{1.7}$$

and by substituting it back into the Euclidean metric, we obtain the *induced metric on the* S^2,

$$\begin{aligned} ds^2_{\text{induced}} &= dx_1^2\left(1 + \frac{x_1^2}{R^2 - x_1^2 - x_2^2}\right) + dx_2^2\left(1 + \frac{x_2^2}{R^2 - x_1^2 - x_2^2}\right) + 2dx_1dx_2\frac{x_1x_2}{R^2 - x_1^2 - x_2^2} \\ &= g_{\mu\nu}(x^\rho)dx^\mu dx^\nu. \end{aligned} \tag{1.8}$$

This was an example of a curved d-dimensional space obtained by embedding it into a flat (Euclidean or Minkowski) $(d+1)$-dimensional space. We can ask: Is this always possible? The answer is no. To see this, first note that $g_{\mu\nu}$ is a symmetric matrix, with $d(d+1)/2$ arbitrary components. Then, the general coordinate transformations $x'^\mu = x'^\mu(x^\rho)$ correspond

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to d arbitrary functions, which can be used to put d components to zero, thus remaining $d(d-1)/2$ independent components of $g_{\mu\nu}$. On the other hand, if we were to embed the manifold M^d into $(d+1)$-dimensional Euclidean space E^{d+1}, there would be a unique coordinate x^{d+1} written as a function of the others, $x^{d+1} = x^{d+1}(x^\rho)$, as in the example of the sphere. We see that $d(d-1)/2 = 1$ is true only in the particular case of $d = 2$.

We note here that general coordinate transformations $x'^\mu = x'^\mu(x^\rho)$ act on *the fields* $g_{\mu\nu}(x)$, that is, the functions of spacetime, allowing us to fix their d components, so we have a redundancy similar to the one in gauge transformations in field theory; thus, we can say that general coordinate invariance is a kind of gauge invariance. We will see that we can turn this observation into a useful tool later on.

If we cannot always embed the manifold M^d into $(d+1)$-dimensional space, can we do it by adding more extra dimensions? At first sight, we would say yes, perhaps by adding not 1, but $d(d-1)/2$ dimensions in general. But actually, the situation is worse than that: We also need to make, *case by case*, a discrete choice of the *signature* of the space into which we are embedding a manifold.

Even in the simplest case of two-dimensional surfaces, we need to make this choice: Do we embed two-dimensional surfaces into a 3-dimensional Euclidean space like in the case of the sphere, with signature $(+,+,+)$, or into a three-dimensional Minkowski space, with signature $(-,+,+)$? Note that, since the multiplication of the metric by a sign changes only the convention, these are the only possibilities in three dimensions (the $(-,-,-)$ and $(-,-,+)$ ones are related by multiplication by a sign).

The example of embedding Lobachevsky space into Minkowski space is a famous one, defined by the constraint

$$x^2 + y^2 - z^2 = -R^2. \tag{1.9}$$

Lobachevsky space cannot be embedded into Euclidean space but only into Minkowski space with the metric

$$ds^2 = dx^2 + dy^2 - dz^2, \tag{1.10}$$

with the minus sign in the same place as in the constraint. We might think that this is because the signature on the two-dimensional Lobachevsky space is Minkowski, $(-,+)$ (equivalent to $(+,-)$), but that is wrong also: The signature on the space is two-dimensional Euclidean, so $(+,+)$ or equivalently, $(-,-)$. That is, $\det g_{\mu\nu} > 0$ and not < 0. Indeed, by differentiating the constraint, like for the sphere, we obtain

$$dz = \frac{xdx + ydy}{z} = \frac{xdx + ydy}{\sqrt{R^2 + x^2 + y^2}}, \tag{1.11}$$

and by replacing in the Minkowski metric, we obtain the induced metric on the Lobachevsky space,

$$ds^2_{\text{induced}} = dx^2 + dy^2 + \frac{(xdx + ydx)^2}{R^2 + x^2 + y^2} \equiv g_{\mu\nu}dx^\mu dx^\nu, \tag{1.12}$$

which is positive definite, so $\det g_{\mu\nu} > 0$.

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Finally, this means that even two-dimensional surfaces of Euclidean signature can be embedded in three dimensions, but either in Euclidean or in Minkowski ones, depending on the surface. In higher dimensions, the number of choices for the signature becomes even larger, so defining spaces by embedding is possible, but very complicated and not very useful.

Instead, we must consider spaces as intrinsically curved, without embedding, and that in turn leads to non-Euclidean, Riemannian, geometry. This observation was believed to be first made by Gauss, who tried to measure if our space is actually curved (but failed, of course; on scales of even kilometers, space is flat to a very high accuracy).

In curved spaces, to define geometry, we must first define the analog of "straight lines" of Euclidean geometry, which are the geodesics, also defined as lines of the shortest distance $\int_a^b ds$ between two points a and b. In non-Euclidean geometry, a triangle made by two geodesics has the sum of its inner angles, $\alpha + \beta + \gamma \neq \pi$. In Euclidean geometry, of course, the sum is *equal to* π by a theorem.

On spaces like S^2 of "positive curvature," $R > 0$, we have $\alpha + \beta + \gamma > \pi$, as we can easily see in the following example: Consider a triangle made by two meridian lines starting from the North Pole and ending on an Equator line. The meridian lines with the Equator line make $\pi/2$ each, so $\alpha + \beta + \gamma > \pi$.

But that is not the only possibility. On a space like Lobachevsky space, we can check that $\alpha + \beta + \gamma < \pi$, and we call this a space of "negative curvature," $R < 0$. We will see in Section 1.3 what $R < 0$ and $R > 0$ means.

In conclusion, we see that for general relativity, we will need intrinsically curved spacetimes, with non-Euclidean geometry, with a general metric $g_{\mu\nu}(x)$, and acted upon by general coordinate transformations that act as gauge transformations.

1.2 Einstein's theory of general relativity

Einstein thought of defining general relativity in order to modify Newton's gravity at high gravitational acceleration $\vec{g}$ and high velocity $\vec{v}$ in order to make it compatible with special relativity. The need for that arose also because of experimental results: The deflection of light by the Sun using only special relativity is a factor of 1/2 off the actual result.

The construction of general relativity was based on two physical assumptions:

(1) **Gravity is geometry**

That is, matter follows geodesics (paths of shortest distance) in curved spacetimes, and to us, it appears as the effect of gravity.

Pictorially, consider a planar rubber sheet and put a heavy ball at a point on it: It will curve the sheet locally. Then, when throwing a light ball on the sheet, the local disturbance deflects it (think of a golfer doing a putt and the golf ball just missing the hole). Of course, this is just a pictorial way of describing the phenomenon; otherwise, it is a cheat: The sheet curves because of the terrestrial gravity it feels, and the curvature is only of space, not of spacetime. But this is a nice way of viewing what happens.

(2) **Matter sources gravity**
This means matter generates the gravitational field that is equated with the curvature of the geometry of spacetime from the first assumption.

These two physical assumptions were then translated into two physical principles with a mathematical formulation, defining the *kinematics* of general relativity, plus one equation for the dynamics, that is, Einstein's equation.

(A) Physics is invariant (or, more generally, covariant) under general coordinate transformations, which generalizes the Lorentz invariance or covariance in the case of special relativity.

For a general coordinate transformation $x'^{\mu} = x'^{\mu}(x^{\nu})$, we obtain

$$ds^2 = g_{\rho\sigma}(x)dx^{\rho}dx^{\sigma} = g'_{\mu\nu}dx'^{\mu}dx'^{\nu}, \tag{1.13}$$

giving the transformation rules for the field $g_{\mu\nu}$ (thought of as a field in spacetime),

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x)\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}. \tag{1.14}$$

This transformation is like a gauge invariance, and physics must be invariant or covariant with respect to it.

(B) The equivalence principle.

In Newtonian theory, there are *a priori* two masses: one is the inertial mass m_i, appearing in Newton's law of force, that is, $\vec{F} = m_i\vec{a}$, and the other is the gravitational mass m_g, appearing in Newton's gravitational law, that is, $\vec{F}_G = m_g\vec{g}$.

The equality of the two masses is the mathematical form of the equivalence principle, that is,

$$m_i = m_g. \tag{1.15}$$

In more physical terms, we say that "there is no difference between gravity and local acceleration." We can also explain this using Einstein's *gedanken (thought) experiment*. Consider a person inside a freely falling elevator with no windows. Then, by performing local experiments inside the elevator, the person cannot distinguish between being weightless and being inside a freely falling elevator. Of course, the *locality* condition is important, because if one is allowed to probe large regions of space, then he or she will note that there are tidal forces – gravity acting at different points in different directions (all pointing toward the center of the Earth). Also, locality in time is important; otherwise, eventually the elevator will hit the hard surface of the Earth, ending the experiment.

On the basis of the above principles, we now turn to constructing the kinematics of general relativity.

First, consider an infinitesimal general coordinate transformation, $x'^{\mu} = x^{\mu} - \xi^{\mu}$, with ξ^{μ} small, and we want to describe it as a gauge transformation. Then,

$$\begin{aligned} g'_{\mu\nu}(x^{\lambda} - \xi^{\lambda}) &= (\delta^{\rho}_{\mu} + \partial_{\mu}\xi^{\rho})(\delta^{\sigma}_{\nu} + \partial_{\nu}\xi^{\sigma})g_{\rho\sigma}(x) \\ &= g'_{\mu\nu}(x^{\lambda}) - (\partial_{\lambda}g'_{\mu\nu}(x))\xi^{\lambda}, \end{aligned} \tag{1.16}$$

where in the first equality, we used the transformation law of $g_{\mu\nu}$, and in the second equality, we used the Taylor expansion.

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Equating the two, we obtain

$$\delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) \simeq \xi^\lambda \partial_\lambda g'_{\mu\nu}(x) + (\partial_\mu \xi^\rho) g_{\rho\nu}(x) + (\partial_\nu \xi^\sigma) g_{\mu\sigma}(x)$$
$$\simeq \xi^\lambda \partial_\lambda g_{\mu\nu}(x) + (\partial_\mu \xi^\rho) g_{\rho\nu}(x) + (\partial_\nu \xi^\rho) g_{\mu\rho}(x). \qquad (1.17)$$

In formula (1.17), the first term was from the Taylor expansion, so it is just a translation, while the last two terms correspond to a generalized gauge transformation with parameter ξ^ρ instead of the usual α of gauge theory (with $\delta A_\mu = \partial_\mu \alpha$). Since there are two indices on $g_{\mu\nu}$, unlike the case of A_μ, there are two terms, one with ∂_μ and the other with ∂_ν, and the extra metric is needed in order to lower the index on ξ^ρ.

Note that in the global case (with ξ^ρ independent of position), there is only the translation term. Therefore, we can say that *general coordinate transformations are a local version of translations*, and moreover, *General relativity is a "gauge theory of translations."*

1.3 Kinematics

We now move on to defining kinematics per se. We first ask: What is a good variable that corresponds to A_μ in our gauge theory analogy? And correspondingly, what is the respective field strength $F_{\mu\nu}$?

Our first guess would be the metric $g_{\mu\nu}$ itself. We saw that it has $(d(d-1)/2)$-*independent* components (or degrees of freedom, off-shell). However, we know that locally (in a small enough neighborhood), every space looks flat (which in our case means locally Minkowski). In mathematical terms, locally we can always find coordinates such that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \mathcal{O}(x^2). \qquad (1.18)$$

This means also that locally we can define Lorentz transformations, and so there is an $SO(1,3)$ (or $SO(1,d-1)$ in general dimension) invariance, called the *local (x-dependent) Lorentz invariance*.

In any case, this means that $g_{\mu\nu}$ is not a good measure of the curvature of space, but also not quite like the gauge field A_μ either, since A_μ can locally be put to 0, whereas $g_{\mu\nu}$ can only be put to $\eta_{\mu\nu}$.

To understand better what happens, defining general relativity tensors through a simple generalization of special relativity tensors, we have:

– Contravariant tensors A^μ, that are the objects that transform as dx^μ,

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \Rightarrow A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu. \qquad (1.19)$$

– Covariant tensors B_μ that are the objects that transform as ∂_μ,

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \Rightarrow B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu. \qquad (1.20)$$

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– Mixed tensors that transform as products, for example,

$$T'^{\mu}{}_{\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} T^{\rho}{}_{\sigma}(x), \tag{1.21}$$

and with an obvious generalization to $T^{\mu_1,\dots,\mu_n}_{\nu_1,\dots,\nu_m}$.

Given these definitions, we turn back to the question of what is a good analog of the gauge field A_μ? We can now rephrase this question. Since in gauge theory the covariant derivative $D_\mu = \partial_\mu - iA_\mu$ transforms covariantly, that is, like a covariant vector, we can ask the same question in general relativity as follows: How do we construct a gravitationally covariant derivative?

Since, as we saw, the local Lorentz group is $SO(1, d-1)$, and this is in some sense the gauge group we are looking for, we note that for an $SO(p, q)$ group, the adjoint representation, for the gauge field, is written in terms of the fundamental indices a, b as (ab) (antisymmetric in them), so the gauge covariant derivative on a generic field in the fundamental representation, ϕ^a, is (lowering one index b on the gauge field to have a match with the general relativity construction)

$$D_\mu \phi^a = \partial_\mu \phi^a + (A^a{}_b)_\mu \phi^b. \tag{1.22}$$

In our case, we define something similar to that, with the only difference being that we identify fundamental gauge and spacetime indices, and write for the gravitationally covariant derivative of a contravariant tensor (so that the index is up, just like a on ϕ^a)

$$D_\mu T^\nu = \partial_\mu T^\nu + (\Gamma^\nu{}_\sigma)_\mu T^\sigma, \tag{1.23}$$

where the object $\Gamma^\nu{}_{\sigma\mu}$ is called the "Christoffel symbol," and in Equation (1.23), we put brackets around Γ, just like for the gauge field, but we did not need to, since the gauge and spacetime indices are the same. This object is then the "gauge field of gravity" that we were looking for.

We can easily generalize its action on tensors, by taking into account the position of the indices (only the sign in front is not defined this way), so that

$$D_\mu T^\rho{}_\nu = \partial_\mu T^\rho{}_\nu + \Gamma^\rho{}_{\sigma\mu} T^\sigma{}_\nu - \Gamma^\sigma{}_{\mu\nu} T^\rho{}_\sigma. \tag{1.24}$$

To calculate $\Gamma^\mu{}_{\nu\rho}$ in terms of the metric $g_{\mu\nu}$, we consider the following: If $\Gamma^\mu{}_{\nu\rho}$ is a gauge field, then it should be possible to put it locally to zero by a general coordinate transformation (a gauge transformation), when the space becomes locally flat. At the same time, we saw that $g_{\mu\nu}$ is locally $\eta_{\mu\nu}$. Then

$$D_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma{}_{\nu\rho} g_{\sigma\rho} - \Gamma^\sigma{}_{\rho\mu} g_{\sigma\nu} = 0 \tag{1.25}$$

locally, but we saw that a tensor transforms by multiplication under general coordinate transformations, so it must be that the result is 0 globally as well (in any coordinate system).

This is an equation whose unique solution is

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right). \tag{1.26}$$

The proof of this is left as an exercise. Note that here we define the inverse metric $g^{\mu\nu}$ as the matrix inverse of $g_{\nu\rho}$, so $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$.

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Further, we define the Riemann tensor as the analog of the field strength of the $SO(p,q)$ gauge field, $F^{ab}_{\mu\nu}$, namely, since

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + A^{ac}_\mu A^{cb}_\nu - A^{ac}_\nu A^{cb}_\mu, \tag{1.27}$$

it follows that we can define

$$(R^\mu{}_\nu)_{\rho\sigma}(\Gamma) = \partial_\rho(\Gamma^\mu{}_\nu)_\sigma - \partial_\sigma(\Gamma^\mu{}_\nu)_\rho + (\Gamma^\mu{}_\lambda)_\rho(\Gamma^\lambda{}_\nu)_\sigma - (\Gamma^\mu{}_\lambda)_\sigma(\Gamma^\lambda{}_\nu)_\rho. \tag{1.28}$$

Here we have put brackets around the “gauge indices” to make the analogy with the gauge case more obvious, but, as in the case of the Christoffel symbol, this is not necessary, since gauge and spacetime indices are the same now.

Unlike the gauge case, now we can define the contractions of the Riemann tensor as the Ricci tensor,

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \tag{1.29}$$

and as the Ricci scalar,

$$R = R_{\mu\nu}g^{\mu\nu}. \tag{1.30}$$

Finally, the Ricci scalar, by virtue of being a scalar, is invariant under general coordinate transformations, so it is a true invariant measure of the curvature of space at a point, the object we were looking for. In particular, when we said that the sphere was an object of positive curvature $R > 0$ and the Lobachevsky space of negative curvature $R < 0$, we were referring to the Ricci scalar.

The symmetry properties of the Riemann tensor are as follows. First, there are a number of properties that are obvious from the gauge field strength analogy:

1. Since for a gauge field we have the Bianchi identity $(D_{[\mu}F_{\nu\rho]})^a = 0$, we now also have the gravitational Bianchi identity

$$D_{[\lambda}(R^\mu{}_\nu)_{\rho\sigma]} = 0, \tag{1.31}$$

where antisymmetry only acts on $[\lambda\rho\sigma]$.

2, 3. From the antisymmetry of the spacetime indices of the field strength, and of the fundamental indices in the adjoint of $SO(p,q)$, we have (note that we have lowered the first index with a metric on the Riemann tensor for simplicity)

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}. \tag{1.32}$$

4. Not a symmetry property but the action on a tensor is defined through two covariant derivatives. Since for a gauge field we have $[D_\mu, D_\nu] = F_{\mu\nu}$, which can act on tensors, we now have

$$[D_\mu, D_\nu]T_\rho = -(R^\sigma{}_\rho)_{\mu\nu}T_\sigma = R_{\rho\sigma\mu\nu}T^\sigma. \tag{1.33}$$

5, 6. But then, we have other properties that are not obtained this way, and we must check them from the definition of the Riemann tensor:

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \quad R_{\mu[\nu\rho\sigma]} = 0. \tag{1.34}$$

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1.4 Actions in general relativity

We now move on to writing actions for general relativity fields. To generalize a special relativity action to a general relativity action, we first change special relativity tensors into general relativity tensors, in particular derivatives ∂_μ into gravitationally covariant derivatives D_μ and the metric $\eta_{\mu\nu}$ into the general $g_{\mu\nu}$. The only thing left is to generalize the integration measure from d^dx. From the transformation rule for dx^μ, we find the one for d^dx,

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu \Rightarrow d^dx = \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right) d^dx'. \tag{1.35}$$

To compensate this, we note that we have the transformation of the metric

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \Rightarrow \det g'_{\mu\nu} \equiv g' = \left[\det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\right]^2 g, \tag{1.36}$$

which means that the invariant measure is (the minus is for the reality of the square root in the Minkowski signature case)

$$\sqrt{-g'}d^dx' = \sqrt{-g}d^dx. \tag{1.37}$$

1.5 The Einstein–Hilbert action

Now we are ready to write an action for the dynamics of general relativity, the Einstein–Hilbert action, as promised. This is an *a priori* independent postulate, which doesn't follow from the previous ones. The role of the action for the dynamics is to match experiment, and there is no fundamental principle behind it, unlike the case of the kinematics.

As is familiar from quantum field theory, to construct actions, we go in increasing order of mass dimension of possible terms in the Lagrangian.

The simplest possibility is to integrate a constant (one times a dimensionful constant) with the invariant measure, so a term of dimension 0,

$$S_0 = \Lambda \int d^dx\sqrt{-g}. \tag{1.38}$$

This doesn't give the correct dynamics. In fact, by varying it, we obtain $\delta g^{\mu\nu} g_{\mu\nu} = 0$, so the equation of motion is nonsensical, $g_{\mu\nu} = 0$. We will see in Section 2.3 that such a term can in fact be added, with a very small constant Λ in front (of dimension d), called a cosmological constant term, but it is not understood as part of the gravity action, but usually as part of the matter action.

The next term, at dimension 2, is the Ricci scalar (since $R \sim \partial\Gamma + \Gamma\Gamma$, and $\Gamma \sim g^{-1}\partial g$, and $g_{\mu\nu}$ is dimensionless, it follows that R is dimension 2, as it has two derivatives), integrated with the invariant measure, with a certain dimensionful constant in

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front. This is in fact the Einstein–Hilbert action, the correct action for gravity,*

$$S_{\text{E–H}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} R. \tag{1.39}$$

Note that the factor in front had to involve G_N, since we need to obtain Newtonian gravity in the weak field, small velocities limit, and the actual factor is taken such that we obtain exactly the Newtonian potential $U_N(r)$. This action matches experiments to a very high accuracy: Every experiment we did until now confirms it. Note that we can write the coefficient in terms of the d-dimensional Planck mass,

$$\frac{1}{16\pi G_N} = \frac{M_{\text{Pl}}^{d-2}}{2}. \tag{1.40}$$

However, note that in principle we can have corrections to this action, coming from terms of higher mass dimension, and such terms in fact do appear because of quantum corrections in string theory or supergravity, for instance. The next possible terms, at mass dimension 4, are (with coefficients that are implicitly coming from quantum corrections, due to the power of M_{Pl})

$$\sim \int d^d x \sqrt{-g} M_{\text{Pl}}^{d-4} R^2, \tag{1.41}$$

where R^2 can mean the Ricci scalar squared, but also $R_{\mu\nu}R^{\mu\nu}$ or $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$.

We are now ready to write Einstein's equations in vacuum, the equations of motion of the Einstein–Hilbert action. Writing the determinant of $g_{\mu\nu}$ as an exponential, we can calculate its variation,

$$g = \det g_{\mu\nu} = e^{\text{Tr}\log\, g_{\mu\nu}} \Rightarrow \frac{\delta\sqrt{-g}}{\sqrt{-g}} = -\frac{1}{2} g_{\mu\nu}\delta g^{\mu\nu}, \tag{1.42}$$

where we have used $g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}$. Since $R = g^{\mu\nu}R_{\mu\nu}$, we must only calculate $\delta R_{\mu\nu}$.

But, as left to prove in Exercise 4, we have

$$g^{\mu\nu}\delta R_{\mu\nu} = D_\mu (g^{\nu\rho}\delta\Gamma^\mu{}_{\nu\rho} - g^{\mu\nu}\delta\Gamma^\rho{}_{\nu\rho}) \equiv D_\mu U^\mu. \tag{1.43}$$

Then, since

$$D_\mu U^\mu = \partial_\mu U^\mu + \Gamma^\mu{}_{\sigma\mu} U^\sigma, \tag{1.44}$$

and

$$\Gamma^\mu{}_{\sigma\mu} = \frac{1}{2} g^{\mu\lambda}\partial_\sigma g_{\mu\lambda} = \frac{\partial_\sigma\sqrt{-g}}{\sqrt{-g}}, \tag{1.45}$$

we have

$$\sqrt{-g} D_\mu U^\mu = \partial_\mu(\sqrt{-g} U^\mu), \tag{1.46}$$

and the term becomes a total derivative, that is, a boundary term.

* Note on conventions: If we use the $+---$ metric, we get a $-$ in front of the action, since $R = g^{\mu\nu}R_{\mu\nu}$ and $R_{\mu\nu}$ is invariant under constant rescalings of $g_{\mu\nu}$.

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Finally then, the variation of the Einstein–Hilbert action is

$$\delta S_{\text{E-H}} = \frac{1}{16\pi G_N}\int d^dx\sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right), \tag{1.47}$$

so the Einstein equations in vacuum are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{1.48}$$

When adding matter, for the energy–momentum tensor in curved space, we have the Belinfante formula,

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{1.49}$$

It is worth noting that, even if we are in flat space, we can formally introduce a nontrivial metric and vary with respect to it as in (1.49), after which we put back $g_{\mu\nu} = \eta_{\mu\nu}$, in order to obtain the Belinfante energy–momentum tensor, uniquely defined (even if, otherwise, for instance for electromagnetism, there are ambiguities in the definition of $T_{\mu\nu}$).

Now reading this formula in reverse, we can find the variation of the matter action as

$$\delta S_{\text{matter}} = -\frac{1}{2}\int d^dx\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu}, \tag{1.50}$$

so that the total variation of the action is

$$\delta(S_{\text{gravity}} + S_{\text{matter}}) = \frac{1}{16\pi G_N}\int d^dx\sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 8\pi G_N T_{\mu\nu}\right) = 0, \tag{1.51}$$

giving the Einstein equations with matter,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}. \tag{1.52}$$

To understand better $T_{\mu\nu}$, we give a couple of examples. The kinetic action for a scalar in Minkowski space is

$$S_{M,\phi} = -\frac{1}{2}\int d^dx(\partial_\mu\phi)(\partial_\nu\phi)\eta^{\mu\nu}. \tag{1.53}$$

In curved space, ∂_μ becomes D_μ; however, on a scalar, they are the same, $\partial_\mu\phi = D_\mu\phi$, so the scalar kinetic action in curved space is

$$S_\phi = -\frac{1}{2}\int d^dx\sqrt{-g}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu}. \tag{1.54}$$

Then the resulting energy momentum tensor for the scalar kinetic term is

$$T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\rho\phi)^2. \tag{1.55}$$

For electromagnetism, the action in flat space is

$$S_{\text{M,e-m}} = -\frac{1}{4}\int d^dxF_{\mu\nu}F_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}, \tag{1.56}$$

which easily translates into curved space as

$$S_{\text{e-m}} = -\frac{1}{4}\int d^dx\sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}, \tag{1.57}$$

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leading to the (Belinfante) energy–momentum tensor

$$T^{\mathrm{e-m}}_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{1.58}$$

Important concepts to remember

- In general relativity, space is intrinsically curved.
- In general relativity, physics is invariant under general coordinate transformations.
- Gravity is the same as curvature of space, or gravity = local acceleration, or $m_i = m_g$.
- General relativity can be thought of as a gauge theory of local translations.
- General relativity tensors are a generalization of special relativity tensors.
- The Christoffel symbol acts like a gauge field of gravity, giving the covariant derivative.
- Its field strength is the Riemann tensor, whose scalar contraction, the Ricci scalar, is an invariant measure of curvature.
- One postulates the action for gravity as $(1/(16\pi G_N))\int d^dx\sqrt{-g}R$, giving Einstein's equations.
- Adding a matter action, obtained from the flat space action by generalization, we obtain the Einstein's equations with matter.

References and further reading

For a very basic (but not too explicit) introduction to general relativity, you can try the general relativity chapter in Peebles [1]. A good and comprehensive treatment is done in [2], which has a very good index, and detailed information, but one needs to be selective in reading only the parts you are interested in. An advanced treatment, with an elegance and concision that a theoretical physicist should appreciate, is found in the general relativity section of Landau and Lifshitz [3], though it might not be the best introductory book. A more advanced and thorough book for the theoretical physicist is Wald [4].

Exercises

(1) Parallel the derivation in the text to find the metric on the two-dimensional sphere in its usual form,

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.59}$$

from the three-dimensional Euclidean metric, using the embedding in terms of θ, ϕ of the three-dimensional Euclidean coordinates.

(2) Show that the metric $g_{\mu\nu}$ is covariantly constant ($D_\mu g_{\nu\rho} = 0$) by substituting the Christoffel symbols.

(3) Prove that we have the relation

$$(D_\mu D_\nu - D_\nu D_\mu)A_\rho = R^\sigma{}_{\rho\mu\nu}A_\sigma, \tag{1.60}$$

if A_σ is a covariant vector.

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(4) The Christoffel symbol $\Gamma^{\mu}_{\nu\rho}$ is not a tensor, and can be put to zero at any point by a choice of coordinates (Riemann normal coordinates, for instance), but $\delta\Gamma^{\mu}_{\nu\rho}$ is a tensor. Show that the variation of the Ricci scalar can be written as

$$\delta R = \delta^{\rho}_{\mu} g^{\nu\sigma} (D_{\rho} \delta\Gamma^{\mu}_{\nu\sigma} - D_{\sigma} \delta\Gamma^{\mu}_{\nu\rho}) + R_{\nu\sigma} \delta g^{\nu\sigma}. \tag{1.61}$$